On the Feasibility and Continuity of Feedback Controllers Defined by Multiple Control Barrier Functions

Axton Isaly, Masoumeh Ghanbarpour, Ricardo G. Sanfelice, Warren E. Dixon

Outline

- Defined a notion of control barrier function (CBF) amenable to problems involving multiple CBFs
  - CBFs guarantee the existence of safety-ensuring controllers
- Developed a constructive method for synthesizing safety-ensuring controllers using optimization
  - Sufficient conditions for continuity of the optimal controller
- Used sum of squares programming to certify the feasibility of the optimal control law
  - Corresponds to verifying that a function is a CBF

Control Objective

- Design a controller so that

$$\mathcal{S} \triangleq \{x \in \Pi(C_u) : B(x) \leq 0\}$$

is forward invariant, where

$$B(x) \triangleq [B_1(x), B_2(x), \ldots, B_d(x)]^T$$

- Safe set described by multiple continuously differentiable functions

$$\dot{x} \in F(x, u) \qquad (x, u) \in C_u$$

$$\Psi(x) \triangleq \{u \in \mathbb{R}^m : (x, u) \in C_u\} \qquad \Pi(C_u) \triangleq \{x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m \ s.t. \ (x, u) \in C_u\}$$

Flow Constraints

$$\dot{x} \in F(x,u) \qquad (x,u) \in C_u$$

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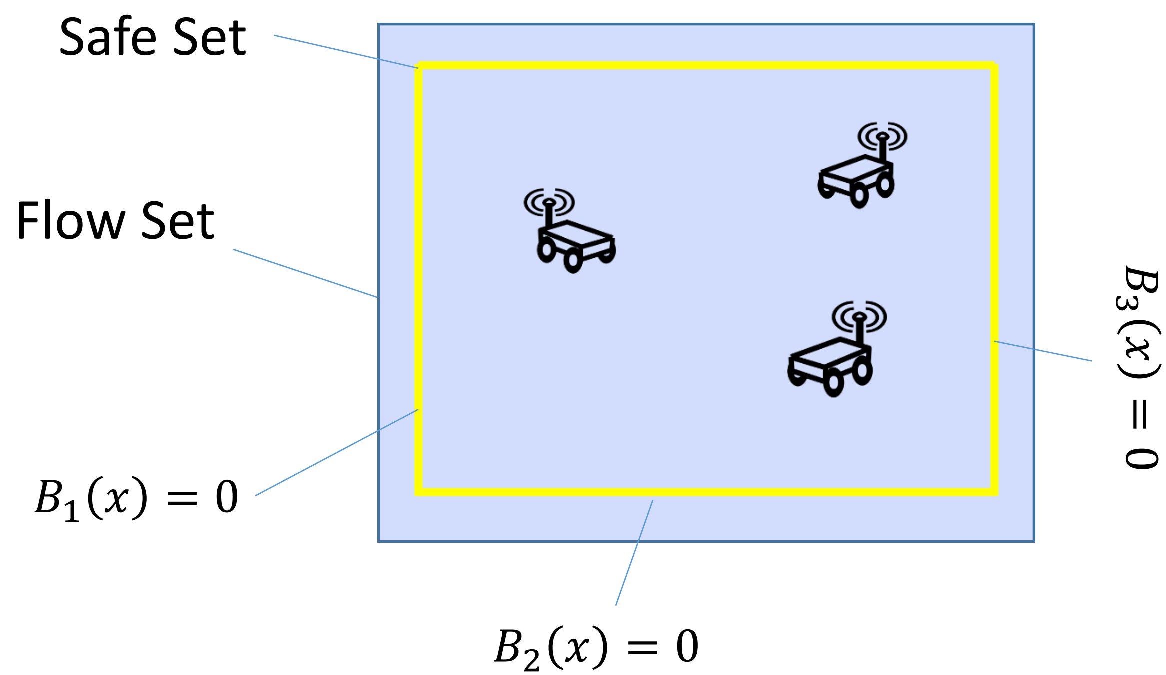

Control Barrier Functions

- A CBF candidate $B:\mathbb{R}^n \to \mathbb{R}^d$ defining the set $\mathcal{S} \subset \Pi\left(C_u\right)$ is a CBF for $\left(F, C_u\right)$ and $\mathcal{S}$ on a set $\mathcal{O} \subset \Pi\left(C_u\right)$ with respect to a function $\gamma : \Pi\left(C_u\right) \to \mathbb{R}^d$ if

