Nonsmooth Systems

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Motivation and Approach

Common features in AFOSR applications:

- Variables changing continuously (e.g., physical quantities) and discretely (e.g., logic variables, resetting timers).
- Abrupt changes in the dynamics (changes in the environment, control decisions, communication events, or failures).

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How can we systematically design such systems featuring **switching** and **intermittency of information** with provable robustness to uncertainties arising in real-world environments?

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Approach:

- Model continuous and discrete behavior using dynamical models that are **hybrid**.
- Develop systematic control theoretical tools for **stability**, **invariance**, **safety**, and **temporal logic**, with **robustness**.

Modeling Hybrid Dynamical Systems

Hybrid dynamical systems include a wide range of systems

Switched systems

$$\dot{z} = f_{\sigma(t)}(z)$$

σ switching signal

Impulsive systems

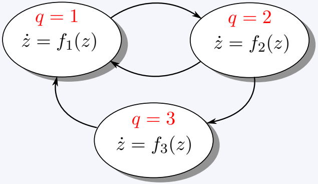
$$\begin{aligned} \dot{z}(t) &= f(z(t)) \\ z(t^+) &= g(z(t)) \qquad t = t_1, t_2, \ldots \end{aligned}$$

Differential-algebraic equations

$$\begin{aligned} \dot{z} &= f(z, w) \\ 0 &= \eta(z, w) \end{aligned}$$

w algebraic variables

Hybrid automata



Prevalent Network Control Applications

Control of Groups of Neurons
[ACC 14, TCNS 16]

Multi-agent Systems with Limited Information [Automatica 16, TAC 18]

Coordination of Underactuated Vehicles
[Automatica 15, TAC 16]

Prevalent Network Control Applications

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Key Features:

- Nonlinearities
- Fast time scales / events
- Limited information

Outline of Recent Results Relevant to the CoE

1. **Optimization**
   - High Performance Optimization via Uniting Control
     ACC19, ACC20 (submitted), + CoE collab. (M. Hale)
   - Model Predictive Control for Hybrid Systems
     ACC19, CDC19, ACC20 (submitted), CDC19 Workshop
     + collab. w/ AFRL/RV (S. Phillips and C. Petersen)
2. **Tools to Satisfy High-level Specifications**
   - Solution-independent Conditions for Invariance and Finite-time Attractivity *Automatica 19, TAC 19, NAHS and IFAC WC20 (submitted) + Collab. w/ NASA (A. Mavridou)*
   - (Necessary and Sufficient) Safety Certificates
     HSCC19, ACC19, and ACC20 (submitted)
3. **Hybrid Control**
   - Global Robust Stabilization on Manifolds
     Automatica 19, TAC19, and ACC19 + CoE collab. (W. Dixon)
   - Synchronization over Networks w/ Intermittent Information *Automatica 19, ACC19, and ACC20 (submitted)*

Safety Certificates

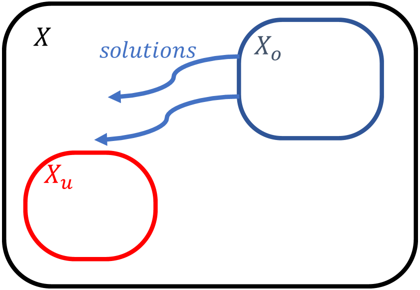
Basic Setting

Consider the system

$$\dot{x} = f(x) \qquad x \in X$$

and the sets

$$X_o \subset X \text{ the initial set,}$$
$$X_u \subset X \backslash X_o \text{ the unsafe set.}$$



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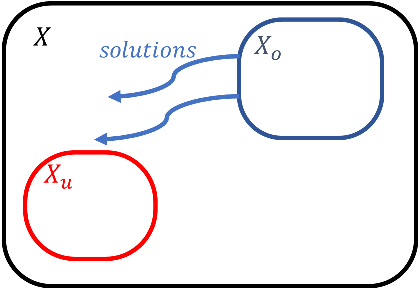


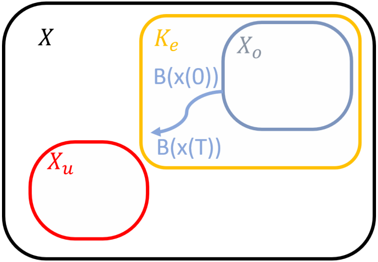
Safety with respect to (X_o, X_u) $\Leftrightarrow$ $\mathrm{reach}(X_o) \cap X_u = \emptyset$

