# Asynchronous Distributed Optimization

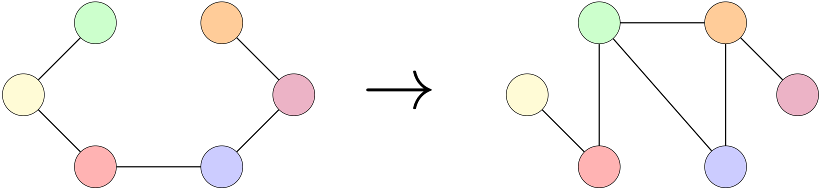
**Matthew Hale**

**Department of Mechanical and Aerospace Engineering**
**University of Florida**

**AFOSR Center of Excellence Kickoff**
**May 14, 2019**

# Agents collaborate on several types of tasks

- Modifying the network itself to improve communications



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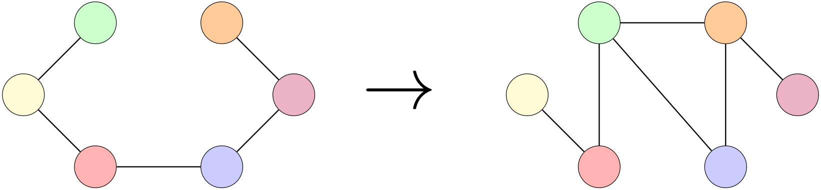

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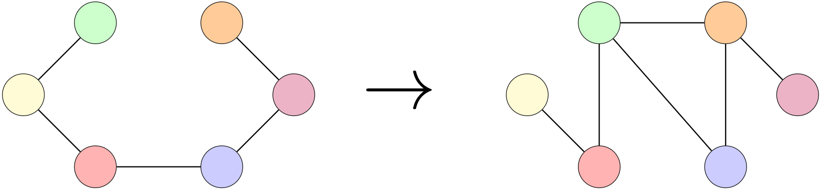

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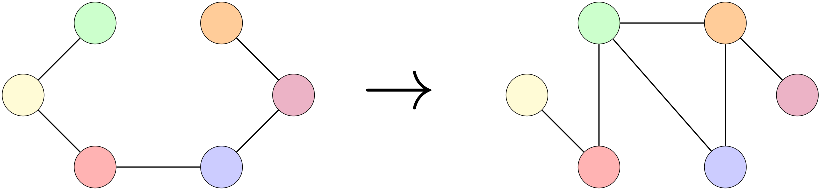

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**Common Threads**

These tasks require (i) information sharing and (ii) optimizing some quantity.

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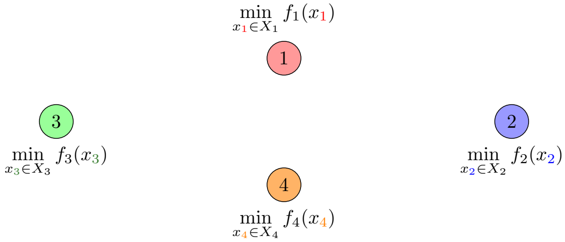

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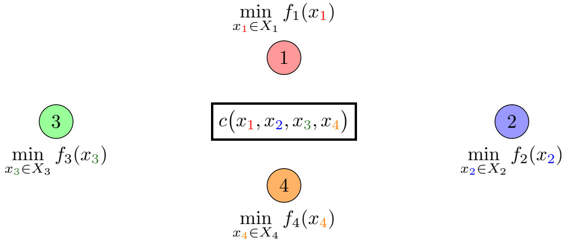

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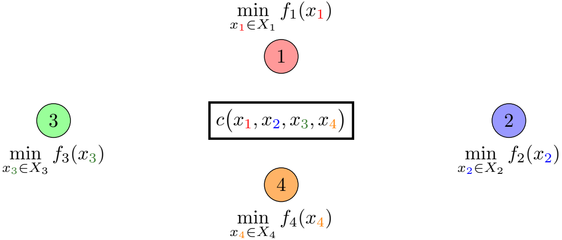


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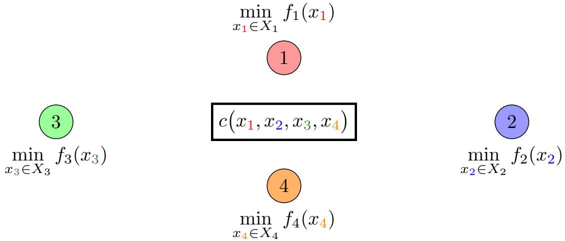
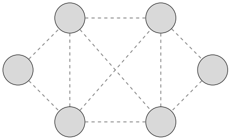



- Define $x = (x_1, x_2, x_3, x_4)$ and $X = X_1 \times X_2 \times X_3 \times X_4$
- Want to solve

$$\underset{x \in X}{\text{minimize}} \; f(x) := c(x) + \sum_{i=1}^{N} f_i(x_i)$$

# Agents must optimize asynchronously

- Communications are unavoidably asynchronous



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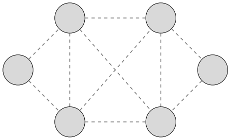

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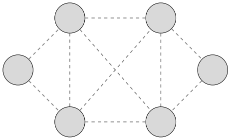

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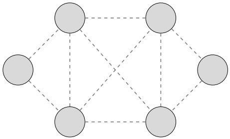

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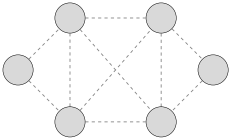

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- This is a forward invariance condition!