1. There exists a neighborhood of the boundary of $\mathcal{S}$ such that $U\left(\partial\mathcal{S}\right) \cap \Pi\left(C_u\right) \subset \mathcal{O}$
2. For each $i \in [d]$, $\gamma_i(x) \geq 0$ for all $x \in \left(U\left(M_i\right) \backslash \mathcal{S}_i\right) \cap \Pi\left(C_u\right)$
3. $K_c\left(x\right)$ is nonempty for all $x \in \mathcal{O}$

$$K_c\left(x\right) \triangleq \left\{u \in \Psi\left(x\right) : \Gamma_i\left(x,u\right) \leq -\gamma_i\left(x\right),\ \forall i \in [d]\right\}$$

$$\Gamma_i\left(x,u\right) \triangleq \sup_{f \in F(x,u)} \left\langle \nabla B_i\left(x\right), f\right\rangle \qquad M_i \triangleq \left\{x \in \partial\mathcal{S} : B_i\left(x\right) = 0\right\}$$

Forward (pre-)Invariance

- Forward pre-invariance is attained using continuous selections

$$\kappa(x) \in K_c(x) \quad \forall x \in \mathcal{O}$$

$$\dot{x} \in F(x, \kappa(x)) \triangleq F_{cl}(x)$$

- Under mild conditions, such selections render

$$\mathcal{S} \triangleq \{x \in \Pi(C_u) : B(x) \leq 0\}$$

forward pre-invariant

Optimal Safety-Ensuring Selections

$$\Psi(x) = \{u \in \mathbb{R}^m : \psi(x,u) \leq 0\}$$

$$\kappa^*(x) \triangleq \underset{u \in \mathbb{R}^m}{\arg\min}\ Q(x,u) \quad s.t.\ \Gamma(x,u) \leq -\gamma(x), \quad \psi(x,u) \leq 0$$

$$= \underset{u \in K_c(x)}{\arg\min}\ Q(x,u)$$

Assumption 5. For every $i \in [d]$ and $j \in [k]$,
A) For all $x \in \mathcal{O}$, the functions $u \mapsto \Gamma_i(x,u)$ and $u \mapsto \psi_j(x,u)$ are convex on $\Psi(x)$.
B) The functions $(x,u) \mapsto \Gamma_i(x,u) + \gamma_i(x)$ and $(x,u) \mapsto \psi_j(x,u)$ are continuous on $C_u \cap (\mathcal{O} \times \mathbb{R}^m)$ and $\mathcal{O} \times \mathbb{R}^m$, respectively.

Optimal Safety-Ensuring Selections

- Given the previous assumptions, assume additionally that
  1. The cost function Q is continuous and, for each $x \in \mathcal{O}$, $u \mapsto Q(x,u)$ is strictly convex.
  2. The following set is nonempty for every $x \in \mathcal{O}$

$$K_c^{\circ}(x) \triangleq \left\{ \begin{array}{c} u \in \mathbb{R}^m : \Gamma(x,u) < -\gamma(x) \\ \psi(x,u) < 0 \end{array} \right\}$$

  3. Either Q is level-bounded in u, locally uniformly in x, or Ψ is locally bounded.

- **Then κ^* is continuous**

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Feasibility Verification with Sum of Squares

- Everything looks good provided the feasible set $K_c(x)$ is nonempty...
- Verifying feasibility is the same as asking if we have a CBF
- Consider

$$K(x) \triangleq \{u \in \mathbb{R}^m : A(x)u + b(x) \leq 0\}$$

where A and b are polynomials.

- Can verify nonemptiness using sum of squares

Feasibility Verification with Sum of Squares

$$K\left(x\right) \triangleq \left\{u \in \mathbb{R}^m : A\left(x\right)u + b\left(x\right) \leq 0\right\}$$

Problem 1. (Global Feasibility) Given polynomials $A \in \mathcal{P}^{n_c \times m}\left[x\right]$ and $b \in \mathcal{P}^{n_c}\left[x\right]$, find a constant $\epsilon \geq 0$ and a polynomial $u \in \mathcal{P}^m\left[x\right]$ such that, for all $i \in \left[n_c\right]$,

$$-A_{i*}\left(x\right)u\left(x\right) - b_i\left(x\right) - \epsilon \in \Sigma\left[x\right],$$

where $A_{i*}\left(x\right)$ denotes that i-th row of $A\left(x\right)$. The parameter ϵ could either be a fixed value or a decision variable. If $\epsilon > 0$, then $K^{\circ}\left(x\right) \triangleq \left\{u \in \mathbb{R}^m : A\left(x\right)u + b\left(x\right) < 0\right\}$ is nonempty.