$\mathrm{reach}(X_o) := \{x \in \mathbb{R}^n : x = \phi(t; x_o)$, with ϕ a solution from $x_o \in X_o$ and any $t \in \mathrm{dom}\,\phi\}$ – namely, the infinite reach set

Sufficient Conditions for Safety when $X = \mathbb{R}^n$

Let B be continuous such that

$$B(x) > 0 \qquad \forall x \in X_u$$
$$B(x) \leq 0 \qquad \forall x \in X_o$$



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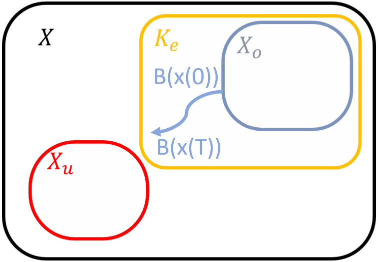

and for each solution ϕ from $x_o \in \mathbb{R}^n \backslash K_e$

$t \mapsto B(\phi(t; x_o))$ is nonincreasing

where $K_e := \{x \in \mathbb{R}^2 : B(x) \leq 0\}$ – the **zero-sublevel** set of B

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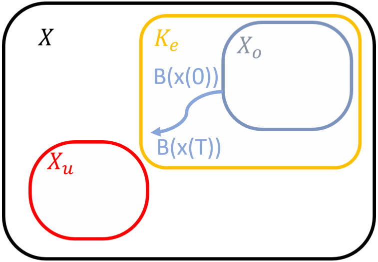

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It follows that the system $\dot{x} = f(x)$ is safe w.r.t. (X_o, X_u)

Converse Safety Problem

Given a safe system $\dot{x} = f(x)$ w.r.t. (X_o, X_u), find a scalar function $B : \mathbb{R}^n \to \mathbb{R}$ (at least continuous) such that

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Two solutions to the **converse safety problem** in the literature are

[Prajna & Rantzer 05] when

1. $f \in \mathcal{C}^1$
2. $\exists V \in \mathcal{C}^1$ s.t.
   $\langle \nabla V(x), f(x) \rangle < 0 \; \forall x \in X$
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[Wisniewski & Sloth 17] when

1. $f \in \mathcal{C}^1$
2. $\exists V$ smooth s.t. $\langle \nabla V(x), f(x) \rangle < 0$ on X except at critical points of V $+$ geometric conditions
3. (X, X_o, X_u) are compact manifolds

Safe System but No Continuous Barrier Exists

Consider the planar continuous-time system [Krasovskii 63]

$$\begin{aligned} \dot{x}_1 &= -x_2 + r x_1 \sin^2(1/r) \\ \dot{x}_2 &= x_1 + r x_2 \sin^2(1/r) \end{aligned}$$

where

$$r = |x|$$

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It is easy to check that the system is safe w.r.t. (X_o, X_u) due to X_o being **forward invariant**

Safe System w/o C^0 State-Dependent Barrier

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There is no continuous barrier function with $B(0) = 0$, $B(x) > 0$ forall $x \in X_u$, and $t \mapsto B(\phi(t; x_o))$ nonincreasing on $\mathbb{R}^n \setminus X_o$

Modeling Hybrid Systems: Closed Loop

Hybrid closed-loop systems are given by *hybrid inclusions*

$$\mathcal{H} \quad \begin{cases} \dot{x} = F(x) & x \in C \\ x^+ = G(x) & x \in D \end{cases}$$

where x is the *state*

- C is the *flow set*
- F is the *flow map*
- D is the *jump set*
- G is the *jump map*

Solutions are functions parameterized by hybrid time (t, j):

- Flows parameterized by $t \in \mathbb{R}_{\geq 0} := [0, +\infty)$
- Jumps parameterized by $j \in \mathbb{N}_{\geq 0} := \{0, 1, 2, \ldots\}$

Then, solutions to $\mathcal{H}$ are given by hybrid arcs x defined on

$$([0, t_1] \times \{0\}) \cup ([t_1, t_2] \times \{1\}) \cup \ldots ([t_j, t_{j+1}] \times \{j\}) \cup \ldots$$

