

Lyapunov-Based Range and Motion Identification For A Nonaffine Perspective Dynamic System

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Abstract—Many applications require the interpretation of the Euclidean motion of features of a 3-dimensional (3D) object through 2D images. In this paper, the range and the Euclidean coordinates of an object undergoing general affine motion are determined for a paraboloid imaging system. Unlike image systems that are based on a planar image surface (or spherical or ellipsoidal surfaces), the perspective dynamic system resulting from the paraboloid projected image does not maintain an affine form. Because of the nonaffine form, existing range identification observers can not be directly utilized. The contribution of the current result is the development of a nonlinear state estimator that can be applied to the nonaffine perspective dynamic system to determine the range and Euclidean coordinates of an object feature without the use of linear approximations. The nonlinear estimator asymptotically determines the range information from a single camera provided some observability conditions are satisfied and that the Euclidean motion parameters are known. The proposed technique is developed through a Lyapunov-based design and stability analysis, and simulation results are provided that illustrate the performance of the state estimator.

I. INTRODUCTION

Autonomous vehicle/robotic guidance, navigation and control applications typically require the interpretation of the Euclidean motion of features of a 3-dimensional (3D) object through 2D images that are perspectively¹ projected from the 3D feature. Determining the Euclidean motion of an object through the image projection is challenging because the distance from the camera to the object along the focal length (i.e., the range to the object) is an unmeasurable time-varying signal that appears nonlinearly in the projected image-space. Although the problem of determining the Euclidean coordinates of a moving object from its 2D images is inherently nonlinear, typical results are based on linearization based methods such as extended Kalman filtering [2] and [6]. Classical structure from motion (SFM) results, such as [8] and [17], are examples of such linearization based methods.

Several researchers have recently investigated the problem of determining the unknown states (i.e., the Euclidean coordinates) when the motion parameters are known. For example, a discontinuous observer was developed in [7] to exponentially identify range information of features from successive images of a camera where the object model is based on known skew-symmetric affine motion parameters. Motivated to generalize the object motion beyond the skew-symmetric form in [7],

Chen and Kano developed a new discontinuous observer in [3] that exponentially forces the state observation error to be uniformly ultimately bounded (UUB) for known motion parameters. In comparison to the UUB result in [3], a continuous observer was constructed in [5] to asymptotically identify the range information for a general affine system with known motion parameters (i.e., the result in [5] eliminated the skew-symmetric assumption and yielded an asymptotic result with a continuous observer). More recently, Ma et al. developed a state estimation strategy in [9] and [10] for affine systems with known motion parameters where only a single homogeneous observation point is provided (i.e., a single image coordinate).

All of the aforementioned results are based on the assumption that a conventional planar imaging surface is utilized. The use of a planar imaging surface is restrictive for some applications due to limitations in the field-of-view (FOV). To improve the FOV researchers have proposed the use of spherical, elliptical, or paraboloidal imaging surfaces [11]; rotating imaging systems [1]; fish-eye lenses [16]; catadioptric lenses [12] or a cluster of cameras [14]. As described in [11], a perspective dynamic system (PDS) retains an affine form for some of these imaging surfaces (e.g., planar, spherical, or ellipsoidal). That is, the projection of the affine Euclidean motion of an object onto some imaging surfaces yields an affine PDS. In previous results, the affine form of the PDS is transformed into the following nonlinear system:

$$\begin{aligned}\dot{x}_1 &= \omega^T(x_1, u)x_2 + \phi(x_1, u) \\ \dot{x}_2 &= g(x_1, x_2, u) \\ y &= x_1\end{aligned}\tag{1}$$

where $\omega(x_1, u)$ and $g(x_1, x_2, u)$ are nonlinear functions of their arguments. Once the PDS is written as in (1), the identifier based observer (IBO) proposed in [7] can be applied directly to determine the object range and motion [11]. However, the PDS is not guaranteed to maintain an affine form in general (e.g., a paraboloid image surface does not preserve the affine form). As stated in [11], the IBO (or most of the existing perspective nonlinear observers) can not be directly applied for projections on a paraboloid. In [11], Ma et.al. suggested that range identification could be achieved for a paraboloid image surface using a linear approximation-based observer where the state estimation of the original nonlinear system is carried out by a sequence of approximate linear time-varying observers originally proposed in [10].

¹Other projective models (e.g., Orthographic Projection) have also been used in literature for vision research; however, the most commonly accepted model is the perspective projection.

The contribution of the current paper is the development of a nonlinear estimator that can be used to identify range information (and hence, the Euclidean coordinates) from a nonaffine PDS without the use of linear approximations as in [11]. The structure of the nonlinear observer is based on our previous observer in [5]; however, new observability conditions are imposed due to the nonaffine form of the projection resulting from the paraboloid image surface. A Lyapunov-based analysis is used to prove the range and the 3D coordinates of an object moving with general affine Euclidean motion (and nonaffine image dynamics) are asymptotically identified. Numerical simulation results are provided that illustrate the performance of the estimator.

II. OBJECT MOTION MODEL

A. Affine Euclidean Motion

Consider an object feature undergoing an affine motion described by [5]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (2)$$

where $x_1(t)$, $x_2(t)$, $x_3(t) \in \mathbb{R}$ denote the unmeasurable Euclidean coordinates of an object feature along the X , Y , and Z axes of an inertial reference frame, respectively, with the Z axis being perpendicular with an image plane formed by a camera (i.e., the coordinate $x_3(t)$ denotes the depth from the image plane to the Euclidean object feature along the optical axis Z). In (2), the parameters $a_{i,j}(t) \in \mathbb{R}$ and $b_i(t) \forall i, j = 1, 2, 3$ denote known motion parameters [3], [15]. The affine motion dynamics introduced in (2) are expressed in a general form that describes an object motion that undergoes a rotation, translation, and linear deformation [15].

Assumption 1: The known motion parameters $a_{i,j}(t)$ and $b_i(t) \forall i, j = 1, 2, 3$ introduced in (2) are bounded functions of time where the parameters $a_{i,j}(t)$ are first order differentiable, and the parameters $b_i(t)$ are second order differentiable.

