Guaranteeing Safety in Control Systems with Intermittent Information

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CoE Review @ Zoom - October 30, 2020

Outline of Recent Results

1. **Estimation**
   - Finite-time Parameter Estimation via Hybrid Methods
     ACC 21a, ACC 21b, ACC 21c (all submitted), + *CoE collab*
   - Observers for Hybrid Systems *ACC 20, CDC 19, CDC 20, Automatica (submitted)*
2. **Safety**
   - Reachable maps for hybrid systems and regularity
     HSCC 20, TAC 19, NAHS 20, HSCC 20, CDC 20 (submitted)
   - (Necessary and Sufficient) Safety Certificates, with Events *ACC 21a, ACC 21d (submitted), TAC 20* + *CoE collab*
3. **Optimization**
   - High Performance Optimization via Uniting Control
     ACC 19, MTNS 20, ACC 20e (submitted) + AFRL/RV collab.
   - Model Predictive Control for Hybrid Systems *ACC 20, CDC 20, IFAC WC 20 Workshop*

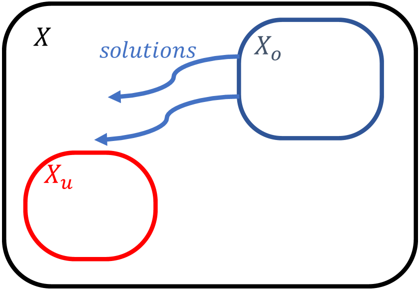
Basic Setting

Consider the system

$$\dot{x} = f(x) \qquad x \in X \subset \mathbb{R}^n$$

and the sets

$$X_o \subset X \text{ the initial set,}$$
$$X_u \subset X \backslash X_o \text{ the unsafe set.}$$



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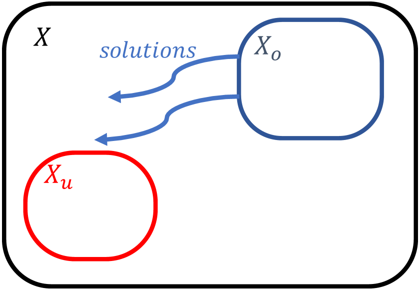

Safety with respect to (X_o, X_u) $\Leftrightarrow$ $\mathrm{reach}(X_o) \cap X_u = \emptyset$

$$\mathrm{reach}(X_o) := \{x \in \mathbb{R}^n : x = \phi(t; x_o), \text{with } \phi \text{ a solution from } x_o \in X_o \text{ and } t \in \mathrm{dom}\,\phi\}$$ $\leftarrow$ the infinite reach set

A solution to $\dot{x} = f(x)$ is denoted $t \mapsto \phi(t)$, and when starts at x_o as $t \mapsto \phi(t; x_o)$

Guaranteeing Safety in Control Systems with Intermittent Information

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Assured Autonomy in Contest Environments

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Fall 2020 Review

October 30, 2020

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Context

Consider a constrained control system $\mathcal{H}_f := (C, F)$ given by

$$\dot{x} \in F(x, u) \qquad x \in C \subset \mathbb{R}^n, \quad u \in \mathbb{R}^m.$$

- Assume that C is closed and $F : C \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded, and with convex images.
- Consider a continuous feedback law $\kappa : C \to \mathbb{R}^m$ that enforces a control objective, e.g. stability, convergence, safety.
- Assume that the control input is updated only at a sequence to times $\{t_i\}_{i=0}^{\infty}$ and the measurements are available only at that sequence. Hence,

$$u(t) = \kappa(x(t_i)) \qquad \forall t \in [t_i, t_{i+1}].$$

How does this implementation affects the control objective?

→ **How to design $\{t_i\}_{i=0}^{\infty}$ such that $(t_{i+1} - t_i)$ is large?**
→ **What conditions guarantee $(t_{i+1} - t_i) \geq T_s^* > 0$?**

Background

- **Classical Digital Implementation** [Franklin et al., 1997]

$$t_{i+1} = t_i + T_s$$

  - Constant inter-event times, analysis using heuristics (20 x the system bandwidth), or viewing the digital implementation as a delayed input.

