Making Connected Unions of Random Graphs

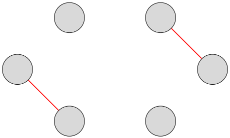
Matthew Hale

Department of Mechanical and Aerospace Engineering
University of Florida

AFOSR Center of Excellence Kickoff
May 13, 2019

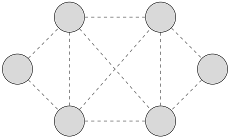
Time-Varying Graphs are Common

- With autonomous agents, communication graphs may look like

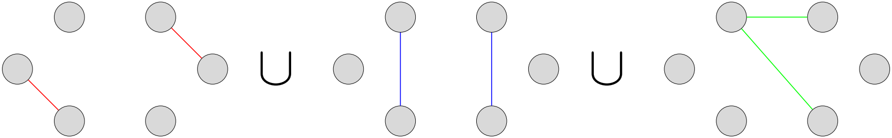


Time-Varying Graphs are Common

- With autonomous agents, communication graphs may look like

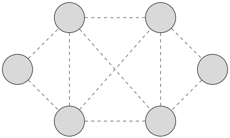


- No single graph is connected

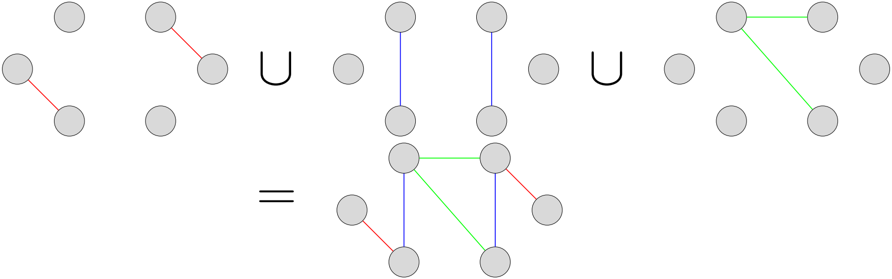


Time-Varying Graphs are Common

- With autonomous agents, communication graphs may look like



- No single graph is connected, but their union is



Common Assumption

There exists an N such that

$$\bigcup_{k=t+1}^{t+N} G(k)$$

is connected for any t.

How to Make Connected Unions of Graphs?

Common Assumption

There exists an N such that

$$\bigcup_{k=t+1}^{t+N} G(k)$$

is connected for any t.

Question 1

Q1: When is the "average union" connected?
A1: $N \geq f(\text{graph parameters})$

How to Make Connected Unions of Graphs?

Common Assumption

There exists an N such that

$$\bigcup_{k=t+1}^{t+N} G(k)$$

is connected for any t.

Question 1

Q1: When is the "average union" connected?

A1: $N \geq f(\text{graph parameters})$

Question 2

Q2: What is the probability that a specific union is connected?

A2: $\mathbb{P}\left[\bigcup_{k=t+1}^{t+N} G(k) \text{ connected}\right] \geq h(\text{graph parameters})$

How to Make Connected Unions of Graphs?

Common Assumption

There exists an N such that

$$\bigcup_{k=t+1}^{t+N} G(k)$$

is connected for any t.

Question 1

Q1: When is the "average union" connected?

A1: $N \geq f(\text{graph parameters})$

Question 2

Q2: What is the probability that a specific union is connected?

A2: $\mathbb{P}\left[\bigcup_{k=t+1}^{t+N} G(k) \text{ connected}\right] \geq h(\text{graph parameters})$

Why Study Random Graphs?

1. Randomness models unknown and uncontrollable adversarial impacts on communications

Why Study Random Graphs?

1. Randomness models unknown and uncontrollable adversarial impacts on communications
2. Randomness captures unpredictable information flows among agents

Why Study Random Graphs?

1. Randomness models unknown and uncontrollable adversarial impacts on communications
2. Randomness captures unpredictable information flows among agents

Erdős-Rényi Graphs

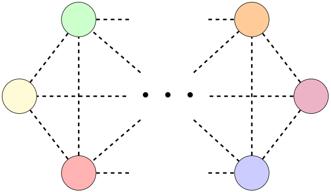
- Defined by parameters $n \in \mathbb{N}$ and $p \in (0, 1)$

Why Study Random Graphs?

