Reproducing Swarm Dynamics for a Three-Satellite System with Generalized Swarm Optimization

Taryn J. Noone & Norman G. Fitz-Coy

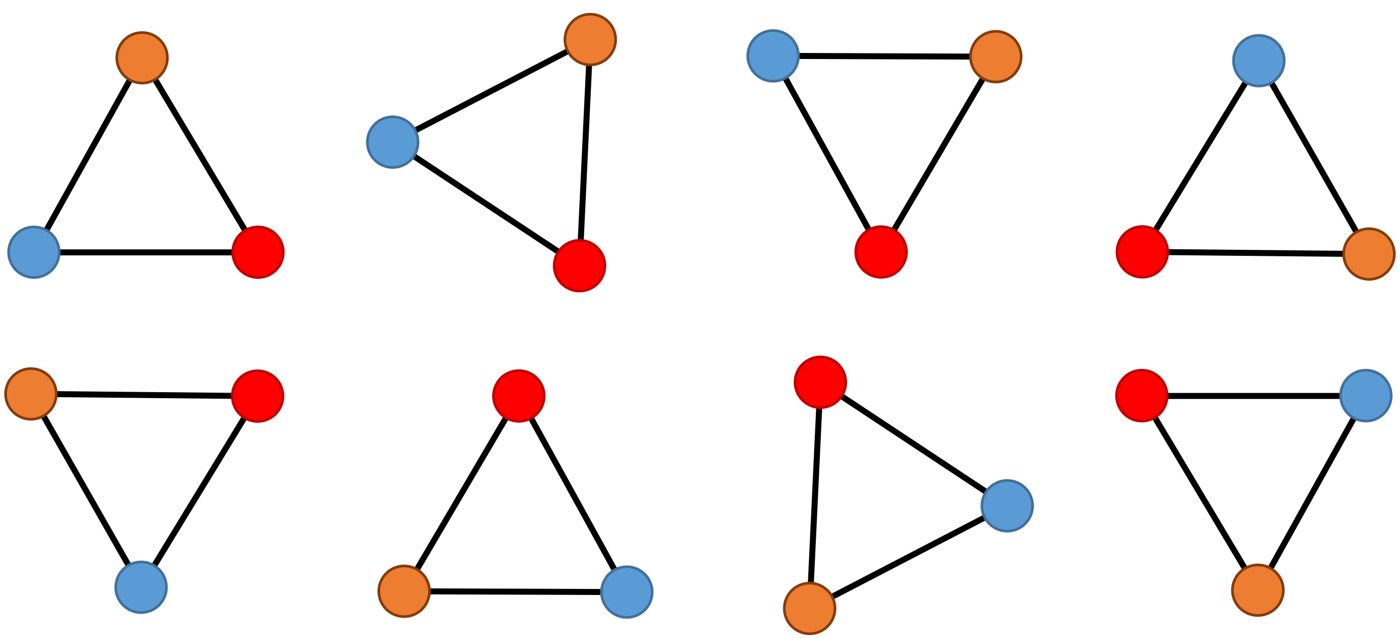
Overview

- We are seeking to solve the Swarm Initialization Problem (SIP) in the special case of a **circular swarm trajectory** (eccentricity = 0, exact).
- We assume invariance of swarm optimality under:
  - Rotation of the swarm within a known space of valid rotations;
  - Translation of the swarm within a known space of valid translations;
  - Transposition, or re-labeling of any two satellites.
- **Quantization of the orbit into evenly-spaced 'checkpoints'**
  - Same number of checkpoints as satellites.
  - Assume no checkpoint is 'special'; i.e., re-orienting global coordinates to set any checkpoint as 'initial' results in an identical problem.
- **Construction of formation chains** based on geometry alone.
- **Assessment of formation chains** for compatibility with dynamics.
  - Uses Munkres' Algorithm to identify orbits compatible with geometry.