- **Event-Triggered Control** [Tabuada, 2007], given $\Gamma : C \mapsto \mathbb{R}$,

$$t_{i+1} = t_i + \max\{t \geq 0 : \Gamma(x(s + t_i)) > 0 \ \forall s \in [0, t]\}.$$

  - Measurements always available.

- **Self-Triggered Control** [Anta and Tabuada, 2010]. Let $T_s : C \mapsto \mathbb{R}_{\geq 0}$ be a **sampling function**; hence,

$$t_{i+1} = t_i + T_s(x(t_i)).$$

  - Measurements available only at the sampling times.

Notions

Definition (Forward pre-Invariance)

A set $X \subset C$ is forward pre-invariant for $\mathcal{H}_f$ if, for each $x_o \in X$ and for each solution x to $\mathcal{H}_f$ starting at x_o, $x(t) \in X$ for all $t \in \operatorname{dom} x$.

Definition (Barrier Function Candidate)

A a scalar function $\rho : C \to \mathbb{R}$ is a barrier function candidate defining the set $X \subset C$ if $X = \{x \in \mathrm{cl}(C) : \rho(x) \geq 0\}$.

Definition (Reachability Map)

Given $x_o \in C$ and $T > 0$, the reachability map R is given by

$$R(T, x_o) := \{\phi(t) : \phi \in \mathcal{S}_{\mathcal{H}_f}(x_o),\ t \in \operatorname{dom} x \cap [0, T]\}$$

where $\mathcal{S}_{\mathcal{H}_f}(x_o)$ be the set of solutions to $\mathcal{H}_f$ from x_o.

Guaranteeing Forward Invariance

Theorem

Consider a system $\mathcal{H}_f := (C, F)$, a closed set $X \subset C$, a continuously differentiable barrier function candidate $\rho : C \to \mathbb{R}$, and a continuous feedback $\kappa : C \mapsto \mathbb{R}^m$. Assume that, for each $x \in C$,

$$\langle \nabla\rho(x), f\rangle \geq \alpha(x) \qquad \forall f \in (F(x, \kappa(x)) \cap T_C(x)).$$

- *When $\alpha(x) \geq 0$ for all $x \in U(X_e) \backslash X \cap C$. Then, X is* **forward pre-invariant** *for the closed-loop of $\mathcal{H}_f$ using κ.*
- *When $\alpha(x) > 0$ for all $x \in \partial X_e \cap C$. Then, X is* **pre-contractive** *for the closed-loop of $\mathcal{H}_f$ using κ.*

Definition: A set $X \subset C$ is **pre-contractive** for $\mathcal{H}_f$ if, for each $x_o \in X$ and for each **nontrivial** solution x to $\mathcal{H}_f$ starting at x_o, $x(t) \in \text{int}(X)$ for all $t \in \text{int}(\text{dom}\, x)$.

• $X_e := \{x \in \mathbb{R}^n : \rho(x) \geq 0\}$ • $U(X)$ is an open neighborhood around X

Guaranteeing Forward Invariance

Theorem

Consider a system $\mathcal{H}_f := (C, F)$, a closed set $X \subset C$, a locally Lipschitz barrier function candidate $\rho : C \to \mathbb{R}$, and a continuous feedback $\kappa : C \mapsto \mathbb{R}^m$. Assume that, for each $x \in C$,

$$\langle \zeta, f \rangle \geq \alpha(x) \qquad \forall (\zeta, f) \in \partial\rho(x) \times (F(x, \kappa(x)) \cap T_C(x)).$$

- *When $\alpha(x) \geq 0$ for all $x \in U(X_e) \backslash X \cap C$. Then, X is* **forward pre-invariant** *for the closed-loop of $\mathcal{H}_f$ using κ.*
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• $X_e := \{x \in \mathbb{R}^n : \rho(x) \geq 0\}$ • $U(X)$ is an open neighborhood around X

Problem Formulation

Consider a constrained control system $\mathcal{H}_f = (C, F)$ and a feedback law $\kappa : C \mapsto \mathbb{R}^m$ that renders X **forward pre-invariant** for

$$\mathcal{H}_f^{cl} : \quad \dot{x} \in F(x, \kappa(x)) \qquad x \in C \subset \mathbb{R}^n.$$

(P1) Find a function $T_s : C \mapsto \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that the sequence

$$t_{i+1} = t_i + T_s(x(t_i)).$$

guarantees forward pre-invariance of the self-triggered closed-loop system.

