Hybrid Algorithms for Parameter Estimation in Static and Dynamical Systems

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Motivation and Approach

Common features in AFOSR applications:

- Variables changing continuously (e.g., physical quantities) and discretely (e.g., logic variables, resetting timers).
- Abrupt changes in the dynamics (changes in the environment, control decisions, communication events, or failures).

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How can we systematically design such systems featuring **switching** and **intermittency of information** with provable robustness to uncertainties arising in real-world environments?

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- Model continuous and discrete behavior using dynamical models that are **hybrid**.

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Approach:

- Model continuous and discrete behavior using dynamical models that are **hybrid**.
- Develop systematic control theoretical tools for **stability**, **invariance**, **safety**, and **temporal logic**, with **robustness**.

Outline of Recent Results

1. **Estimation**
   - Finite-time Parameter Estimation via Hybrid Methods
     ACC 21a, ACC 21b, ACC 21c (all submitted), + *CoE collab*
   - Observers for Hybrid Systems *ACC 20, CDC 19, CDC 20, Automatica (submitted)*
2. **Safety**
   - Reachable maps for hybrid systems and regularity *HSCC 20, TAC 19, NAHS 20, HSCC 20, CDC 20 (submitted)*
   - (Necessary and Sufficient) Safety Certificates, with Events *ACC 21a, ACC 21d (submitted), TAC 20* + *CoE collab*
3. **Optimization**
   - High Performance Optimization via Uniting Control
     ACC 19, MTNS 20, ACC 20e (submitted) + AFRL/RV collab.
   - Model Predictive Control for Hybrid Systems *ACC 20, CDC 20, IFAC WC 20 Workshop*

Modeling Hybrid Systems: Closed Loop

Hybrid closed-loop systems are given by *hybrid inclusions*

$$\mathcal{H} \quad \begin{cases} \dot{x} & = & F(x) & \qquad x \in C \\ x^+ & = & G(x) & \qquad x \in D \end{cases}$$

where x is the *state*

- C is the *flow set*
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Solutions are functions parameterized by hybrid time (t, j):

- Flows parameterized by $t \in \mathbb{R}_{\geq 0} := [0, +\infty)$
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The state x can have logic, memory, and timer components.

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The hybrid system $\mathcal{H}$ satisfies the hybrid basic conditions if C, D are closed and F, G are "continuous"

Parameter Estimation via Hybrid Methods

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Hybrid methods enable the following:

- **Finite-time estimation** of θ using a **hybrid algorithm** that triggers jumps in the estimates
- **Asymptotic estimation** of θ using a **hybrid algorithm** for the case when the regressor model or the dynamic model is hybrid

with robustness and safety, under appropriate PE conditions.

Robust Finite-Time Parameter Estimation Using Hybrid Dynamical Systems

Ryan S. Johnson and Ricardo G. Sanfelice

Hybrid Systems Laboratory
University of California, Santa Cruz

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October 29, 2020

Outline

1. Context and Motivation
   - Preliminaries
   - Problem statement
2. A Hybrid Algorithm for Finite-Time Parameter Estimation
   - Hybrid model
   - Main results
   - Numerical examples
3. References

Parameter estimation algorithms seek to determine the values of unknown parameters in a dynamical system whose data is partially known.

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For example, consider the nonlinear system of the form

$$\dot{x} = f(x) + g(x)\theta \tag{1}$$

where $t \mapsto x(t) \in \mathbb{R}^n$ is the known state vector, $f(x) \in \mathbb{R}^n$ and $g(x) \in \mathbb{R}^{n\times p}$ are known continuous functions of the state, and $\theta \in \mathbb{R}^p$ is an unknown vector of parameters.

Assumptions [1-4]

- The unknown parameter vector θ is constant or piecewise constant.
- The function g is *persistently exciting*, that is, there exist $\sigma_1, \sigma_2 > 0$ such that for any $t_0 \geq 0$ and any solution $t \mapsto \phi_x(t)$ to (1),

$$\int_{t_0}^{t_0+\sigma_1} g^\top(\phi_x(s))g(\phi_x(s))ds \geq \sigma_2 I. \tag{2}$$

Problem

Given a nonlinear system of the form

$$\dot{x} = f(x) + g(x)\theta \tag{3}$$

where $t \mapsto x(t) \in \mathbb{R}^n$ is the known state vector, $f(x) \in \mathbb{R}^n$ and $g(x) \in \mathbb{R}^{n\times p}$ are known continuous functions of the state, and $\theta \in \mathbb{R}^p$ is an unknown vector of parameters.