B. Nonaffine Perspective Projection

The projection of the Euclidean coordinates onto a paraboloid imaging surface (see Fig. 1) is defined as follows:

$$X_p = x_1 x_3 / L, \quad Y_p = x_2 x_3 / L, \quad Z_p = x_3^2 / L \quad (3)$$

where $X_p(t)$, $Y_p(t)$, $Z_p(t) \in \mathbb{R}$ denote the projected coordinates of $x_1(t)$, $x_2(t)$, $x_3(t)$, and $L \in \mathbb{R}$ is defined as

$$L = x_1^2 + x_2^2. \quad (4)$$

The image-space coordinates, denoted by $y(t) \in \mathbb{R}^4$, can then be defined as follows:

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix}^T = \begin{bmatrix} X_p & Y_p & Z_p & 1/L \end{bmatrix}^T \quad (5)$$

where $y_1(t)$ and $y_2(t)$ are the measured pixel coordinates from the camera, $y_3(t)$ is computed from the measured pixel coordinates as

$$y_3 = y_1^2 + y_2^2, \quad (6)$$

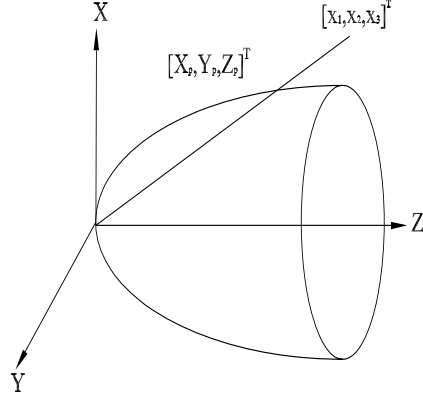


Fig. 1. A paraboloid imaging surface centered at $[0,0,0]^T$.

and $y_4(t)$ contains the unknown range information [11]. The projection onto the paraboloid can be expressed by the following nonaffine PDS [11]:

$$\dot{y}_1 = a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + a_{31}\frac{y_1^2}{y_3} \quad (7)$$

$$+ a_{32}\frac{y_1y_2}{y_3} + a_{33}y_1 - 2a_{11}\frac{y_1^3}{y_3} - 2(a_{12} + a_{21})\frac{y_1^2y_2}{y_3} - 2a_{22}\frac{y_1y_2^2}{y_3} - 2a_{13}y_1^2 - 2a_{23}y_1y_2 + f_1$$

$$\dot{y}_2 = a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + a_{31}\frac{y_1y_2}{y_3} \quad (8)$$

$$+ a_{32}\frac{y_2^2}{y_3} + a_{33}y_2 - 2a_{11}\frac{y_1^2y_2}{y_3} - 2(a_{12} + a_{21})\frac{y_1y_2^2}{y_3} - 2a_{22}\frac{y_2^3}{y_3} - 2a_{13}y_1y_2 - 2a_{23}y_2^2 + f_2$$

$$\dot{y}_3 = 2a_{31}y_1 + 2a_{32}y_2 + 2a_{33}y_3 \quad (9)$$

$$- 2a_{11}y_1^2 - 2a_{22}y_2^2 - 2(a_{12} + a_{21})y_1y_2 - 2a_{13}y_1y_3 - 2a_{23}y_2y_3 + f_3$$

$$\dot{y}_4 = -\frac{2a_{11}y_1^2y_4}{y_3} - 2a_{23}y_2y_4 \quad (10)$$

$$- 2b_1y_4y_5 - \frac{2a_{22}y_2^2y_4}{y_3} - 2a_{13}y_1y_4 - 2(a_{12} + a_{21})\frac{y_1y_2y_4}{y_3} - 2b_2y_4y_6$$

where $f_1(y_1, y_5, y_6, y_7)$, $f_2(y_2, y_5, y_6, y_7)$, $f_3(y_3, y_5, y_6, y_7) \in \mathbb{R}$ are unmeasurable signals defined as

$$f \triangleq \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}^T = \Omega_1 \begin{bmatrix} y_5 & y_6 & y_7 \end{bmatrix}^T. \quad (11)$$

In (11), $\Omega_1(y_1, y_2, y_3) \in \mathbb{R}^{3 \times 3}$ denotes the following matrix of known and measurable signals:

$$\Omega_1 \triangleq \begin{bmatrix} (b_3 - 2b_1y_1) & -2b_2y_1 & b_1 \\ -2b_1y_2 & (b_3 - 2b_2y_2) & b_2 \\ -2b_1y_3 & -2b_2y_3 & 2b_3 \end{bmatrix}, \quad (12)$$

and the unmeasurable auxiliary signals $y_5(y_1, y_3, y_4)$, $y_6(y_2, y_3, y_4)$, $y_7(y_3, y_4) \in \mathbb{R}$ are defined as

$$\begin{bmatrix} y_5 & y_6 & y_7 \end{bmatrix}^T = \frac{1}{L} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \\ = \begin{bmatrix} y_1 \sqrt{\frac{y_4}{y_3}} & y_2 \sqrt{\frac{y_4}{y_3}} & \sqrt{y_3 y_4} \end{bmatrix}^T. \quad (13)$$

To facilitate the subsequent development, the PDS in (7)-(9) can be rewritten as

$$\begin{bmatrix} \dot{y}_1 & \dot{y}_2 & \dot{y}_3 \end{bmatrix}^T = \Omega_2 + f \quad (14)$$

where $\Omega_2(y_1, y_2, y_3) \in \mathbb{R}^3$ denotes a vector of measurable and known signals.

Assumption 2: The image-space feature coordinates $y_1(t)$, $y_2(t)$ and $y_3(t)$ are bounded functions of time.

Assumption 3: The object feature motion avoids the degenerate case where the feature intersects the image plane (i.e., $x_3(t) \neq 0$).

Assumption 4: The object feature does not collide with the imaging surface (i.e., $L > Z_p$). Hence, from (3) and (5) $y_4(t) \in \mathcal{L}_\infty$. Moreover, the object feature is not a vanishing point since $L \in \mathcal{L}_\infty$.

Assumption 5: The matrix $\Omega_1(y_1, y_2, y_3)$ introduced in (12) is invertible provided

$$b_3 \neq 0 \quad (15)$$

and

$$b_3^2 - 2b_2b_3y_2 + b_2^2y_3 - 2b_1b_3y_1 + b_1^2y_3 \neq 0. \quad (16)$$

After utilizing (6), the condition in (16) can be written as

$$(y_1 - k_1)^2 + (y_2 - k_2)^2 \neq 0 \quad (17)$$

where $k_1(t)$ and $k_2(t) \in \mathbb{R}$ are auxiliary terms defined as

$$k_1 = \frac{b_1b_3}{b_1^2 + b_2^2} \quad k_2 = \frac{b_2b_3}{b_1^2 + b_2^2} \quad (18)$$

Geometrically, the observability condition in (17) indicates that the projection of the object feature cannot intersect the point (k_1, k_2) . For the special case when $b_1 = b_2 = 0$, (16) reduces to (15). Based on (17) and (18) the observability condition in (15) is equivalent to the physical property given in Assumption 3. That is if $b_3 = 0$, then (3)-(6) indicate that $x_3(t) = 0$.

Remark 1: Assumptions 1-4 are standard assumptions that are practical properties of the physical system rather than assumptions.

Remark 2: Based on Assumptions 1-4, the expressions given in (7)-(10) and (21) can be used to determine that $\dot{y}_i(t)$, $\Omega_1(y_1, y_2, y_3)$, and $f(y_1, y_2, y_3, y_5, y_6, y_7) \in \mathcal{L}_\infty \forall i = 1, 2, \dots, 4$. Given that these signals are bounded, Assumptions

1-4 can be used to determine that $\dot{f}(\cdot)$ and $\ddot{f}(\cdot) \in L_\infty$.

