Totally Asynchronous Distributed Quadratic Programming

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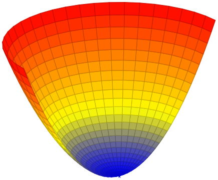
QPs Arise Across Control and Optimization

- QPs take the general form

$$\underset{x}{\text{minimize}}\ \frac{1}{2}x^TQx + r^Tx$$



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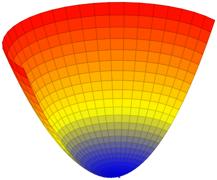

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- Appear explicitly in
  - Quadrotor trajectory generation
  - Numerical optimal control
  - Statistical learning

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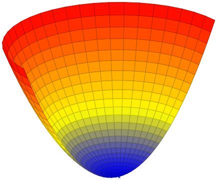

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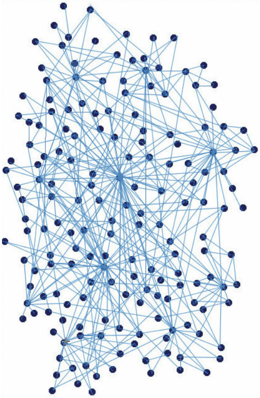
Overall Motivation

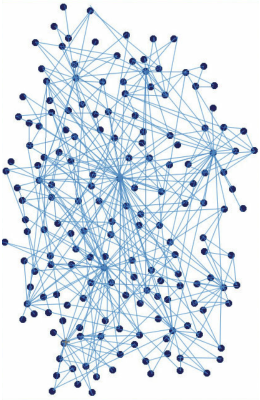
Design a multi-agent framework for solving QPs.

- Adversaries can disrupt agents' communications



Solving QPs in Contested Environments

- Adversaries can disrupt agents' communications
  - We don't want agents to wait and synchronize between computations
  - We don't want to require bounded delays



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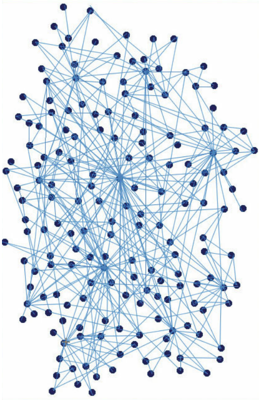

Algorithmic Goal

We want to allow *totally* asynchronous operations by agents.

Distributing Computations Across Agents

- Agent i stores a local copy of all decision variables, denoted x^i

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  - Promotes scalability
  - Amenable to control problems in which agents compute trajectories/control decisions

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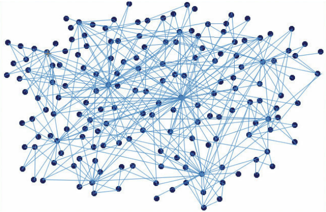
- Agent i updates only $x^i_{[i]}$
- Agent i waits to receive $x^i_{[j]}$ from agent j

$$x^i = \begin{pmatrix} x^i_{[1]} \\ \vdots \\ x^i_{[i]} \\ \vdots \\ x^i_{[n]} \end{pmatrix}$$

Gradients are Robust to Asynchrony

- Asynchrony requires a sufficiently robust update law
- It should also be simple to decentralize and scale up



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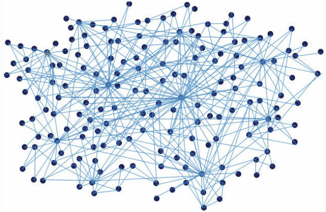

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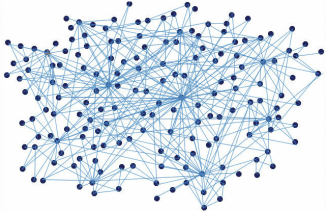

- Further want agents to execute updates as fast as possible
- We will use gradient descent as the template update law:

$$\begin{aligned} x(k+1) &= x(k) - \gamma \nabla f\big(x(k)\big) \\ &= x(k) - \gamma\big(Qx(k) + r\big) \end{aligned}$$

Proposed Distributed QP Algorithm

- Split up Q and r via

$$Q = \begin{pmatrix} Q^{[1]} \\ Q^{[2]} \\ \vdots \\ Q^{[N]} \end{pmatrix} \qquad r = \begin{pmatrix} r^{[1]} \\ r^{[2]} \\ \vdots \\ r^{[N]} \end{pmatrix}$$

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Convergence of Distributed QP Algorithm