- We have found

$$u(x) \in K(x), \quad \forall x \in \mathbb{R}^n$$

- However, global feasibility will often not be possible.
- We also develop a program for verifying feasibility on sublevel sets

$$\mathcal{L}_{\tilde{B}}\left(\beta\right) \triangleq \left\{x \in \mathbb{R}^n : \tilde{B}\left(x\right) \leq \beta\right\}$$

Verification of CBF

- Recall, a CBF candidate $B:\mathbb{R}^n \to \mathbb{R}^d$ defining the set $\mathcal{S} \subset \Pi\left(C_u\right)$ is a CBF for $\left(F, C_u\right)$ and $\mathcal{S}$ on a set $\mathcal{L}_{\tilde{B}}\left(\beta\right)$ with respect to a function $\gamma : \Pi\left(C_u\right) \to \mathbb{R}^d$ if

1. There exists a neighborhood of the boundary of $\mathcal{S}$ such that $U\left(\partial\mathcal{S}\right) \cap \Pi\left(C_u\right) \subset \mathcal{L}_{\tilde{B}}\left(\beta\right)$
2. For each $i \in [d]$, $\gamma_i(x) \geq 0$ for all $x \in \left(U\left(M_i\right) \backslash \mathcal{S}_i\right) \cap \Pi\left(C_u\right)$
3. $$K_c\left(x\right) \triangleq \left\{ \begin{array}{c} u \in \mathbb{R}^m : \Gamma\left(x,u\right) \leq -\gamma\left(x\right) \\ \psi\left(x,u\right) \leq 0 \end{array} \right\}$$

   is nonempty for all $x \in \mathcal{L}_{\tilde{B}}\left(\beta\right)$

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Verification of CBF

Problem 2. (Feasibility on Level Sets) Given $A \in \mathcal{P}^{n_c \times m}[x]$, $b \in \mathcal{P}^{n_c}[x]$, $\tilde{B} \in \mathcal{P}^{n_b}[x]$, and $\beta \in \mathbb{R}$, find polynomials $u \in \mathcal{P}^m[x]$, $s_0, s_1, \ldots, s_{n_b} \in \Sigma[x]$, and a constant $\epsilon \geq 0$ such that, for all $i \in [n_c]$,

$$-A_{i*}(x)\,u(x) - b_i(x) - \epsilon - s_0(x) - \sum_{j=1}^{n_b} s_j(x)\left(\beta - \tilde{B}_j(x)\right) \in \Sigma[x]. \quad (11)$$

- Assume there exists polynomials A, b such that

$$A(x)\,u + b(x) \geq (\Gamma(x,u) + \gamma(x), \psi(x,u)) \quad \forall (x,u) \in C_u$$

- If Problem 2 has a solution, then K_c is nonempty on $\mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$ and B is a CBF
- And if $\epsilon > 0$, K_c° is nonempty on $\mathcal{L}_{\tilde{B}}(\beta) \cap \Pi(C_u)$

Adaptive Safety for Hybrid Systems

Ricardo Sanfelice
Department Electrical and Computer Engineering
University of California

CoE Review @ Zoom - November 9, 2021

Outline of Recent Results

1. **Estimation**
   - Finite-time Parameter Estimation via Hybrid Methods

     ACC 21a, ACC 21b, ACC 21c + CoE collab
   - Observers for Hybrid Systems

     CDC 21a, CDC 21b, Automatica
2. **Safety**
   - Safety Certificates, with Optimality

     ACC 22a, ACC 22b (submitted), TAC 20 + CoE collab
   - Applications of Safety

     ACC 22c (submitted), Frontiers in AI + CoE collab
3. **Optimization**
   - High Performance and Distributed Optimization

     ACC 21d, CDC 21c + CoE collab + AFRL/RV collab.
   - Model Predictive Control for Hybrid Systems

     CDC 21d, CPSWeek 21 Workshop, CPSWeek 21 + AFRL collab

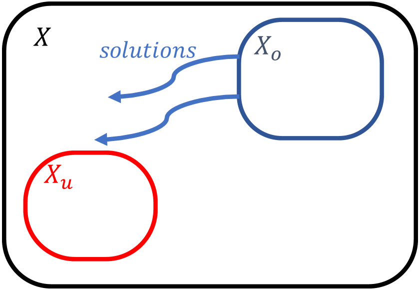
Consider the system

$$\dot{x} = f(x) \qquad x \in X \subset \mathbb{R}^n$$

and the sets

$X_o \subset X$ the initial set,

$X_u \subset X \backslash X_o$ the unsafe set.