1. Randomness models unknown and uncontrollable adversarial impacts on communications
2. Randomness captures unpredictable information flows among agents

Erdős-Rényi Graphs

- Defined by parameters $n \in \mathbb{N}$ and $p \in (0, 1)$
- Graphs are on n nodes:

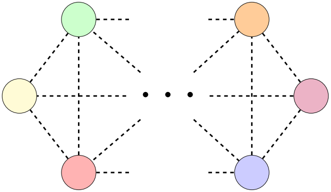


Why Study Random Graphs?

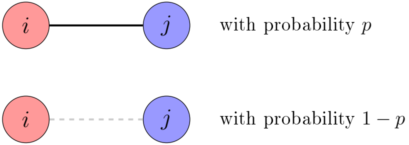
1. Randomness models unknown and uncontrollable adversarial impacts on communications
2. Randomness captures unpredictable information flows among agents

Erdős-Rényi Graphs

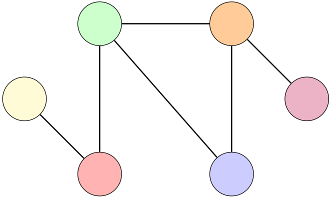
- Defined by parameters $n \in \mathbb{N}$ and $p \in (0, 1)$
- Graphs are on n nodes:



- Each edge appears with probability p:

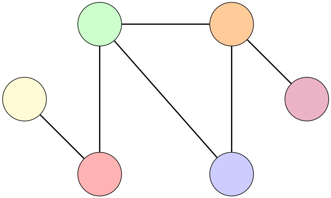


- We require innovations beyond the existing literature for two reasons:
  - Common to take $n \to \infty$, but results do not apply to small networks



New Analyses Required for Multi-Agent Systems

- We require innovations beyond the existing literature for two reasons:
  - Common to take $n \to \infty$, but results do not apply to small networks



  - We also avoid assuming $p \sim \frac{\ln n}{n}$ or $p = f(n)$ because p is outside our control

We Study Graphs using Algebraic Connectivity

- Define $L = D - A$, and denote its i^{th} eigenvalue by $\ell_i(L)$
- Then

$$\lambda_2(L) = \min_{2 \le i \le n} \ell_i(L) \quad \backslash \quad \ell_1(L) = 0$$

We Study Graphs using Algebraic Connectivity

- Define $L = D - A$, and denote its i^{th} eigenvalue by $\ell_i(L)$
- Then

$$\lambda_2(L) = \min_{2 \leq i \leq n} \ell_i(L) \quad \setminus \quad \ell_1(L) = 0$$

- When is

$$\mathbb{E}\left[\lambda_2\left(\bigcup_{k=t+1}^{t+N} G(k)\right)\right] \geq \lambda_{min}(n)?$$

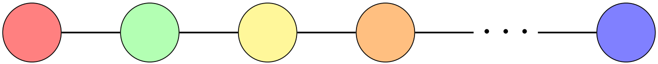
We Study Graphs using Algebraic Connectivity

- Define $L = D - A$, and denote its i^{th} eigenvalue by $\ell_i(L)$
- Then

$$\lambda_2(L) = \min_{2 \leq i \leq n} \ell_i(L) \quad \backslash \quad \ell_1(L) = 0$$

- When is

$$\mathbb{E}\left[\lambda_2\left(\bigcup_{k=t+1}^{t+N} G(k)\right)\right] \geq \lambda_{min}(n)?$$



- $\lambda_{min}(n) = \lambda_2(L_n) = 2\left(1 - \cos\frac{\pi}{n}\right)$

We can Bound $\mathbb{E}[\lambda_2]$ with Known Constants

- For L the Laplacian of $\bigcup_{k=t+1}^{t+N} G(k)$, we want

$$\mathbb{E}\left[\min_{2 \leq i \leq n} \ell_i(L)\right]$$

We can Bound $\mathbb{E}[\lambda_2]$ with Known Constants

- For L the Laplacian of $\bigcup_{k=t+1}^{t+N} G(k)$, we want

$$\mathbb{E}\left[\min_{2 \leq i \leq n} \ell_i(L)\right] \neq \min_{2 \leq i \leq n} \mathbb{E}\left[\ell_i(L)\right]$$

We can Bound $\mathbb{E}[\lambda_2]$ with Known Constants

- For L the Laplacian of $\bigcup_{k=t+1}^{t+N} G(k)$, we want

$$\mathbb{E}\left[\min_{2\leq i\leq n} \ell_i(L)\right] \neq \min_{2\leq i\leq n} \mathbb{E}\left[\ell_i(L)\right]$$