(P2) Find conditions under which there exists $T_s^* > 0$ such that

$$t_{i+1} - t_i \geq T_s^* \qquad \forall i \in \mathbb{N}.$$

Overview

- **Event-Triggered Stability** [Tabuada, 2007], of the origin, F globally Lipschitz. [Chai et al., 2017], of a compact set.
- **Self-Triggered Stability** [Anta and Tabuada, 2010], of the origin, F homogeneous. [Tiberi and Johansson, 2012], of the origin, F globally Lipschitz.
- **Event-Triggered Forward Invariance** [Taylor et al., 2020], a general closed set X, F globally bounded, ρ continuously differentiable.
- **Self-Triggered Forward-Invariance** [Di Benedetto et al., 2013], X compact, F smooth. [Kogel and Findeisen, 2014], X convex and compact, linear systems.

Contributions: - We assume mild regularities on F, - we allow X to be unbounded, - we allow ρ to be nonsmooth, - we avoid using a global bounds on F.

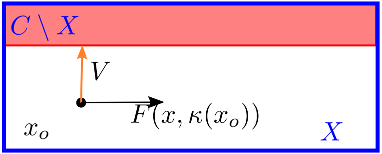
Example

Consider a control system $\mathcal{H}_f = (\mathbb{R}^2, F)$ with and

$$F(x,u) := (0,u)^\top.$$

Assume that for some $V > 0$, $|F(x,u)| \leq V$ for all $(x,u) \in \operatorname{dom} F$.

Let $X := \{x \in C : x_1 < \delta\}$ and the corresponding barrier function candidate $\rho(x) := \delta - x_1$.



Then, the sampling function $T_s(x_o) := \rho(x_o)/V$ [Fainekos et al., 2009] guarantees forward invariance of X for the self-triggered closed loop system.

- The sampling function $T_s(x_o)$ is too conservative.

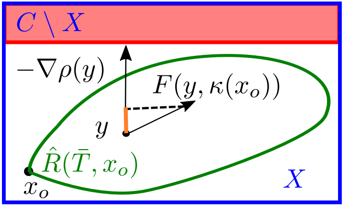
Indeed, solutions never approach ∂X, but $T_s(x_o)$ is calculated as if they are.

A Reachabiliy Based Approach

Consider a control system $\mathcal{H}_f = (\mathbb{R}^n, F)$ with F single valued. Let a set X be defined by a smooth barrier candidate ρ. Assume a continuous feedback law κ renders X forward invariant for $\mathcal{H}_f^{cl}$.

The speed of the solutions, starting from $x_o \in X$, towards ∂X on the interval $[0, \bar{T}]$ is upper bounded by

$$M(\bar{T}, x_o) := \sup\{\langle -\nabla\rho(y), F(y, \kappa(x_o))\rangle, \\ y \in \hat{R}(\bar{T}, x_o))\}.$$



Hence, a sampling function guaranteeing forward invariance of X for the self-triggered closed loop system is given by

$$T_s(x_o) := \begin{cases} \bar{T} & \text{if } M(\bar{T}, x_o) \leq 0 \\ \min\left\{\bar{T}, \frac{\rho(x_o)}{M(\bar{T}, x_o)}\right\} & \text{otherwise.} \end{cases}$$

- $\hat{R}(\bar{T}, x_o)$ overestimates $R(\bar{T}, x_o)$ along the solutions to $\dot{x} = F(x, \kappa(x_o))$.
- $\bar{T}$ is the forward propagation interval of the reachable set.

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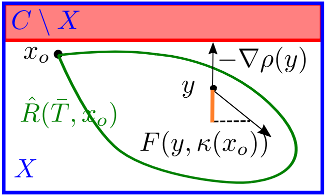

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Proposed Self-Triggered Control Strategy

Consider a system $\mathcal{H}_f^{cl} := (C, F)$, a closed set $X \subset C$ defined by a continuously differentiable barrier function candidate ρ, and a feedback law $\kappa : C \mapsto \mathbb{R}^m$ such that, for each $(x, \eta) \in C \times C$,

$$\langle \nabla \rho(x), f \rangle \geq \alpha(x) - \gamma(x, \eta) \qquad \forall f \in (F(x, \kappa(\eta)) \cap T_C(x)),$$

where $\alpha : \mathbb{R}^n \mapsto \mathbb{R}$ and $\gamma : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ are continuously differentiable functions with $\alpha(x) > 0$ for all $x \in \partial X$, and $\gamma(x, x) = 0$.