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For this class of systems, we have the following:

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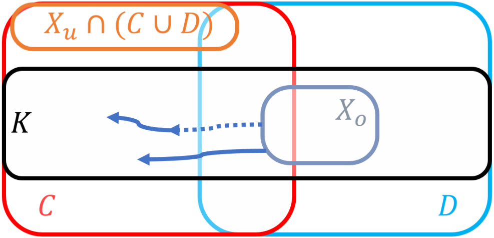
- An initial set $X_o \subset C \cup D$ and an unsafe set $X_u \subset \mathbb{R}^n$.
- Assume that $\mathbb{R}^n \backslash (C \cup D) \subset X_u$.
- A **barrier function candidate** $B : \mathbb{R}^n \to \mathbb{R}$ is defined as $B(x) > 0 \ \forall x \in X_u \cap (C \cup D)$ and $B(x) \leq 0 \ \forall x \in X_o$.
- A **barrier function candidate** B defines the set

$$K := \{x \in C \cup D : B(x) \leq 0\}$$

Sufficient Conditions for Safety

Theorem: The system $\mathcal{H}$ is safe w.r.t. (X_o, X_u) if a **barrier candidate** B exists such that $K = \{x \in C \cup D : B(x) \leq 0\}$ is closed and forward pre-invariant for $\mathcal{H}$.



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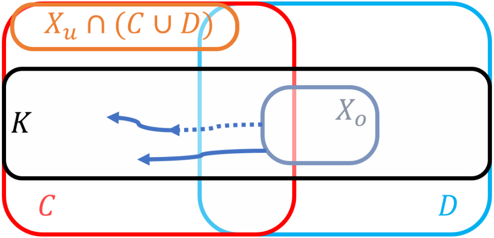

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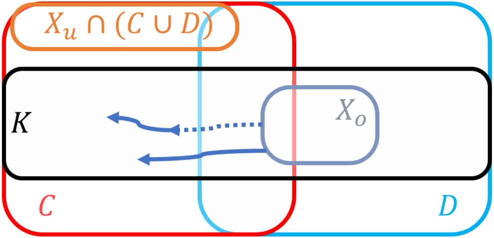

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1. $B(\eta) \leq 0 \quad \forall \eta \in G(x) \; \forall x \in D \cap K$
2. $G(D \cap K) \subset C \cup D$

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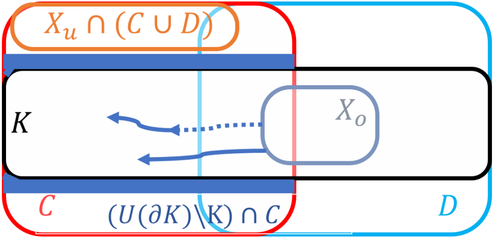

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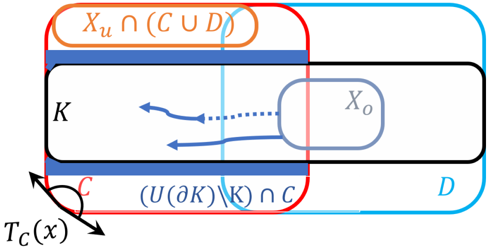
2. $G(D \cap K) \subset C \cup D$
3. $t \mapsto B(\phi(t,0))$ is nonincreasing for flowing solutions $t \mapsto \phi(t,0)$ in $(U(\partial K)\backslash K) \cap C$

where $U(S)$ is any neighborhood around the set S

Monotone Behavior Along the Flows



Proposition:

- When B is $\mathcal{C}^1$, 3. is satisfied if $\langle \nabla B(x), \eta \rangle \leq 0 \; \forall x \in (U(\partial K_e) \setminus K_e) \cap C \; \forall \eta \in F(x) \cap T_C(x)$.
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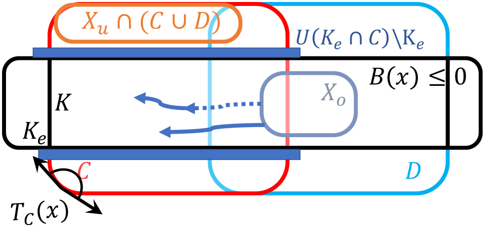

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- When B is loc. Lip., we replace ∇B by ∂B.
- When B is lower semicontinuous and F locally bounded we replace ∇B by $\partial_p B$ and $(U(\partial K_e)\backslash K_e) \cap C$ by $U(K_e \cap C)\backslash K_e$, $K_e := \{x \in \mathbb{R}^n : B(x) \leq 0\}$.