Design a hybrid algorithm to estimate θ in finite-time.

Hybrid Modeling

As in [2], we define a state estimate given by

$$\dot{\hat{x}} = f(x) + g(x)\hat{\theta} + k(x - \hat{x}) + wh(x)$$

then the state prediction error $e = x - \hat{x}$ has dynamics

$$\dot{e} = g(x)\tilde{\theta} - k(x - \hat{x}) - wh(x)$$

where $\tilde{\theta} = \theta - \hat{\theta}$ and $\hat{\theta}$ is generated the parameter update law given by

$$\dot{\hat{\theta}} = \gamma(w^\top + g(x)^\top)(x - \hat{x}) =: h(x).$$

where $\gamma = \gamma^\top > 0$.

Next, define $t \mapsto \eta(t)$ and $t \mapsto w(t)$ with dynamics

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It can be shown that integration of $\dot{\eta}$ over an interval $[t_0,\ t_1]$ from an initial condition with $\hat{x}_0 = x$, $w_0 = 0$, and $\eta_0 = 0$ yields

$$\eta = e - w\tilde{\theta}.$$

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Premultiplying each side of the above by $w^\top$ yields

$$w^\top \eta = w^\top e - w^\top w\tilde{\theta}. \tag{4}$$

Then, (4) may be rearranged to solve for $\tilde{\theta}$ as

$$\tilde{\theta} = (w^\top w)^{-1} w^\top (e - \eta) =: \psi$$

Thus, whenever the inverse of $w^\top w$ is well defined, we may solve for θ as

$$\hat{\theta} + \tilde{\theta} = \hat{\theta} + \psi = \theta$$

Closed-loop system

The hybrid system is denoted $\mathcal{H} = (C, F, D, G)$ with state $z = (x, \hat{x}, \hat{\theta}, \eta, w) \in \mathcal{X}$ where

- x: plant state vector
- $\hat{x}$: plant state vector estimate
- $\hat{\theta}$: parameter vector estimate
- η, w: auxiliary state variables

and data

$$\begin{aligned} \dot{z} &\in F(z) \qquad z \in C \\ z^+ &\in G(z) \qquad z \in D \end{aligned} \tag{5}$$

Closed-loop system

where

$$
F(z) := \begin{bmatrix} f(x) + g(x)\theta \\ f(x) + g(x)\hat{\theta} + k(x - \hat{x}) + wh(z) \\ h(z) \\ -k(x - \hat{x}) \\ g(x) \end{bmatrix} \qquad z \in C
$$

$$
G(z) := (x, x, \theta + \psi, 0, 0) \qquad z \in D
$$

$$
\psi := \left(w^\top w\right)^{-1} w^\top (x - \hat{x} - \eta)
$$

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C := \left\{ z \in \mathcal{X} \ : \ \det(w^\top w) \leq \varepsilon \right\}
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D := \left\{ z \in \mathcal{X} \ : \ \det(w^\top w) \geq \varepsilon \right\}
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and $h(z) = \gamma(w^\top + g(x)^\top)(x - \hat{x})$, $k = k_1 + \frac{1}{4} g\gamma g$, $k_1 = k_1^\top > 0$, $\gamma > 0$, and $\varepsilon > 0$.

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Main Result

Theorem

Suppose g *in* (3) *is persistently exciting as in* (2)*. Consider the hybrid system* $\mathcal{H}$ *in* (5) *with*

$$h(z) = \gamma(w^\top + g(x)^\top)(x - \hat{x}),$$

$$k = k_1 + \frac{1}{4} g\gamma g, \quad k_1 = k_1^\top > 0,$$

and $\varepsilon > 0$*. Then, the set*

$$\mathcal{A} = \left\{ z \in \mathcal{X} \ : \ \hat{x} = x, \ \hat{\theta} = \theta, \ \eta = 0 \right\} \tag{6}$$

is globally finite time stable for $\mathcal{H}$*.*

Proof sketch

To prove this result, we show:

- Since g is persistently exciting, the set

$$\mathcal{A}_0 = \{z \in \mathcal{X} \ : \ \eta = 0, \ w = 0 \ \}$$

is globally finite-time attractive for $\mathcal{H}$.