# Accounting for Heterogeneity

- We track agents' different information:

$$x^i(k) = \begin{pmatrix} x_1^i(k) \\ \vdots \\ x_N^i(k) \end{pmatrix}$$

- Always contains most recent information held by agent $i$

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- Let agents independently regularize

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| $f(x) + \alpha_1 \|x_1\|^2$ | $f(x) + \alpha_2 \|x_2\|^2$ | $f(x) + \alpha_3 \|x_3\|^2$ | $f(x) + \alpha_4 \|x_4\|^2$ |

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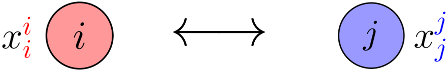

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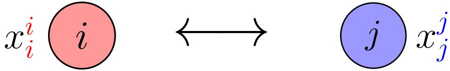


**Desired Update Law**

$$x_i^i(k+1) = \begin{cases} \Pi_{X_i}\left[x_i^i(k) - \gamma\left(\frac{\partial f}{\partial x_i}(x^i(k)) + \alpha_i x_i^i(k)\right)\right] & i \text{ updates at } k \\ \end{cases}$$

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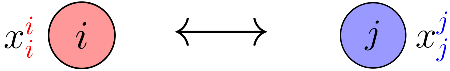


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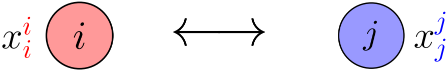


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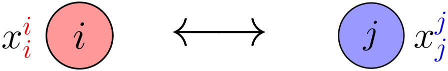


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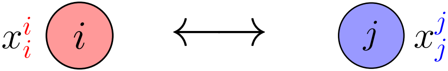


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"Do gradient descent with whatever you have"

# Lyapunov Convergence Analysis

- Define

$$V(k) = \max_{i \in [N]} \|x^i(k) - \hat{x}\|_2$$

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**Theorem 1: Asymptotic Convergence**

Suppose $\gamma$ is small enough and $\alpha_i > 0$ for all $i \in [N]$. Then $V(k) \to 0$ and

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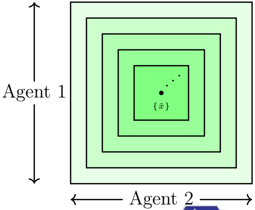
converges to $\hat{x}$ asymptotically.

- And each sub-level set is forward-invariant!

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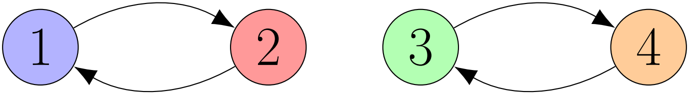
- Definition: A **communication cycle** occurs after each agent has (i) computed a state update and (ii) shared it with others that need it



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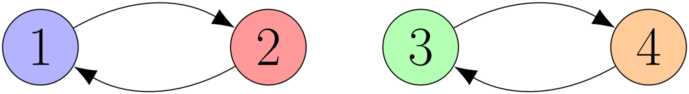


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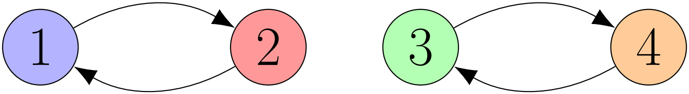


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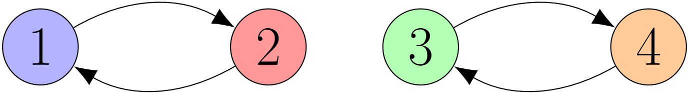


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**Theorem 2: Convergence Rate**

We have $q \in (0, 1)$ and

$$V(k) \leq q^{c(k)} V(0)$$

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**Direct Generalization of Classic Result**

Make centralized, set all parameters equal. Then

$$\|x(k) - \hat{x}\|_2^2 \leq q^k \|x(0) - \hat{x}\|_2^2.$$

# Next Steps

- Add in constraints!

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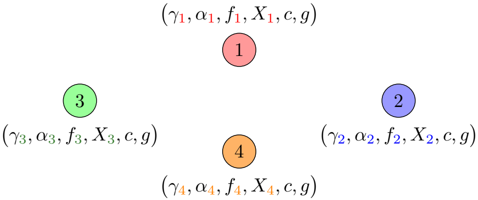
- End goal:



# Thank you