III. RANGE AND MOTION IDENTIFICATION

A. Objective

The objective in this paper is to identify the unmeasurable state $y_4(t)$ of the PDS described by (7)-(14). From (13) and the fact that $y_1(t)$, $y_2(t)$ and $y_3(t)$ are measurable, if $y_4(t)$ could be identified then the complete Euclidean coordinates of the feature can be determined. To achieve this objective, an estimator is constructed based on the unmeasurable image-space dynamics for $y(t)$. To quantify the objective, a measurable estimation error, denoted by $e(t) \in \mathbb{R}^3$, is defined as follows:

$$e \triangleq \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}^T \\ = \begin{bmatrix} y_1 - \hat{y}_1 & y_2 - \hat{y}_2 & y_3 - \hat{y}_3 \end{bmatrix}^T \quad (19)$$

where $\hat{y}_i(t) \in \mathbb{R} \forall i = 1, 2, 3$ denotes a subsequently designed estimator. An unmeasurable² filtered estimation error, denoted by $r(t) \in \mathbb{R}^3$, is also defined as

$$r \triangleq \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}^T = \dot{e} + \alpha e \quad (20)$$

where $\alpha \in \mathbb{R}^{3 \times 3}$ denotes a diagonal matrix of positive constant gains.

The error systems in (19) and (20) are defined based on the goal to prove that the PDS in (7)-(9) can be identified (i.e., that $f(y_1, y_2, y_3, y_5, y_6, y_7)$ can be identified). If $f(y_1, y_2, y_3, y_5, y_6, y_7)$ can be identified, and assuming the conditions in (15) and (16) are satisfied, then (11) can be used to identify $y_5(y_1, y_3, y_4)$, $y_6(y_2, y_3, y_4)$, and $y_7(y_3, y_4)$. If $y_5(y_1, y_3, y_4)$, $y_6(y_2, y_3, y_4)$, and $y_7(y_3, y_4)$ can be identified, then (13) can be used to compute $y_4(t)$ as follows:

$$y_4 = \frac{y_3(y_5^2 + y_6^2 + y_7^2)}{y_1^2 + y_2^2 + y_3^2}. \quad (21)$$

B. Estimator Design and Error System

Based on the PDS in (14), the estimation objective, and the subsequent analysis, the following estimation signals are defined

$$\begin{bmatrix} \dot{\hat{y}}_1 & \dot{\hat{y}}_2 & \dot{\hat{y}}_3 \end{bmatrix}^T = \Omega_2 + \hat{f}(t) \quad (22)$$

where $\hat{f}(t) \triangleq [\hat{f}_1(t), \hat{f}_2(t), \hat{f}_3(t)]^T \in \mathbb{R}^3$ denotes a subsequently designed estimate for $f(y_1, y_2, y_3, y_5, y_6, y_7)$ introduced in (11). The following error dynamics are obtained after taking the time derivative of $e(t)$ and utilizing (14) and (22):

$$\dot{e} = f - \hat{f}. \quad (23)$$

The time-derivative of (20) can be expressed as

$$\dot{r} = \dot{f} - \dot{\hat{f}} + \alpha (f - \hat{f}) \quad (24)$$

after utilizing (23) and the time derivative of (23). Based on the structure of (24) and the subsequent analysis, $\hat{f}(t)$ is designed

²The filtered estimation signal is unmeasurable due to a dependence on the unmeasurable terms $f_1(y_1, y_5, y_6, y_7)$, $f_2(y_2, y_5, y_6, y_7)$, $f_3(y_3, y_5, y_6, y_7)$.

as follows [5]:

$$\dot{\hat{f}} = -(k_s + \alpha)\hat{f} + \gamma \text{sgn}(e) + \alpha k_s e \quad (25)$$

where $k_s, \gamma \in \mathbb{R}^{3 \times 3}$ denote diagonal matrices of positive constant estimation gains, and the notation $\text{sgn}(\cdot)$ is used to indicate a vector with the standard signum function applied to each element of the argument. After substituting (25) into (24) and then adding and subtracting the terms $k_s f(y_1, y_2, y_3, y_5, y_6, y_7)$, the following expression can be obtained:

$$\dot{r} = \eta - k_s r - \gamma \text{sgn}(e) \quad (26)$$

where $\eta(t) \triangleq [\eta_1 \ \eta_2 \ \eta_3]^T \in \mathbb{R}^3$ is defined as

$$\eta \triangleq \dot{\hat{f}} + (k_s + \alpha) \hat{f}. \quad (27)$$

Remark 3: The structure of the estimator in (25) contains discontinuous terms; however, as discussed in [5], the overall structure of the estimator is continuous (i.e., $\hat{f}(t)$ is continuous).

Remark 4: Once $y_4(t)$ is identified, the complete 3D Euclidean coordinates of the object feature can be determined using (5) and (13). Provided the observability conditions given in (15) and (16) are satisfied $y_4(t)$ can be identified if $\hat{f}(t)$ approaches $f(t)$ as $t \rightarrow \infty$ (i.e., $\hat{y}_1(t)$, $\hat{y}_2(t)$ and $\hat{y}_3(t)$ approach $y_1(t)$, $y_2(t)$ and $y_3(t)$ as $t \rightarrow \infty$) since the parameters $b_i(t) \ \forall i = 1, 2, 3$ are assumed to be known, and $y_1(t)$, $y_2(t)$ and $y_3(t)$ are measurable. To prove that $\hat{f}(t)$ approaches $f(t)$ as $t \rightarrow \infty$, the subsequent development will focus on proving that $\|\dot{e}(t)\| \rightarrow 0$ and $\|e(t)\| \rightarrow 0$ as $t \rightarrow \infty$ based on (19) and (23).

IV. ANALYSIS

The following theorem and associated proof can be used to conclude that the observer design of (22) and (25) can be used to identify the unmeasurable state $y_4(t)$ (i.e., the object feature range and motion can be determined) provided the observability conditions in (15) and (16) are satisfied.

Theorem 1: Given the PDS in (7)-(13), the unmeasurable state $y_4(t)$ (and hence, the Euclidean coordinates of the object feature) can be asymptotically determined from the estimator in (22) and (25) provided the elements of the constant diagonal matrix γ introduced in (25) are selected according to the sufficient conditions

$$\begin{aligned} \gamma_1 &\geq |\eta_1| + |\dot{\eta}_1| & \gamma_2 &\geq |\eta_2| + |\dot{\eta}_2| \\ \gamma_3 &\geq |\eta_3| + |\dot{\eta}_3| \end{aligned} \quad (28)$$

where $\eta(t)$ is defined in (27), and the observability conditions introduced in (15) and (16) are satisfied.