Lemma *(Arnold & Groeneveld, 1976)*

Let $X_1, \ldots, X_m$ be a collection of random variables. If they all have mean μ and variance σ^2, then

$$\mu - \sigma\sqrt{\frac{m-k}{k}} \leq \mathbb{E}[X_{k:m}] \leq \mu + \sigma\sqrt{\frac{k-1}{m-k+1}}$$

We can Bound $\mathbb{E}[\lambda_2]$ with Known Constants

- For L the Laplacian of $\bigcup_{k=t+1}^{t+N} G(k)$, we want

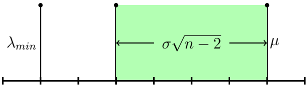
$$\mathbb{E}\left[\min_{2\le i\le n} \ell_i(L)\right] \neq \min_{2\le i\le n} \mathbb{E}\left[\ell_i(L)\right]$$

Lemma *(Arnold & Groeneveld, 1976)*

Let $X_1, \dots, X_m$ be a collection of random variables. If they all have mean μ and variance σ^2, then

$$\mu - \sigma\sqrt{\frac{m-k}{k}} \le \mathbb{E}[X_{k:m}] \le \mu + \sigma\sqrt{\frac{k-1}{m-k+1}}$$

- For $\lambda_2(L)$: $\mathbb{E}[\lambda_2] \ge \mu - \sigma\sqrt{n-2}$
- We want



Lemma

$\mu := \mathbb{E}[\ell_i]$ is an eigenvalue of $\mathbb{E}[L]$

Lemma

$\mu := \mathbb{E}[\ell_i]$ is an eigenvalue of $\mathbb{E}[L]$

- $\mathbb{E}[L]$ takes the form

$$\mathbb{E}[L] = \left(1-(1-p)^N\right)\begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$$

What is μ for ℓ_i?

Lemma

$\mu := \mathbb{E}[\ell_i]$ is an eigenvalue of $\mathbb{E}[L]$

- $\mathbb{E}[L]$ takes the form

$$\mathbb{E}[L] = \left(1 - (1-p)^N\right) \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$$

Finding μ

- For $i \in \{2, \ldots, n\}$, ℓ_i has mean

$$\mu = n\left(1 - (1-p)^N\right)$$

What is σ for ℓ_i?

- We know that $\sigma^2[\ell_i] = \mathbb{E}[\ell_i^2] - \mathbb{E}[\ell_i]^2 = \mathbb{E}[\ell_i^2] - \mu^2$

What is σ for ℓ_i?

- We know that $\sigma^2[\ell_i] = \mathbb{E}[\ell_i^2] - \mathbb{E}[\ell_i]^2 = \mathbb{E}[\ell_i^2] - \mu^2$

Lemma

$\mathbb{E}[\ell_i^2]$ is an eigenvalue of $\mathbb{E}[L^2]$

- We know that $\sigma^2[\ell_i] = \mathbb{E}[\ell_i^2] - \mathbb{E}[\ell_i]^2 = \mathbb{E}[\ell_i^2] - \mu^2$

Lemma

$\mathbb{E}[\ell_i^2]$ is an eigenvalue of $\mathbb{E}[L^2]$

- $\mathbb{E}[L^2]$ takes the form

$$\mathbb{E}[L^2] = \left((n-2)\left[1-(1-p)^N\right]^2 + 2\left(1-(1-p)^N\right)\right) \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$$

What is σ for ℓ_i?

- We know that $\sigma^2[\ell_i] = \mathbb{E}[\ell_i^2] - \mathbb{E}[\ell_i]^2 = \mathbb{E}[\ell_i^2] - \mu^2$

Lemma

$\mathbb{E}[\ell_i^2]$ is an eigenvalue of $\mathbb{E}[L^2]$

- $\mathbb{E}[L^2]$ takes the form

$$\mathbb{E}[L^2] = \Big((n-2)\big[1-(1-p)^N\big]^2 + 2\big(1-(1-p)^N\big)\Big) \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$$

Computing σ^2

For $i \in \{2, \ldots, n\}$, ℓ_i has variance

$$\sigma^2 = 2n\big(1-(1-p)^N\big)(1-p)^N$$

Answer to Question 1

- We now have the bound

$$\mathbb{E}\left[\lambda_2\left(\bigcup_{k=t+1}^{t+N} G(k)\right)\right] \geq \underbrace{n\left(1-(1-p)^N\right)}_{\mu} - \sqrt{n-2}\,\underbrace{\sqrt{2n\left(1-(1-p)^N\right)(1-p)^N}}_{\sigma}$$