Then, the sampling sequence given by

$$t_{i+1} = t_i + T_s(x(t_i)) \quad T_s(x(t_i)) := \max\{T_1(x(t_i)), T_2(x(t_i))\},$$

where T_1, T_2 are defined next, solves (P1).

(P1) Find a function $T_s : C \mapsto \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that the sequence $t_{i+1} = t_i + T_s(x(t_i))$. guarantees forward pre-invariance of the self-triggered closed-loop system.

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$$T_1(x_o) := \begin{cases} \bar{T} & \text{if } M_\alpha(\bar{T}, x_o) - M_\gamma(\bar{T}, x_o) \geq 0 \\ \min\left\{\bar{T}, \frac{2\alpha(x_o)}{M_\alpha(\bar{T},x_o) - M_\gamma(\bar{T},x_o)}\right\} & \text{otherwise,} \end{cases}$$

$$T_2(x_o) := \begin{cases} \bar{T} & \text{if } M_2(\bar{T}, x_o) \leq 0 \\ \min\left\{\bar{T}, \rho(x_o)/M_2(x_o, \bar{T})\right\} & \text{otherwise,} \end{cases}$$

$$M_\gamma(\bar{T}, x) := \sup\{\langle \nabla\gamma(y), \eta\rangle : \eta \in F(y, \kappa(x)) \cap T_C(y), y \in \hat{R}(\bar{T}, x)\},$$
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$$\langle \zeta, f \rangle \geq \alpha(x) - \gamma(x, \eta)$$
$$\forall (\zeta, f) \in \partial_C \rho(x) \times (F(x, \kappa(\eta)) \cap T_C(x)),$$

where $\alpha : \mathbb{R}^n \mapsto \mathbb{R}$ and $\gamma : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ are locally Lipschitz functions with $\alpha(x) > 0$ for all $x \in \partial X$, and $\gamma(x, x) = 0$.

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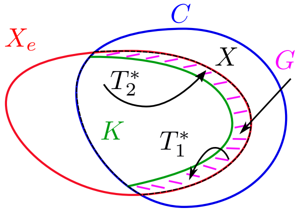
$$\begin{aligned} M_\gamma(\bar{T}, x) &:= \sup\{\langle \gamma_1, \eta \rangle : \gamma_1 \in \partial_C \gamma(y, x), \\ &\qquad \eta \in F(y, \kappa(x)) \cap T_C(y), y \in \hat{R}(\bar{T}, x)\}, \\ M_\alpha(\bar{T}, x) &:= \sup\{\langle -\gamma_2, \eta \rangle : \gamma_2 \in \partial_C \alpha(y), \\ &\qquad \eta \in F(y, \kappa(x)) \cap T_C(y),\ y \in \hat{R}(\bar{T}, x)\}, \\ M_2(x_o, \bar{T}) &:= \sup\{\langle -\gamma, \eta \rangle : \gamma \in \partial_C \rho(y), \\ &\qquad \eta \in F(y, \kappa(x_o)) \cap T_C(y), y \in \hat{R}(\bar{T}, x_o)\} \end{aligned}$$

- $\hat{R}(\bar{T}, x_o)$ overestimates $R(\bar{T}, x_o)$ along the solutions to $\dot{x} = F(x, \kappa(x_o))$.

Guaranteeing a Uniform Lower Bound

Assume that, given $\bar{T} > 0$ and $\beta > 0$, the following hold:

(A1) There exists $T_1^* > 0$ such that
$\min\{T_1(x) : x \in G\} \geq T_1^*$,
$G := \{x \in X,\ |x|_{\partial X_e} \leq \beta\}$.



(A2) There exists $T_2^* > 0$ such that, for each solution x to
$\dot{x} \in F(x, \kappa(x_o))\ x \in C$ from $x_o \in K := \{x \in X : |x|_{\partial X_e} \geq \beta\}$, we
have $x(t) \in X$ for all $t \in [0, T_2^*]$.