Proof: Consider a non-negative function $V(t) \in \mathbb{R}$ as follows (i.e., a Lyapunov function candidate):

$$V \triangleq \frac{1}{2} r^T r. \quad (29)$$

After taking the time derivative of (29) and substituting for the error system dynamics given in (26), the following expression

can be obtained:

$$\dot{V} = -r^T k_s r + (\dot{e} + \alpha e)(\eta - \gamma \text{sgn}(e)). \quad (30)$$

After integrating (30) and exploiting the fact that

$$\xi_i \cdot \text{sgn}(\xi_i) = |\xi_i| \ \forall \xi_i \in \mathbb{R},$$

the following inequality can be obtained:

$$\begin{aligned} V(t) &\leq V(t_0) - \int_{t_0}^t (r^T(\sigma) k_s r(\sigma)) d\sigma \\ &\quad + \alpha_1 \int_{t_0}^t |e_1(\sigma)| (|\eta_1(\sigma)| - \gamma_1) d\sigma + \chi_1 \\ &\quad + \alpha_2 \int_{t_0}^t |e_2(\sigma)| (|\eta_2(\sigma)| - \gamma_2) d\sigma + \chi_2 \\ &\quad + \alpha_3 \int_{t_0}^t |e_3(\sigma)| (|\eta_3(\sigma)| - \gamma_3) d\sigma + \chi_3 \end{aligned} \quad (31)$$

where the auxiliary terms $\chi_1(t)$, $\chi_2(t)$, $\chi_3(t) \in \mathbb{R}$ are defined as

$$\begin{aligned} \chi_1 &\triangleq \int_{t_0}^t \dot{e}_1(\sigma) \eta_1(\sigma) d\sigma \\ &\quad - \gamma_1 \int_{t_0}^t \dot{e}_1(\sigma) \text{sgn}(e_1(\sigma)) d\sigma \end{aligned} \quad (32)$$

$$\begin{aligned} \chi_2 &\triangleq \int_{t_0}^t \dot{e}_2(\sigma) \eta_2(\sigma) d\sigma \\ &\quad - \gamma_2 \int_{t_0}^t \dot{e}_2(\sigma) \text{sgn}(e_2(\sigma)) d\sigma \end{aligned} \quad (33)$$

$$\begin{aligned} \chi_3 &\triangleq \int_{t_0}^t \dot{e}_3(\sigma) \eta_3(\sigma) d\sigma \\ &\quad - \gamma_3 \int_{t_0}^t \dot{e}_3(\sigma) \text{sgn}(e_3(\sigma)) d\sigma. \end{aligned} \quad (34)$$

The integral expressions in (32)-(34) can be evaluated as

$$\begin{aligned} \chi_1 &= e_1(\sigma) \eta_1(\sigma) \Big|_{t_0}^t d\sigma \\ &\quad - \int_{t_0}^t e_1(\sigma) \dot{\eta}_1(\sigma) d\sigma - \gamma_1 |e_1(\sigma)| \Big|_{t_0}^t \\ &= e_1(t) \eta_1(t) - \int_{t_0}^t e_1(\sigma) \dot{\eta}_1(\sigma) d\sigma - \gamma_1 |e_1(t)| \\ &\quad - e_1(t_0) \eta_1(t_0) + \gamma_1 |e_1(t_0)| \end{aligned} \quad (35)$$

$$\begin{aligned} \chi_2 &= e_2(t) \eta_2(t) - \int_{t_0}^t e_2(\sigma) \dot{\eta}_2(\sigma) d\sigma \\ &\quad - \gamma_2 |e_2(t)| - e_2(t_0) \eta_2(t_0) + \gamma_2 |e_2(t_0)| \end{aligned} \quad (36)$$

$$\begin{aligned} \chi_3 &= e_3(t) \eta_3(t) - \int_{t_0}^t e_3(\sigma) \dot{\eta}_3(\sigma) d\sigma \\ &\quad - \gamma_3 |e_3(t)| - e_3(t_0) \eta_3(t_0) + \gamma_3 |e_3(t_0)|. \end{aligned} \quad (37)$$

Substituting (35)-(37) into (31) and performing some algebraic manipulation yields

$$V(t) \leq V(t_0) - \int_{t_0}^t (r^T(\sigma) k_s r(\sigma)) d\sigma + \chi_4 + \zeta_0$$

where the auxiliary terms $\chi_4(t), \zeta_0 \in \mathbb{R}$ are defined as

$$\begin{aligned} \chi_4 &\triangleq \alpha_1 \int_{t_0}^t |e_1(\sigma)| (|\eta_1(\sigma)| + |\dot{\eta}_1(\sigma)| - \gamma_1) d\sigma \\ &\quad + \alpha_2 \int_{t_0}^t |e_2(\sigma)| (|\eta_2(\sigma)| + |\dot{\eta}_2(\sigma)| - \gamma_2) d\sigma \\ &\quad + \alpha_3 \int_{t_0}^t |e_3(\sigma)| (|\eta_3(\sigma)| + |\dot{\eta}_3(\sigma)| - \gamma_3) d\sigma \\ &\quad + |e_1(t)| (|\eta_1(t)| - \gamma_1) \\ &\quad + |e_2(t)| (|\eta_2(t)| - \gamma_2) \\ &\quad + |e_3(t)| (|\eta_3(t)| - \gamma_3) \\ \zeta_0 &\triangleq -e_1(t_0) \eta_1(t_0) + \gamma_1 |e_1(t_0)| \\ &\quad - e_2(t_0) \eta_2(t_0) + \gamma_2 |e_2(t_0)| \\ &\quad - e_3(t_0) \eta_3(t_0) + \gamma_3 |e_3(t_0)|. \end{aligned}$$

Provided the diagonal entries of γ are selected according to the inequalities introduced in (28), $\chi_4(t)$ will always be negative or zero; hence, the following upper bound can be developed:

$$V(t) \leq V(t_0) - \int_{t_0}^t (r^T(\sigma) k_s r(\sigma)) d\sigma + \zeta_0. \quad (38)$$

From (29) and (38), the following inequalities can be determined:

$$V(t_0) + \zeta_0 \geq V(t) \geq 0;$$

hence, $r(t) \in \mathcal{L}_\infty$. The expression in (38) can be used to determine that

$$\int_{t_0}^t (r^T(\sigma) k_s r(\sigma)) d\sigma \leq V(t_0) + \zeta_0 < \infty. \quad (39)$$