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Suppose that

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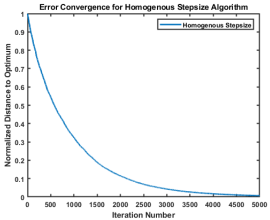
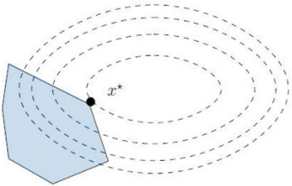
- Convergence is (imperfectly) geometric



Convergence Can Be Slow

- Lots of "reasonable" QPs are not well-conditioned
- Formally, if $k_Q = \frac{\lambda_1(Q)}{\lambda_n(Q)}$ is large, then convergence is slow



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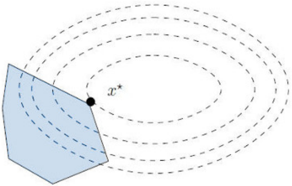

- Convergence rate is dictated by $q \in (0,1)$ here, which is

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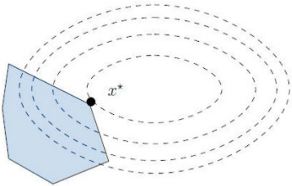

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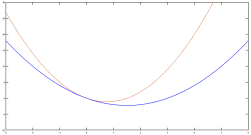
- As $k_Q \to \infty$, find $q \to 1$ and convergence comes to a halt

Heterogeneous Parameter Selection

- We can regularize to make them better: $Q + A$ replaces Q, now solve

$$\underset{x \in X}{\text{minimize}} \ \frac{1}{2} x^T (Q + A) x + r^T x$$



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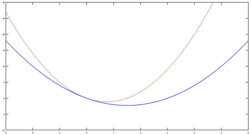

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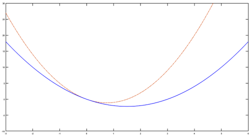
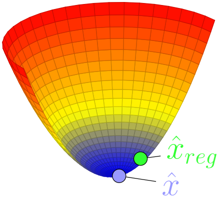


- Conventionally, $A = \alpha I$, but this requires agreeing on α
- We'd rather not have to agree on γ either

Independently Regularizing

- Regularizing Q changes the solution
- Small regularizations $\Rightarrow$ small error



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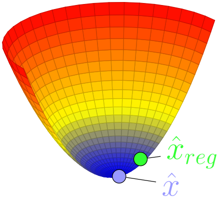

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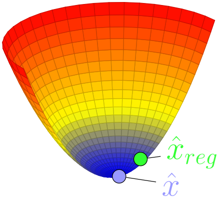

Theorem: Regularization Error

For $\epsilon \in (0,1)$, get $e(A) \leq \epsilon$ if

$$\alpha_i \leq \frac{\sqrt{\epsilon}}{1-\sqrt{\epsilon}} \underbrace{\left(\left|Q_{ii}^{[i]}\right| - \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|Q_{ij}^{[i]}\right| \right)}_{\text{How diagonally dominant row } i \text{ is}}$$

- Only requires knowledge of $Q^{[i]}$!

Independently Choosing Stepsizes

- Want stepsize rules that also depend only upon $Q^{[i]}$ and $r^{[i]}$

$$\gamma_i := \gamma_i\left(\boxed{\quad Q^{[i]} \quad}\,,\,\boxed{r^{[i]}}\right)$$

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- For ℓ^{th} variable, choose

$$\gamma_\ell < \frac{2}{\sum_{j=1}^{N} \left| Q_{kj} \right| + \alpha_\ell}$$

Algorithm 2

For all i and all k:

$$x^i_{[i]}(k+1) = \begin{cases} x^i_{[i]}(k) - \Gamma_i\left(Q^{[i]}x^i(k) + r^{[i]} + A_i x^i_{[i]}(k)\right) & i \text{ updates at } k \\ x^i_{[i]}(k) & \text{otherwise} \end{cases}$$

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- Solving

$$\underset{x\in X}{\text{minimize}}\ \frac{1}{2}x^TQx + r^Tx$$

- Have $k_Q = 100$
- Want $e(A) \leq 0.05$

Numerical Convergence

- Solving

$$\underset{x \in X}{\text{minimize}} \; \frac{1}{2} x^T Q x + r^T x$$

- Have $k_Q = 100$
- Want $e(A) \leq 0.05$
- Then $\alpha_i \leq 0.29$, use stepsize rule
- $k_Q = 77.8$ now