Then, (P1) and (P2) are solved with

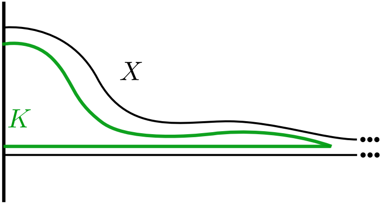
$$t_{i+1} = t_i + T_s(x(t_i))$$
$$T_s(x(t_i)) := \max\{T_2^*, T_1(x(t_i)), T_2(x(t_i))\}$$

Particular Scenarios

Assume, that, given $\bar{T} > 0$ and $\beta > 0$, the following hold:

(A1) $\min\{T_1(x) : x \in G\} > 0$,
$G := \{x \in X,\ |x|_{\partial X_e} \leq \beta\}$.

(A3) The set
$K := \{x \in X : |x|_{\partial X_e} \geq \beta\}$ is
compact.



(A4) The set-valued map $_o \mapsto \hat{R}(\bar{T}, x_o)$ is outer semicontinuous and locally bounded on K.

Then, the sampling sequence given by

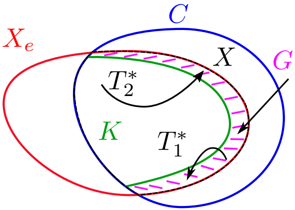
$$t_{i+1} = t_i + T_s(x(t_i)) \quad T_s(x(t_i)) := \max\{T_1(x(t_i)), T_2(x(t_i))\}$$

solves (P1) and (P2).
In Fact, under (A3) and (A4), we prove that, there exists $T_2^* > 0$ such that $\min\{T_2(x) : x \in K\} \geq T_2^*$.

Particular Scenarios

Assume, that, given $\bar{T} > 0$ and $\beta > 0$,

(A4) The set-valued map $_o \mapsto \hat{R}(\bar{T}, x_o)$ is outer semicontinuous and locally bounded on $G := \{x \in X : |x|_{\partial X_e} \le \beta\}$.



Furthermore, one of the following holds:

(A5) The set G is compact.

(A5') The maps $x_o \mapsto \partial_C \gamma(\hat{R}(\bar{T}, x_o), x_o)$, $x_o \mapsto \partial_C \alpha(\hat{R}(\bar{T}, x_o))$, $x_o \mapsto F(\hat{R}(\bar{T}, x_o), \kappa(x_o))$, and $x_o \mapsto \alpha(x_o)$ are uniformly upper semicontinuous on G and bounded on $\partial X_e \cap X$, and $\inf\{\alpha(z) : z \in \partial X_e \cap X\} > 0$.

Then, we can replace (A1) by

(A1') $\min \{T_1(x) : x \in \partial X_e \cap C\} > 0$.

Ex: Forward Invariance of a Sub-Level Set

Consider the control system $\mathcal{H}_f^u = (\mathbb{R}^2, F)$, where

$$F(x,u) := \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

$x := (x_1, x_2) \in \mathbb{R}^2$, and $u \in \mathbb{R}$. Furthermore, consider the feedback law

$$\kappa(x) := Kx := [1 \;\; -4]x.$$

The origin of the closed-loop of $\mathcal{H}_f^u$ using $u = \kappa(x)$, denoted $\mathcal{H}_f^{cl}$, is asymptotically stable. Indeed, using the Lyanpunov function

$$V(x) := x^\top P x, \;\; P := \begin{bmatrix} 1 & 0.25 \\ 0.25 & 1 \end{bmatrix},$$

we conclude that

$$\langle \nabla V(x), Ax + BKx \rangle = -x^\top Q x, \quad Q := \begin{bmatrix} 0.5 & 0.25 \\ 0.25 & 1.5 \end{bmatrix}.$$

Ex: Forward Invariance of a Sub-Level Set

Consider the set X given by

$$X := \{x \in \mathbb{R}^2 : V(x) \leq 0.1\}.$$

For

$$\rho(x) := 0.1 - V(x),$$

we obtain $\alpha(x) := x^\top Qx$ and $\gamma(x, \eta) := \frac{1}{2}x^\top PBK(x - \eta)$.
We also take

$$\hat{R}(\bar{T}, x_o) := R(\bar{T}, x_o) + 0.025\mathbb{B},$$

$$R(\bar{T}, x_a) = \{y \in \mathbb{R}^2 : \exists t \in [0, \bar{T}] : y = x(t)\}.$$