By definition, (39) can now be used to prove that $r(t) \in \mathcal{L}_2$. From the fact that $r(t) \in \mathcal{L}_\infty$, (19) and (20) can be used to prove that $e(t)$, $\dot{e}(t)$, $\hat{y}(t)$, and $\hat{\dot{y}}(t) \in \mathcal{L}_\infty$. The expressions in (22) and (25) can be used to determine that $\hat{f}(t)$ and $\hat{\dot{f}}(t) \in \mathcal{L}_\infty$. Assumptions 1-4 can be used to determine that $\hat{f}(\cdot)$ and $\hat{\dot{f}}(\cdot) \in \mathcal{L}_\infty$. Based on the facts that $f(\cdot), \dot{f}(\cdot) \in \mathcal{L}_\infty$, the expressions in (26) and (27) can be used to prove that $\eta(t)$, $\dot{\eta}(t)$, $\dot{r}(t) \in \mathcal{L}_\infty$. Based on the fact that $r(t), \dot{r}(t) \in \mathcal{L}_\infty$ and that $r(t) \in \mathcal{L}_2$, Barbalat's Lemma [13] can be used to prove that $\|r(t)\| \rightarrow 0$ as $t \rightarrow \infty$; hence, Lemma 1.6 of [4] can be used to prove that $\|e(t)\| \rightarrow 0$ and $\|\dot{e}(t)\| \rightarrow 0$ as $t \rightarrow \infty$. Based on the fact that $\|e(t)\| \rightarrow 0$ and $\|\dot{e}(t)\| \rightarrow 0$ as $t \rightarrow \infty$, the expression given in (19) can be used to determine that $\hat{y}_1(t)$, $\hat{y}_2(t)$ and $\hat{y}_3(t)$ approach $y_1(t)$, $y_2(t)$ and $y_3(t)$ as $t \rightarrow \infty$, respectively. Therefore, the expression in (23) can be used to determine that $\hat{f}$ approaches f as $t \rightarrow \infty$. If the observability conditions given in (15) and (16) are satisfied, then the result that $\hat{f}(t)$ approaches $f(t)$ as $t \rightarrow \infty$,

the fact that the parameters $b_i(t) \forall i = 1, 2, 3$ are assumed to be known, and the fact that the image-space signals $y_1(t)$, $y_2(t)$ and $y_3(t)$ are measurable can be used to identify the unknown Euclidean parameter $y_4(t)$ from (21). Once $y_4(t)$ is identified, the complete Euclidean coordinates of the object feature can be determined using (11)-(13).

V. NUMERICAL SIMULATION

In this section, numerical simulation results are provided to illustrate the performance of the range identification observer given a paraboloid imaging surface. The object feature is governed by the following affine motion dynamics:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.4 & -0.6 \\ 0.1 & -0.2 & 0.3 \\ 0.3 & -0.4 & 0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0.5 & 0.25 & 0.3 \end{bmatrix}^T$$

with the following initial Euclidean coordinates

$$\begin{bmatrix} x_1(0) & x_2(0) & x_3(0) \end{bmatrix}^T = \begin{bmatrix} 0.4 & 0.6 & 1 \end{bmatrix}^T.$$

Based on (3), the relationship in (3) - (5) can be used to determine the following initial conditions in the image-space

$$\begin{aligned} \hat{y}_1(t_0) &= y_1(t_0) = 0.769 \\ \hat{y}_2(t_0) &= y_2(t_0) = 1.150 \\ \hat{y}_3(t_0) &= y_3(t_0) = 1.920 \\ y_4(t_0) &= 1.920 \end{aligned}$$

The estimates for $f(t)$ were initialized as follows:

$$\hat{f}_1(t_0) = 1 \quad \hat{f}_2(t_0) = 1 \quad \hat{f}_3(t_0) = 1.$$

After adjusting the observer gains as

$$\begin{aligned} k_s &= \text{diag}\{10, 10, 10\} \quad \alpha = \text{diag}\{8, 8, 8\} \\ \gamma &= \text{diag}\{1, 1, 1\} \times 10^{-5} \end{aligned}$$

the resulting mismatch between $f_1(t)$ and $\hat{f}_1(t)$, $f_2(t)$ and $\hat{f}_2(t)$, and $f_3(t)$ and $\hat{f}_3(t)$ (i.e., $\dot{e}_1(t)$, $\dot{e}_2(t)$, and $\dot{e}_3(t)$ respectively) is provided in Fig. 2. The mismatch between $y_4(t)$ and $\hat{y}_4(t)$ is provided in Fig. 3. The $y_4(t)$ term is obtained from numerical integration of the $\hat{y}_4(t)$ term, while the estimated value is obtained by replacing $f(t)$ with $\hat{f}(t)$ in (11), solving for $y_5(t)$, $y_6(t)$, and $y_7(t)$, and then utilizing equation (21).

Additive-white-gaussian-noise (AWGN) was injected into the measurable image-space signals $y_1(t)$, $y_2(t)$ via the `awgn()` function in MATLAB, while maintaining a constant signal-to-noise-ratio of 20. Without changing any of the other simulation parameters, the mismatch between $f_1(t)$ and $\hat{f}_1(t)$, $f_2(t)$ and $\hat{f}_2(t)$, and $f_3(t)$ and $\hat{f}_3(t)$ (i.e., $\dot{e}_1(t)$, $\dot{e}_2(t)$, and $\dot{e}_3(t)$ respectively) is provided in Fig. 4, while the mismatch between $y_4(t)$ and $\hat{y}_4(t)$ is provided in Fig. 5.

The results depicted in Figs. 2-5 indicate that the proposed observer can be used to identify the range and hence, the Euclidean coordinates of an object feature moving with affine motion dynamics and a nonaffine PDS, provided the observability conditions are satisfied. These results are comparable

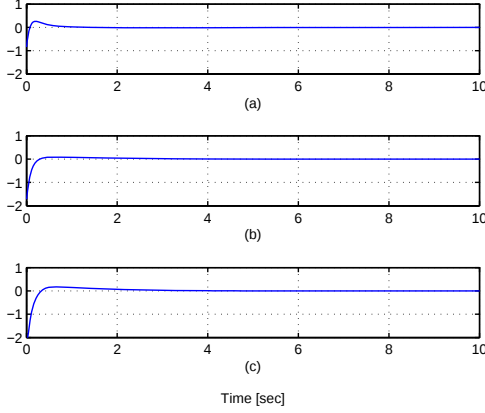


Fig. 2. Estimation error of auxiliary signals (a) $\dot{e}_1(t)$ (b) $\dot{e}_2(t)$ and (c) $\dot{e}_3(t)$ in [pixels/sec].

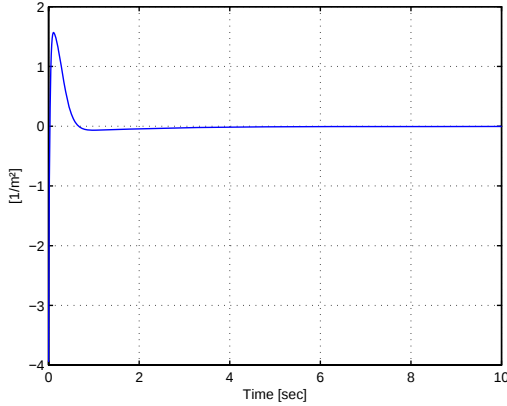


Fig. 3. Mismatch between $y_4(t)$ and $\hat{y}_4(t)$.