Answer to Question 1

- We now have the bound

$$\mathbb{E}\left[\lambda_2\left(\bigcup_{k=t+1}^{t+N} G(k)\right)\right] \geq \underbrace{n\big(1-(1-p)^N\big)}_{\mu} - \sqrt{n-2}\underbrace{\sqrt{2n\big(1-(1-p)^N\big)(1-p)^N}}_{\sigma}$$

Theorem (Answer to Question 1)

A union of N Erdős-Rényi graphs is expected to be connected if

$$N \geq \frac{1}{\ln(1-p)} \ln\left(\frac{4n^2 + 4n\cos\frac{\pi}{n} - 8n - \tau(n)}{6n^2 - 8n}\right),$$

$$\tau(n) := \sqrt{16n^2(n-2)\left(1-\cos\frac{\pi}{n}\right) + 32n(2-n)\left(1-\cos\frac{\pi}{n}\right)^2 + 4n^2(n-2)^2}$$

Answer to Question 1

- We now have the bound

$$\mathbb{E}\left[\lambda_2\left(\bigcup_{k=t+1}^{t+N} G(k)\right)\right] \geq \underbrace{n\left(1-(1-p)^N\right)}_{\mu} - \sqrt{n-2}\underbrace{\sqrt{2n\left(1-(1-p)^N\right)(1-p)^N}}_{\sigma}$$

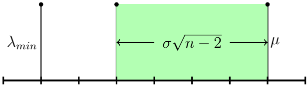
Theorem (Answer to Question 1)

A union of N Erdős-Rényi graphs is expected to be connected if

$$N \geq \frac{1}{\ln(1-p)} \ln\left(\frac{4n^2 + 4n\cos\frac{\pi}{n} - 8n - \tau(n)}{6n^2 - 8n}\right),$$

$$\tau(n) := \sqrt{16n^2(n-2)\left(1-\cos\frac{\pi}{n}\right) + 32n(2-n)\left(1-\cos\frac{\pi}{n}\right)^2 + 4n^2(n-2)^2}$$

- These values of N give



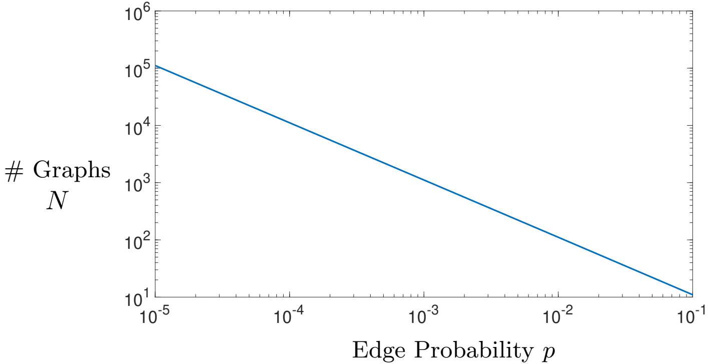
UF UNIVERSITY of FLORIDA · Duke UNIVERSITY · TEXAS The University of Texas at Austin · UC SANTA CRUZ

Numerical Results for Question 1

- Dependence on p often dominates dependence on n
- For $n = 50$:

Numerical Results for Question 1

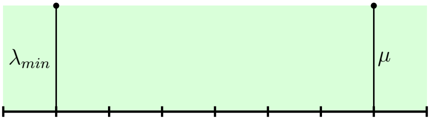
- Dependence on p often dominates dependence on n
- For $n = 50$:



- Order of magnitude increase in p causes $\sim$order of magnitude decrease in N

Question 2

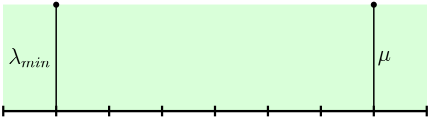
Lower bound $\mathbb{P}\left[\lambda_2\left(\bigcup_{k=t+1}^{t+N} G(k)\right) \geq \lambda_{min}(n)\right]$ in terms of n and p



Working on Question 2

Question 2

Lower bound $\mathbb{P}\left[\lambda_2\left(\bigcup_{k=t+1}^{t+N} G(k)\right) \geq \lambda_{min}(n)\right]$ in terms of n and p