Robustness conditions: For each $(x, \eta) \in C \times C$,

$$\langle \zeta, f \rangle \geq \alpha(x) - \gamma(x, \eta)$$
$$\forall (\zeta, f) \in \partial_C \rho(x) \times (F(x, \kappa(\eta)) \cap T_C(x)),$$

The effect of $\bar{T}$ on the Sampling Strategy

- A large value of $\bar{T}$ seems to allow for a large $T_s(x(t_i))$.
- By increasing $\bar{T}$, we increase the size of $\hat{R}(\bar{T}, x(t_i))$.
- An adequate scaling of $\bar{T}$ as a function of $x(t_i)$ can encourage a large $\bar{T}$ when F is slow and vise versa.

Two Strategies for Selecting $\bar{T}$

1. Adapting $\bar{T}$ to the norm of $F(x(t_i), \kappa(x(t_i)))$.
2. Evaluating multiple values of $\bar{T}$ over a receding horizon.

$$T_s(x_o) := \max\{T_1(x_o), T_2(x_o)\}$$

$$T_2(x_o) := \begin{cases} \bar{T} & \text{if } M_2(\bar{T}, x_o) \leq 0 \\ \min\{\bar{T}, \rho(x_o)/M_2(x_o, \bar{T})\} & \text{otherwise,} \end{cases}$$

$$M_2(x_o, \bar{T}) := \sup\{\langle -\gamma, \eta\rangle : \gamma \in \partial_C \rho(y), \eta \in F(y, \kappa(x_o)) \cap T_C(y), y \in \hat{R}(\bar{T}, x_o)\}.$$

Adapting $\bar{T}$ to the norm of $F(x_o, \kappa(x_o))$

- Nonlinear relationship between $\bar{T}$ and $|F(x_o, \kappa(x_o))|$.

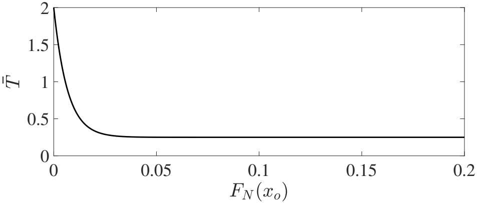
Indeed, consider the map $\bar{T} : \mathbb{R}^n \mapsto [T_{min}, T_{max}]$ given by

$$\bar{T}(x_o) := (T_{max} - T_{min})(1 - F_N(x_o))^{c_s} + T_{min},$$

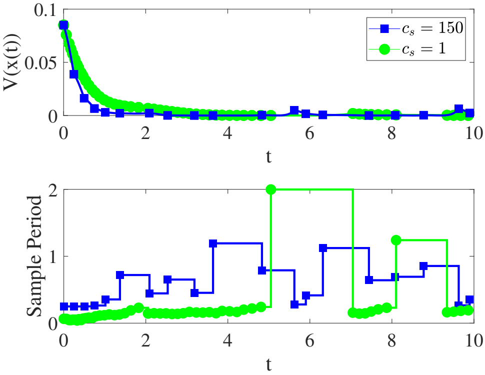
where $c_s \in (0, \infty)$, $T_{max} > T_{min} > 0$ and

$$F_N(x_o) := |F(x_o, \kappa(x_o))| / \sup\{|F(y, \kappa(y))| : y \in X\}.$$

For example, for $T_{max} = 2$, $T_{min} = 0.25$, the figure below illustrates the nonlinear scaling when $c_s = 150$.



Adapting $\bar{T}$ to the norm of $F(x_o, \kappa(x_o))$



Simulation for linear ($c_s = 1$) and nonlinear ($c_s = 150$) scaling.

Average Sampling Period: 0.19 ($c_s = 1$), 0.53 ($c_s = 150$). Minimum Sampling Period: 0.04 ($c_s = 1$), 0.25 ($c_s = 150$).

Evaluating $\bar{T}$ over a receeding horizon

- We can select $\bar{T} \in [T_{min}, T_{max}]$ by maximizing the trade off between current and future sampling periods.