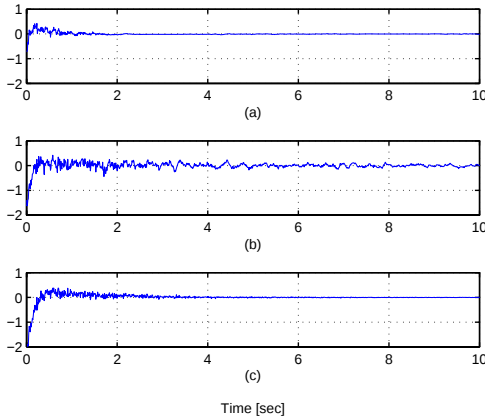


Fig. 4. Estimation error of auxiliary signals in the presence of noise (a) $\dot{e}_1(t)$ (b) $\dot{e}_2(t)$ and (c) $\dot{e}_3(t)$ in [pixels/sec].

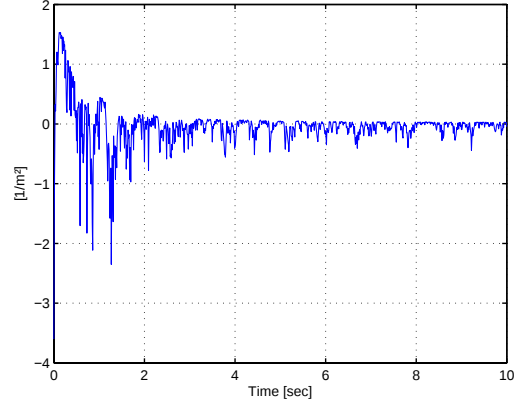


Fig. 5. Mismatch between $y_4(t)$ and $\hat{y}_4(t)$ in the presence of noise.

to the results obtained in [5] which applied the range identification observer to a planar imaging surface.

VI. CONCLUSION

The range and the Euclidean coordinates of an object undergoing general affine motion are determined for a paraboloid imaging system. Unlike image systems that are based on a planar image surface (or spherical or ellipsoidal surface), the perspective dynamic system resulting from the paraboloid projected image does not maintain an affine form. A nonlinear state estimator is developed for the nonaffine PDS without the use of linear approximations. The nonlinear estimator was proven and numerically demonstrated to asymptotically determine the range information from a single camera provided some observability conditions are satisfied and that the Euclidean motion parameters are known.

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APPENDIX

To prove that $\dot{f}_1(\cdot), \dot{f}_2(\cdot), \dot{f}_3(\cdot) \in \mathcal{L}_\infty$, the time derivative of (11) is determined as follows

$$\begin{aligned} \dot{f}_1 = & \dot{b}_3 y_5 + b_3 \dot{y}_5 - 2\dot{b}_1 y_1 y_5 \\ & - 2b_1 y_1 \dot{y}_5 - 2b_1 \dot{y}_1 y_5 - 2\dot{b}_2 y_1 y_6 \\ & - 2b_2 y_1 \dot{y}_6 - 2b_2 y_1 \dot{y}_6 + \dot{b}_1 y_7 + b_1 \dot{y}_7 \end{aligned} \quad (40)$$

$$\begin{aligned} \dot{f}_2 = & -2\dot{b}_1 y_2 y_5 - 2b_1 \dot{y}_2 y_5 - 2b_1 y_2 \dot{y}_5 \\ & + \dot{b}_3 y_6 + b_3 \dot{y}_6 - 2\dot{b}_2 y_2 y_6 - 2b_2 y_2 \dot{y}_6 \\ & - 2b_2 y_2 \dot{y}_6 + \dot{b}_2 y_7 + b_2 \dot{y}_7 \end{aligned} \quad (41)$$

$$\begin{aligned} \dot{f}_3 = & -2\dot{b}_1 y_3 y_5 - 2b_1 \dot{y}_3 y_5 - 2b_1 y_3 \dot{y}_5 \\ & - 2\dot{b}_2 y_3 y_6 - 2b_2 \dot{y}_3 y_6 - 2b_2 y_3 \dot{y}_6 \\ & + 2\dot{b}_3 y_7 + 2b_3 \dot{y}_7. \end{aligned} \quad (42)$$

The facts that $\dot{y}(t) \in \mathcal{L}_\infty$ can be used along with Assumptions 1-4 to conclude from (40) - (42) that $\dot{f}_1(\cdot)$ to $\dot{f}_3(\cdot) \in \mathcal{L}_\infty$.

To prove that $\ddot{f}_1(\cdot), \ddot{f}_2(\cdot)$ and $\ddot{f}_3(\cdot) \in \mathcal{L}_\infty$, the time derivative of (40) - (42) can be determined as follows

$$\begin{aligned} \ddot{f}_1 = & \ddot{b}_3 y_5 + 2\dot{b}_3 \dot{y}_5 + b_3 \ddot{y}_5 - 2\ddot{b}_1 y_1 y_5 \\ & - 4\dot{b}_1 y_1 \dot{y}_5 - 4\dot{b}_1 \dot{y}_1 y_5 - 4b_1 \ddot{y}_1 y_5 \\ & - 2b_1 y_1 \ddot{y}_5 - 2b_1 \ddot{y}_1 y_5 - 2\ddot{b}_2 y_1 y_6 \\ & - 4\dot{b}_2 y_1 \dot{y}_6 - 4\dot{b}_2 y_1 \dot{y}_6 - 2b_2 \ddot{y}_1 y_6 \\ & - 4b_2 y_1 \ddot{y}_6 - 2b_2 y_1 \ddot{y}_6 + \ddot{b}_1 y_7 \\ & + 2\dot{b}_1 \dot{y}_7 + b_1 \ddot{y}_7 \end{aligned} \quad (43)$$

$$\begin{aligned} \ddot{f}_2 = & -2\ddot{b}_1 y_2 y_5 - 4\dot{b}_1 \dot{y}_2 y_5 - 4\dot{b}_1 y_2 \dot{y}_5 \\ & - 2b_1 \ddot{y}_2 y_5 - 4b_1 \dot{y}_2 \dot{y}_5 - 2b_1 y_2 \ddot{y}_5 \\ & + \ddot{b}_3 y_6 + 2\dot{b}_3 \dot{y}_6 + b_3 \ddot{y}_6 - 2\ddot{b}_2 y_2 y_6 \\ & - 4b_2 \dot{y}_2 y_6 - 4\dot{b}_2 y_2 \dot{y}_6 - 2b_2 \ddot{y}_2 y_6 \\ & - 4b_2 y_2 \ddot{y}_6 - 2b_2 y_2 \ddot{y}_6 + \ddot{b}_2 y_7 \\ & + 2\dot{b}_2 \dot{y}_7 + b_2 \ddot{y}_7 \end{aligned} \quad (44)$$