$(\partial B, \partial_p B, T_C)$ are the generalized gradient, the proximal subdifferential, and the contingent cone w.r.t. C [Clarke & al 08].

Safety for Bouncing Ball

Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ modeling a ball bouncing vertically on the ground, with $x = (x_1, x_2) \in \mathbb{R}^2$ given by

$$F(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} \qquad \forall x \in C,\ \gamma > 0$$

$$G(x) := \begin{bmatrix} 0 \\ -\lambda x_2 \end{bmatrix} \qquad \forall x \in D,\ \lambda \in [0, 1]$$

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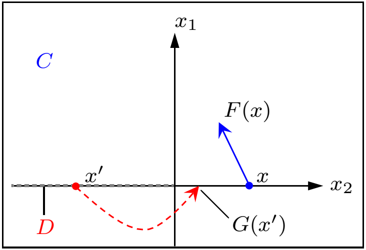

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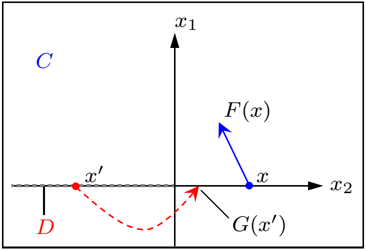

- Let $X_o := \{x \in C : |x| \leq 1/(4\gamma)\}$ and $X_u := \{x \in C : x_1 > 1/\gamma, \ x_2 = 0\}$.
- Consider the barrier candidate $B(x) := 2\gamma x_1 + (x_2 - 1)(x_2 + 1)$.

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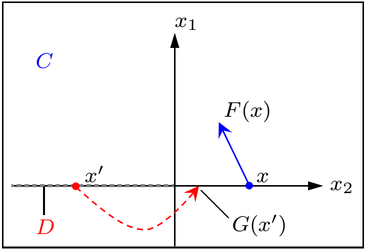

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- Condition 1) holds since $B(G(x)) = 2\gamma x_1 + \lambda^2 x_2^2 - 1 \leq 2\gamma x_1 + x_2^2 - 1 \leq 0 \ \forall x \in K \cap D$.
- Condition 2) holds since $G(D) = \{0\} \times \mathbb{R}_{\geq 0} \subset C \cup D$.
- Condition 3) holds since $\langle \nabla B(x), F(x) \rangle = 0 \ \forall x \in C$.

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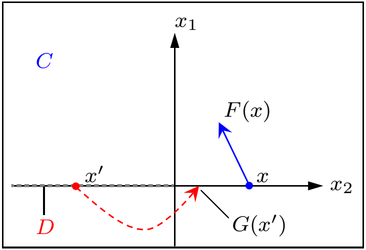

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- Let $X_o := \{x \in C : |x| \leq 1/(4\gamma)\}$ and $X_u := \{x \in C : x_1 > 1/\gamma, \ x_2 = 0\}$.
- Consider the barrier candidate $B(x) := 2\gamma x_1 + (x_2 - 1)(x_2 + 1)$.
- Condition 1) holds since $B(G(x)) = 2\gamma x_1 + \lambda^2 x_2^2 - 1 \leq 2\gamma x_1 + x_2^2 - 1 \leq 0 \ \forall x \in K \cap D$.
- Condition 2) holds since $G(D) = \{0\} \times \mathbb{R}_{\geq 0} \subset C \cup D$

The system $\mathcal{H}$ is safe with respect to (X_o, X_u)

Safety for System with Discrete States

Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ modeling a thermostat system, with the state $x = (q, z) \in \mathcal{X} := \{0, 1\} \times \mathbb{R}$ given by

$$F(x) := \begin{bmatrix} 0 \\ -z + z_0 + z_\Delta q \end{bmatrix} \qquad \forall x \in C$$

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$$C := (\{0\} \times C_0) \cup (\{1\} \times C_1), \quad D := (\{0\} \times D_0) \cup (\{1\} \times D_1).$$

$$C_0 := \{z \in \mathbb{R} : z \geq z_{min}\} \qquad D_0 := \{z \in \mathbb{R} : z \leq z_{min}\}$$