Basic Setting

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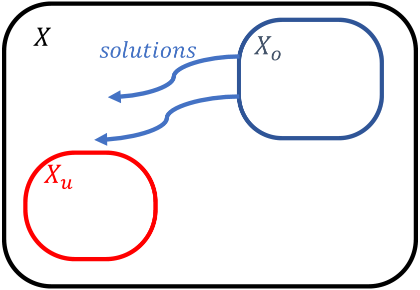


Safety with respect to (X_o, X_u) $\Leftrightarrow$ $\mathrm{reach}(X_o) \cap X_u = \emptyset$

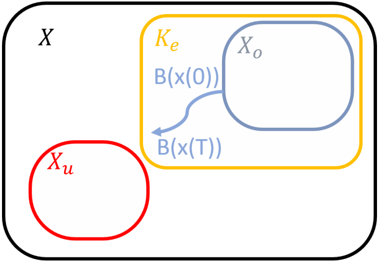
$\mathrm{reach}(X_o) := \{x \in \mathbb{R}^n : x = \phi(t; x_o)$, with ϕ a solution from $x_o \in X_o$ and $t \in \mathrm{dom}\,\phi\}$ $\leftarrow$ the infinite reach set

A solution to $\dot{x} = f(x)$ is denoted $t \mapsto \phi(t)$, and when starts at x_o as $t \mapsto \phi(t; x_o)$

Sufficient Conditions for Safety when $X = \mathbb{R}^n$

Consider $X = \mathbb{R}^n$ and let the function B satisfy

$$B(x) > 0 \qquad \forall x \in X_u$$
$$B(x) \leq 0 \qquad \forall x \in X_o$$



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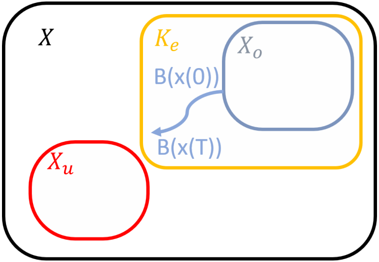

and

the set K_e is "forward invariant" for $\dot{x} = f(x)$

where

$K_e := \{x \in \mathbb{R}^n : B(x) \leq 0\}$ $\leftarrow$ the zero-sublevel set of B

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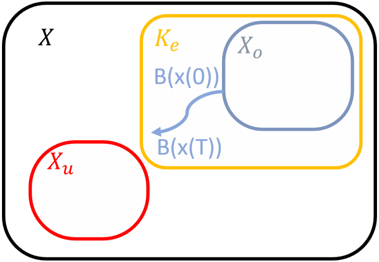

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It follows that the system $\dot{x} = f(x)$ is safe w.r.t. (X_o, X_u)

Questions Driving Research Agenda

These observations motivate the following questions:

- How to guarantee the monotonicity condition

$$t \mapsto B(\phi(t; x_o)) \text{ is nonincreasing}$$

and

$$\text{the set } K_e \text{ is "forward invariant" for } \dot{x} = f(x)$$

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  without checking/computing every solution?
  - How to deal with nonuniqueness, finite escape time, and solutions ending prematurely?
- What are necessary conditions for safety (and invariance)?
  - How regular should one expect a barrier function to be?

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... for dynamical systems given by

$$\mathcal{H} \begin{cases} \dot{x} & \in & F(x) & & x \in C \\ x^+ & \in & G(x) & & x \in D \end{cases}$$

Basic Definitions

- When B is continuously differentiable, our sufficient conditions take the form

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Hence, we require

$$\langle \nabla B(x), f(x) \rangle \leq \ 0 \qquad \forall x \in (U(\partial K_e) \backslash K_e)$$

where $U(\partial K_e)$ is a neighborhood of K_e, so $(U(\partial K_e) \backslash K_e)$ are points outside right outside K_e!

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Nonsmooth barrier certificates naturally emerge in applications, in particular, in **obstacle avoidance problems** where the unsafe set is typically given by the intersection of half spaces.