$$\begin{aligned} \ddot{f}_3 = & -2\ddot{b}_1 y_3 y_5 - 4\dot{b}_1 \dot{y}_3 y_5 - 4\dot{b}_1 y_3 \dot{y}_5 \\ & - 2b_1 \ddot{y}_3 y_5 - 4b_1 \dot{y}_3 \dot{y}_5 - 2b_1 y_3 \ddot{y}_5 \\ & - 2\ddot{b}_2 y_3 y_6 - 4\dot{b}_2 \dot{y}_3 y_6 - 4\dot{b}_2 y_3 \dot{y}_6 \\ & - 2b_2 \ddot{y}_3 y_6 - 4b_2 \dot{y}_3 \dot{y}_6 - 2b_2 y_3 \ddot{y}_6 \\ & + 2\dot{b}_3 y_7 + 4\dot{b}_3 \dot{y}_7 + 2b_3 \ddot{y}_7 \end{aligned} \quad (45)$$

where

$$\begin{aligned} \dot{y}_5 = & \dot{y}_1 \sqrt{\frac{y_4}{y_3}} + \frac{y_1 \dot{y}_4}{2y_3} \sqrt{\frac{y_3}{y_4}} \\ & - \frac{y_1 y_4 \dot{y}_3}{2y_3^2} \sqrt{\frac{y_3}{y_4}} \end{aligned} \quad (46)$$

$$\begin{aligned} \dot{y}_6 = & \dot{y}_2 \sqrt{\frac{y_4}{y_3}} + \frac{y_2 \dot{y}_4}{2y_3} \sqrt{\frac{y_3}{y_4}} \\ & - \frac{y_2 y_4 \dot{y}_3}{2y_3^2} \sqrt{\frac{y_3}{y_4}} \end{aligned} \quad (47)$$

$$\dot{y}_7 = \frac{(\dot{y}_3 y_4 + y_3 \dot{y}_4)}{2\sqrt{y_3 y_4}} \quad (48)$$

$$\begin{aligned} \ddot{y}_1 = & \dot{a}_{11} y_1 + a_{11} \dot{y}_1 + \dot{a}_{12} y_2 \\ & + a_{12} \dot{y}_2 + \dot{a}_{13} y_3 + a_{13} \dot{y}_3 \\ & + \frac{y_3(\dot{a}_{31} y_1^2 + 2a_{31} y_1 \dot{y}_1) - a_{31} y_1^2 \dot{y}_3}{y_3^2} \\ & + \frac{y_3(\dot{a}_{32} y_1 y_2 + a_{32}(\dot{y}_1 y_2 + y_1 \dot{y}_2))}{y_3^2} \\ & - \frac{a_{32} y_1 y_2 \dot{y}_3}{y_3^2} + \dot{a}_{33} y_1 + a_{33} \dot{y}_1 \\ & - \frac{2(y_3(\dot{a}_{11} y_1^3 + 3a_{11} y_1^2 \dot{y}_1) - a_{11} y_1^3 \dot{y}_3)}{y_3^2} \\ & - \frac{2y_3(\dot{a}_{12} y_1^2 y_2 + a_{12}(2y_1 \dot{y}_1 y_2 + y_1^2 \dot{y}_2))}{y_3^2} \\ & + \frac{2a_{12} y_1^2 y_2 \dot{y}_3}{y_3} - 2 \left[\frac{y_3(\dot{a}_{21} y_1^2 y_2}{y_3^2} \right. \\ & + \frac{a_{21}(2y_1 \dot{y}_1 y_2 + y_1^2 \dot{y}_2) - a_{21} y_1^2 y_2 \dot{y}_3}{y_3^2} \left. \right] \\ & - \frac{2(\dot{a}_{22} y_1 y_2^2 + a_{22}(\dot{y}_1 y_2^2 + 2y_1 y_2 \dot{y}_2))}{y_3} \\ & + \frac{2a_{22} y_1 y_2^2 \dot{y}_3}{y_3^2} - 2(\dot{a}_{13} y_1^2 + 2a_{13} y_1 \dot{y}_1) \\ & - 2(\dot{a}_{23} y_1 y_2 + a_{23}(\dot{y}_1 y_2 + y_1 \dot{y}_2)) + \dot{f}_1 \end{aligned} \quad (49)$$

$$\begin{aligned}
\ddot{y}_2 = & \dot{a}_{21}y_1 + a_{21}\dot{y}_1 + \dot{a}_{22}y_2 \\
& + a_{22}\dot{y}_2 + \dot{a}_{23}y_3 + a_{23}\dot{y}_3 \\
& + \frac{y_3(\dot{a}_{32}y_2^2 + 2a_{32}y_2\dot{y}_2) - a_{32}y_2^2\dot{y}_3}{y_3^2} \\
& + \frac{\dot{a}_{31}y_1y_2 + a_{31}(\dot{y}_1y_2 + y_1\dot{y}_2)}{y_3} \\
& - \frac{a_{31}y_1y_2\dot{y}_3}{y_3^2} + \dot{a}_{33}y_2 + a_{33}\dot{y}_2 \\
& - 2 \left[\frac{y_3(\dot{a}_{22}y_2^3 + 3a_{22}y_2^2\dot{y}_2) - a_{22}y_2^3\dot{y}_3}{y_3^2} \right] \\
& - 2 \left[\frac{\dot{a}_{12}y_1y_2^2 + a_{12}(2y_1\dot{y}_2y_2 + y_2^2\dot{y}_1)}{y_3} \right. \\
& \left. + \frac{2a_{12}y_2^2y_1\dot{y}_3}{y_3^2} \right] - 2 \left[\frac{\dot{a}_{21}y_1y_2^2}{y_3} \right. \\
& \left. + \frac{y_3a_{21}(2y_1\dot{y}_2y_2 + y_2^2\dot{y}_1) - a_{21}y_2^2y_1\dot{y}_3}{y_3^2} \right] \\
& - 2 \left[\frac{(\dot{a}_{11}y_2y_1^2 + a_{11}(\dot{y}_2y_1^2 + 2y_1y_2\dot{y}_1))}{y_3} \right. \\
& \left. - \frac{a_{11}y_2y_1^2\dot{y}_3}{y_3^2} \right] - 2(\dot{a}_{23}y_2^2 + 2a_{23}y_2\dot{y}_2) \\
& - 2(\dot{a}_{13}y_1y_2 + a_{13}(\dot{y}_1y_2 + y_1\dot{y}_2)) + \dot{f}_2
\end{aligned} \tag{50}$$

$$\begin{aligned}
\ddot{y}_3 = & 2(\dot{a}_{31}y_1 + a_{31}\dot{y}_1) + 2(\dot{a}_{32}y_2 + a_{32}\dot{y}_2) \\
& + 2(\dot{a}_{33}y_3 + a_{33}\dot{y}_3) - 2(\dot{a}_{11}y_1^2 + 2a_{11}y_1\dot{y}_1) \\
& - 2(\dot{a}_{12}y_1y_2 + a_{12}(\dot{y}_1y_2 + y_1\dot{y}_2)) \\
& - 2(\dot{a}_{21}y_1y_2 + a_{21}(\dot{y}_1y_2 + y_1\dot{y}_2)) \\
& - 2(\dot{a}_{22}y_2^2 + 2a_{22}y_2\dot{y}_2) - 2(\dot{a}_{13}y_1y_3 \\
& + a_{13}(\dot{y}_1y_3 + y_1\dot{y}_3) - 2(\dot{a}_{23}y_2y_3 \\
& + a_{23}(\dot{y}_2y_3 + y_2\dot{y}_3)) + \dot{f}_3
\end{aligned} \tag{51}$$