Given $T_{max} > T_{min} > 0$, $N \in \mathbb{N}$, $\Delta := (T_{max} - T_{min})/N$, and $c_h \in [0, 1]$ compute

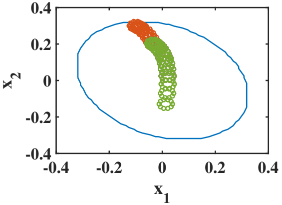
$$n^* := \underset{n \in \{0,1,\dots,N\}}{\text{argmax}} \left\{ c_h T(n\Delta, x_o) + (1 - c_h) T^1(n\Delta, x_o) \right\},$$

$$T(n\Delta, x_o) := T_s(T_{min} + n\Delta, x_o),$$

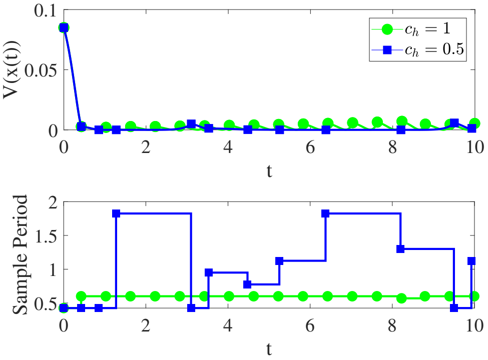
$$T_1(n\Delta, x_o) := \max\{T(m\Delta, x_1) : m \in \{0, \dots N\},\ x_1 \in \hat{R}^b(T_{min} + n\Delta, x_o)\}.$$

- Select $\bar{T} = n^* \Delta$.

$\bullet R^b(T, x_0) := \{\phi(t) : \phi \in \mathcal{S}_{\mathcal{H}_f}(x_o), t \in \mathrm{dom}\, x \cap [0, T],\ \nexists t' \in [0, T] \cap \mathrm{dom}\, \phi : t' > t\}$



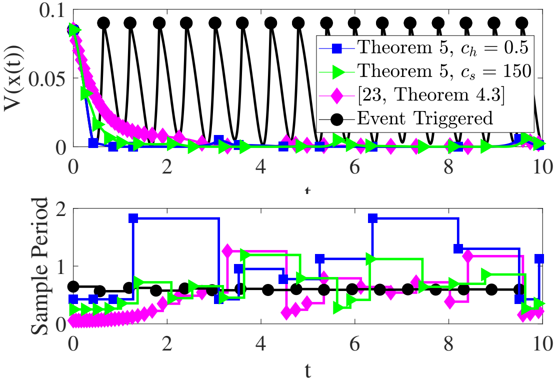
Evaluating $\bar{T}$ over a receding horizon



Simulation results for $c_h = 1$ and $c_h = 0.5$ scaling.

Average Sampling Period: 0.58 ($c_h = 1$), 0.92 ($c_h = 0.5$). Minimum Sampling Period: 0.425 ($c_h = 1$), 0.425 ($c_h = 0.5$).

Comparison to other Methods



Simulation results using various methods

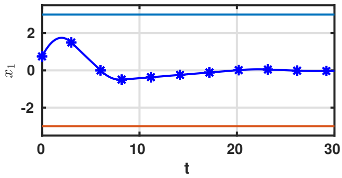
Comparison to other methods

	Average Period	Minimum Period
Scaled $\bar{T}$		
$c_s = 1$	0.19	0.04
$c_s = 150$	0.53	0.25
Receding horizon		
$c_h = 1$	0.58	0.425
$c_h = 0.5$	0.92	0.425
[Di Benedetto et al., 2013]	0.26	0.06
Event Triggered	0.59	0.56

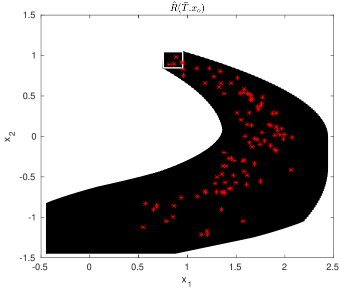
Inter-event properties

Ongoing Work

- Experimenting with off the shelf reachability libraries
- Applications involving more complex systems and tasks



Double Integrator Within Bounds



[CORA 2020, M. Althoff]

Conclusion

- We proposed a self-triggered control strategy that preserves forward invariance.
- We considered general closed sets for a constrained control differential inclusions.
- Future work: Connection to MPC. Connection to non-Zeno behaviors in hybrid systems.