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- z is the room temperature, z_o the room temperature when the heater is OFF
- z_Δ the capacity of the heater to raise the temperature
- q the state of the heater 1 (ON) or 0 (OFF)

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- z stays between z_{min} and z_{max} satisfying

  $$z_o < z_{min} < z_{max} < z_o + z_\Delta.$$
- Let $X_o := \{(q, z) \in \mathcal{X} : z \in [z_{min}/2, z_{max}/2]\}$.
- Let $X_u := \{(q, z) \in \mathcal{X} : z \in (-\infty, z_{min}) \cup (z_{max}, +\infty)\}$.
- Consider the barrier candidate $B(x) := (z - z_{min})(z - z_{max})$ and let $K_e := \{x \in \mathbb{R}^2 : B(x) \leq 0\}$.

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- $C \cup D = \{0, 1\} \times \mathbb{R}$; hence, condition 2) holds since
  $$G(x) = [1 - q \;\; z]^\top \in C \cup D \quad \forall x \in C \cup D.$$
- Condition 1) holds since
  $$B(G(x)) = B([(1 - q) \;\; z]^\top) = B(x) \leq 0 \quad \forall x \in K_e \cap D.$$

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- $K_e = \mathbb{R} \times [z_{min}, z_{max}]$.
- Furthermore, for some $\epsilon > 0$, $(U(K_e) \backslash K_e) \cap C =$ $(\{0\} \times (z_{max}, z_{max} + \epsilon)) \cup (\{1\} \times (z_{min}, z_{min} - \epsilon))$. Hence, condition 3) holds since

$$\langle \nabla B(x), F(x) \rangle = (z_{min} + z_{max} - 2z)(z - z_o - z_\Delta q) \leq 0$$

for all $x \in (U(K_e) \backslash K_e) \cap C$.

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$\Delta q) \leq 0$

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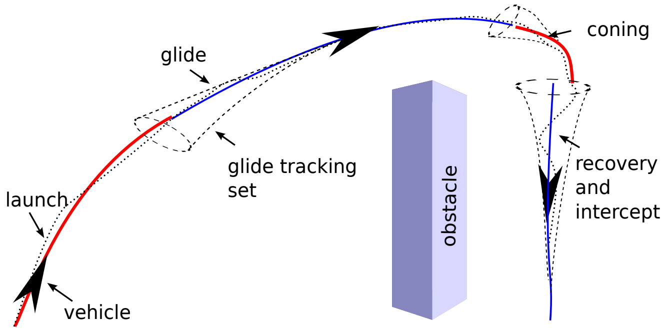
Safety-Based Control for Agile Evasion

Presented at 2019 CASE Robotics Conference – Best Paper Award Finalist

Hybrid Control for Aggresive Maneuvering



Achieving non-parabolic ballistic trajectories for a guided munition using aggressive maneuvers using multiple controllers.

Necessary and Sufficient Conditions for Safety

Theorem: Assume that $\mathcal{H}$ is satisfies the hybrid basic conditions* and that the backward solutions are either bounded or complete.

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- If X_o is **compact**, then the **pre-completeness** condition on the backward solutions is not needed.

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$$
\begin{aligned}
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&G(x) \subset C \cup D && \forall x \in D \ : \ (\tau, k, x) \in K_e, (\tau, k) \in \mathbb{R}_{\geq 0} \times \mathbb{N}
\end{aligned}
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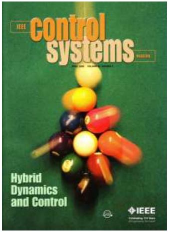
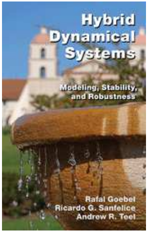
- If the solutions to $\mathcal{H}$ are not **Zeno**, then the result holds with B independent of k.

Summary:

- Overview of Recent Results
- Introduction to Safety
- Sufficient Conditions
- Necessary and Sufficient Conditions



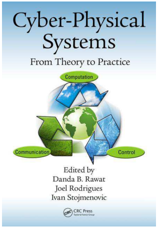
IEEE 2009



Princeton U. Press 2012

Next steps:

- Estimation
- Reachability
- Approximations
- Robustness



CRC Press 2015

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