- Applying the Paley-Zygmund inequality, we find

$$\mathbb{P}[\lambda_2(L) \geq \lambda_{min}(n)] \geq \left(1 - \frac{\lambda_{min}(n)}{\mathbb{E}[\lambda_2(L)]}\right)^2 \frac{\mathbb{E}[\lambda_2(L)]^2}{\mathbb{E}[\lambda_2^2(L)]}$$

Answer to Question 2

- Fortunately, we have

$$\mathbb{E}[\lambda_2^2] \leq \mathbb{E}[\ell_i^2] = n(n-2)\left[1-(1-p)^N\right]^2 + 2n\left(1-(1-p)^N\right)$$

Answer to Question 2

- Fortunately, we have

$$\mathbb{E}[\lambda_2^2] \leq \mathbb{E}[\ell_i^2] = n(n-2)\left[1-(1-p)^N\right]^2 + 2n\left(1-(1-p)^N\right)$$

- Question 2: $\mathbb{P}\left[\lambda_2(L) \geq \lambda_{min}(n)\right] \geq$?

Answer to Question 2

- Fortunately, we have

$$\mathbb{E}[\lambda_2^2] \leq \mathbb{E}[\ell_i^2] = n(n-2)\left[1-(1-p)^N\right]^2 + 2n\left(1-(1-p)^N\right)$$

- Question 2: $\mathbb{P}\left[\lambda_2(L) \geq \lambda_{min}(n)\right] \geq$?

Answer to Question 2

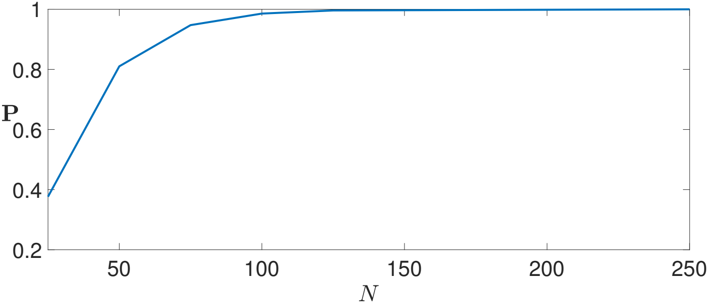
For L the Laplacian of a union of N random graphs, we have

$$\mathbb{P}\left[\lambda_2(L) \geq \lambda_{min}(n)\right] \geq$$

$$\left(1-\frac{2(1-\cos\frac{\pi}{n})}{n\left(1-(1-p)^N\right)}\right)^2 \left(\frac{\left[n\left(1-(1-p)^N\right)-\sqrt{2n(n-2)\left(1-(1-p)^N\right)(1-p)^N}\right]^2}{n(n-2)\left(1-(1-p)^N\right)^2+2n\left(1-(1-p)^N\right)}\right)$$

Numerical Results for Question 2

- For $n = 50$ and $p = 0.1$:



- $\mathbb{P}[\lambda_2(L) \geq \lambda_{min}(n)]$ increases rapidly with N

- What about time-varying probabilities? What conditions do we need on $\{p(k)\}_{k\in\mathbb{N}}$?

Extensions and Next Steps

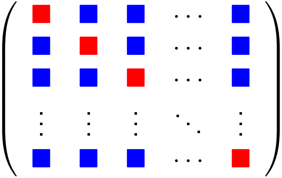
- What about time-varying probabilities? What conditions do we need on $\{p(k)\}_{k\in\mathbb{N}}$?
- What about directed graphs? What if $p_{ij} \neq p_{ji}$?

Extensions and Next Steps

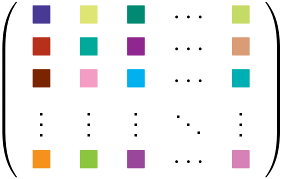
- What about time-varying probabilities? What conditions do we need on $\{p(k)\}_{k\in\mathbb{N}}$?
- What about directed graphs? What if $p_{ij} \neq p_{ji}$?
- How about both? What about $\{p_{ij}(k)\}_{k\in\mathbb{N}}$ and $\{p_{ji}(k)\}_{k\in\mathbb{N}}$?

Extensions and Next Steps

- What about time-varying probabilities? What conditions do we need on $\{p(k)\}_{k \in \mathbb{N}}$?
- What about directed graphs? What if $p_{ij} \neq p_{ji}$?
- How about both? What about $\{p_{ij}(k)\}_{k \in \mathbb{N}}$ and $\{p_{ji}(k)\}_{k \in \mathbb{N}}$?
- Key challenge: for all three, symmetry is lost:



becomes



Thank you