$$\begin{aligned}
\ddot{y}_4 = & -2 \left[\frac{\dot{a}_{11}y_1^2y_4 + a_{11}(2y_1\dot{y}_1y_4 + y_1^2\dot{y}_4)}{y_3} \right. \\
& \left. - \frac{a_{11}y_1^2y_4\dot{y}_3}{y_3^2} \right] - 2 \left[\frac{\dot{a}_{22}y_2^2y_4}{y_3} \right. \\
& \left. + \frac{a_{22}y_3(2y_2\dot{y}_2y_4 + y_2^2\dot{y}_4) - a_{22}y_2^2y_4\dot{y}_3}{y_3^2} \right] \\
& - 2 \left[\frac{\dot{a}_{12}y_1y_2y_4 + a_{12}\dot{y}_1y_2y_4}{y_3} \right. \\
& \left. + \frac{a_{12}y_1y_3(\dot{y}_2y_4 + y_2\dot{y}_4) - a_{12}y_1y_2y_4\dot{y}_3}{y_3^2} \right] \\
& - 2 \left[\frac{\dot{a}_{21}y_1y_2y_4 + a_{21}\dot{y}_1y_2y_4}{y_3} \right. \\
& \left. + \frac{a_{21}y_1y_3(\dot{y}_2y_4 + y_2\dot{y}_4) - a_{21}y_1y_2y_4\dot{y}_3}{y_3^2} \right] \\
& - 2(\dot{a}_{13}y_1y_4 + a_{13}(\dot{y}_1y_4 + y_1\dot{y}_4)) \\
& - 2(\dot{a}_{23}y_2y_4 + a_{23}(\dot{y}_2y_4 + y_2\dot{y}_4)) \\
& - 2(\dot{b}_1y_4y_5 + b_1(\dot{y}_4y_5 + y_4\dot{y}_5)) \\
& - 2(\dot{b}_2y_4y_6 + b_2(\dot{y}_4y_6 + y_4\dot{y}_6))
\end{aligned} \tag{52}$$

$$\begin{aligned}
\ddot{y}_5 = & \ddot{y}_1\sqrt{\frac{y_4}{y_3}} + \frac{\dot{y}_1}{2y_3^2}\sqrt{\frac{y_3}{y_4}}(y_3\dot{y}_4 - y_4\dot{y}_3) \\
& + \frac{1}{2} \left\{ \sqrt{\frac{y_3}{y_4}} \frac{\dot{y}_1\dot{y}_4}{y_3} \right. \\
& + y_1 \left[-\sqrt[3]{\frac{y_3}{y_4}}(\dot{y}_4y_3 - \dot{y}_3y_4) \frac{\dot{y}_4}{2y_3^3} \right. \\
& \left. + \sqrt{\frac{y_3}{y_4}} \frac{(y_3\ddot{y}_4 - \dot{y}_4\dot{y}_3)}{y_3^2} \right] \left. \right\} + \frac{1}{2} \left\{ -\sqrt{\frac{y_3}{y_4}} \right. \\
& \frac{\dot{y}_1\dot{y}_3y_4}{y_3^2} + y_1 \left[\sqrt[3]{\frac{y_3}{y_4}}(y_3\dot{y}_4 - y_4\dot{y}_3) \frac{y_4\dot{y}_3}{2y_3^4} \right. \\
& + \sqrt{\frac{y_3}{y_4}} \left(\frac{-\dot{y}_4\dot{y}_3 - y_4\ddot{y}_3}{y_3^2} \right. \\
& \left. \left. + \frac{2y_4y_3\dot{y}_3^2}{y_3^4} \right) \right] \left. \right\}
\end{aligned} \tag{53}$$

$$\begin{aligned}
\ddot{y}_6 = & \ddot{y}_2\sqrt{\frac{y_4}{y_3}} + \frac{\dot{y}_2}{2y_3^2}\sqrt{\frac{y_3}{y_4}}(y_3\dot{y}_4 - y_4\dot{y}_3) \\
& + \frac{1}{2} \left\{ \sqrt{\frac{y_3}{y_4}} \frac{\dot{y}_2\dot{y}_4}{y_3} \right. \\
& + y_2 \left[-\sqrt[3]{\frac{y_3}{y_4}}(\dot{y}_4y_3 - \dot{y}_3y_4) \frac{\dot{y}_4}{2y_3^3} \right. \\
& \left. + \sqrt{\frac{y_3}{y_4}} \frac{(y_3\ddot{y}_4 - \dot{y}_4\dot{y}_3)}{y_3^2} \right] \left. \right\} + \frac{1}{2} \left\{ -\sqrt{\frac{y_3}{y_4}} \right. \\
& \frac{\dot{y}_2\dot{y}_3y_4}{y_3^2} + y_2 \left[\sqrt[3]{\frac{y_3}{y_4}}(y_3\dot{y}_4 - y_4\dot{y}_3) \frac{y_4\dot{y}_3}{2y_3^4} \right. \\
& + \sqrt{\frac{y_3}{y_4}} \left(\frac{-\dot{y}_4\dot{y}_3 - y_4\ddot{y}_3}{y_3^2} \right. \\
& \left. \left. + \frac{2y_4y_3\dot{y}_3^2}{y_3^4} \right) \right] \left. \right\}
\end{aligned} \tag{54}$$

$$\begin{aligned}
\ddot{y}_7 = & \frac{1}{2} \left\{ \left[-(y_3\dot{y}_4 + y_4\dot{y}_3) \frac{\dot{y}_3y_4}{2\sqrt[3]{y_3y_4}} \right. \right. \\
& \left. + \frac{(\dot{y}_3\dot{y}_4 + y_4\ddot{y}_3)}{\sqrt{y_3y_4}} \right] + \left[\frac{-(y_3\dot{y}_4)^2}{2\sqrt[3]{y_3y_4}} \right. \\
& \left. - \frac{y_4\dot{y}_3y_3\dot{y}_4}{2\sqrt[3]{y_3y_4}} + \frac{(\dot{y}_3\dot{y}_4 + \ddot{y}_4y_3)}{\sqrt{y_3y_4}} \right] \left. \right\}.
\end{aligned} \tag{55}$$

The expressions given in (43) - (55), Assumptions 1-4, and the facts that, $\dot{y}(t)$, $f(y_5, y_6, y_7) \in \mathcal{L}_\infty$ can now be used to prove that $\ddot{f}_1(\cdot)$, $\ddot{f}_2(\cdot)$, $\ddot{f}_3(\cdot) \in \mathcal{L}_\infty$.