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  is finite-time attractive from $\mathcal{A}_0$ for $\mathcal{H}$. Therefore, the set $\mathcal{A}$ is globally finite-time attractive for $\mathcal{H}$.
- It can be shown that for any $\phi \in \mathcal{S}_{\mathcal{H}}$ there exist $\varepsilon > 0$ and $\delta > 0$ such that

  $$|\phi(0,0)|_{\mathcal{A}} \leq \delta \implies |\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$$

  Thus, the set $\mathcal{A}$ is globally finite time stable for $\mathcal{H}$.

Comparison with the Literature

A parameter estimation algorithm is proposed in [1-2] which provides finite-time estimation of the parameter vector θ in (1). However, this scheme suffers from the following issues:

Issues with current implementation

- The algorithm in [1-2] may require a large amount of memory implement.
- The algorithm in [1-2] may take a long time to converge for some functions $g(\cdot)$.

Example 1.1

Consider the frequency estimation problem for a signal $t \mapsto y(t) = \gamma \sin(\nu t)$ where $\gamma \in \mathbb{R}_{>0}$ is the magnitude and $\nu \in \mathbb{R}_{>0}$ is the frequency. Globally estimate ν from measurements of y.

This problem may be rewritten as the problem of estimating θ for a system with state $x = (x_1, x_2)$ such that $x_1 = y$ and $\dot{x}_1 = x_2$ and dynamics

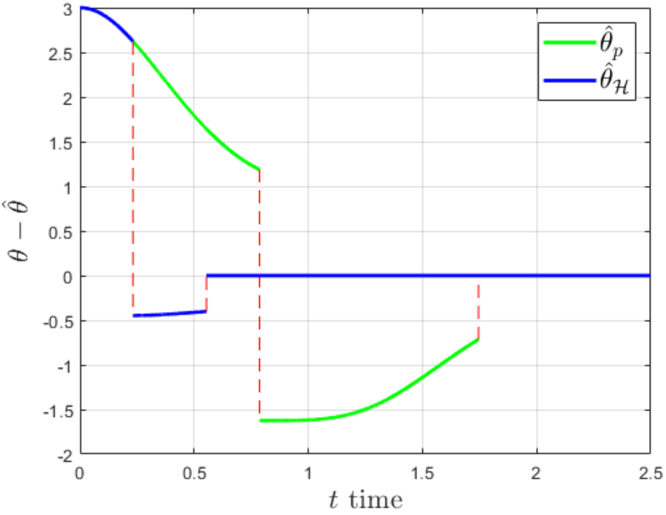
$$\dot{x} = Ax + g(x)\theta$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ -x_1 \end{bmatrix}$$

and $\theta = \nu^2 - 1$.

Numerical Example



Example 1.2

Next, consider the problem of estimating θ for the following perturbed model

$$\dot{x} = Ax + g(x)\theta + d$$

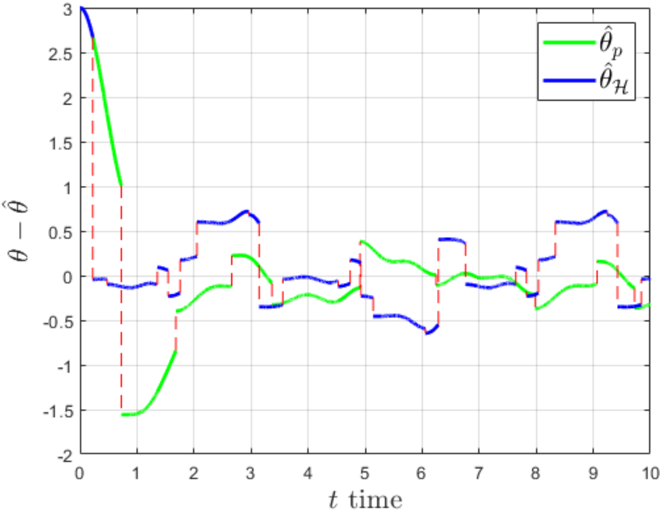
where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ -x_1 \end{bmatrix}$$

and d is an unknown perturbation on the state x given by

$$d = \begin{bmatrix} sin(5t) \\ cos(5t) \end{bmatrix}.$$

Numerical Example with Noise



Conclusion

Summary:

- We proposed a finite-time estimation algorithm for a class of nonlinear systems.
- The proposed algorithm may provide faster convergence compared to the schemes available in the literature.
- The proposed algorithm utilizes a state vector with $> 30\%$ fewer elements compared to schemes in the literature, thereby requiring less memory.

Future work:

- Extend the proposed estimation scheme to switched systems and hybrid systems.

Thank You!

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