Swarm-Preserving Operations

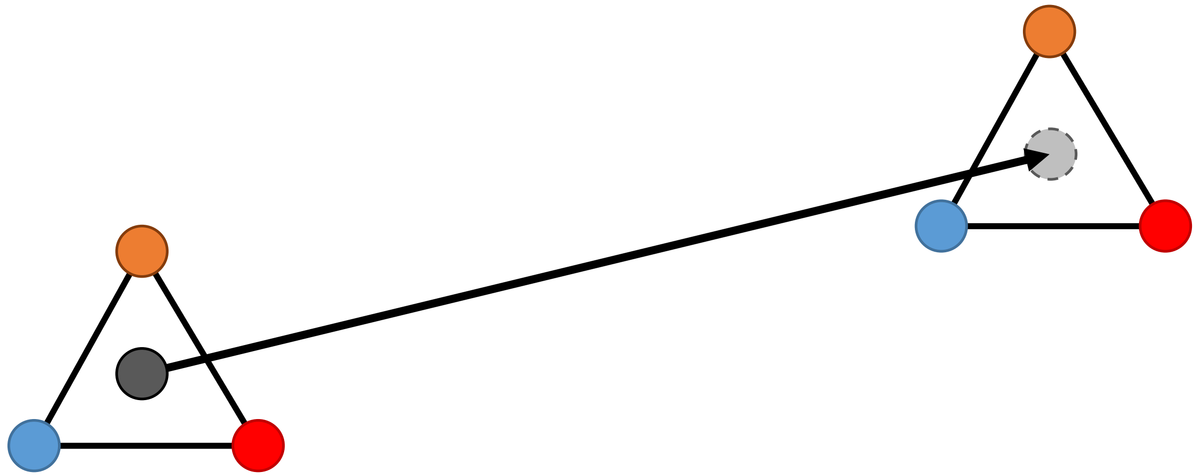
Swarm-Preserving Operations

- Invariance of swarm optimality under rotation of the swarm within a known space of valid rotations:
  - Example: Planar formations rotated about the normal of their plane.



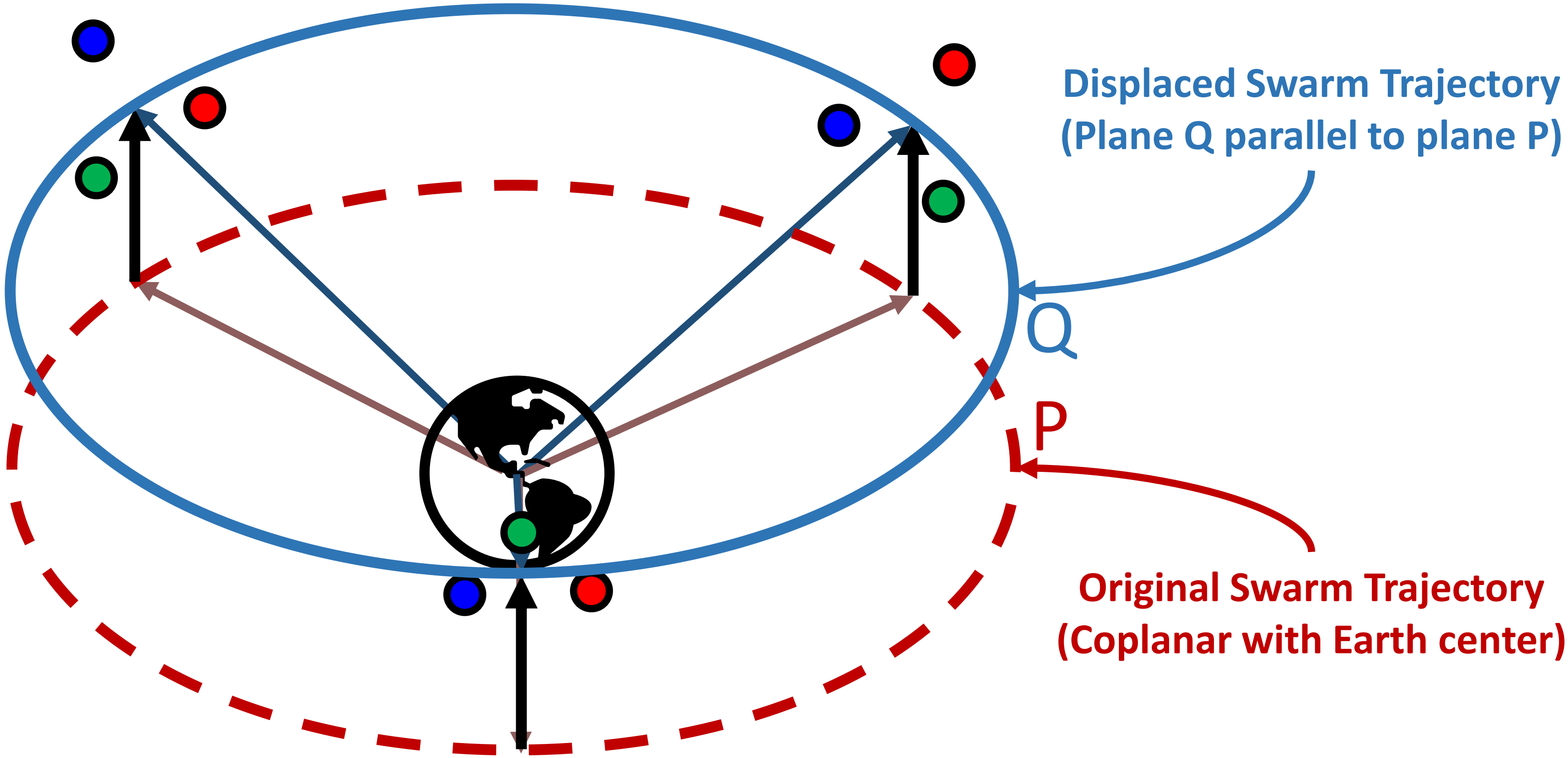
Swarm-Preserving Operations

- Invariance of swarm optimality under translation of the swarm within a known space of valid translations.
  - Complicated by requirement that no one checkpoint is 'special'.
    - Supports two 'modes' of translation:
      1. Globally fixed displacement applied to all checkpoints.
      2. Swarm fixed displacement normal to axis of swarm rotation.



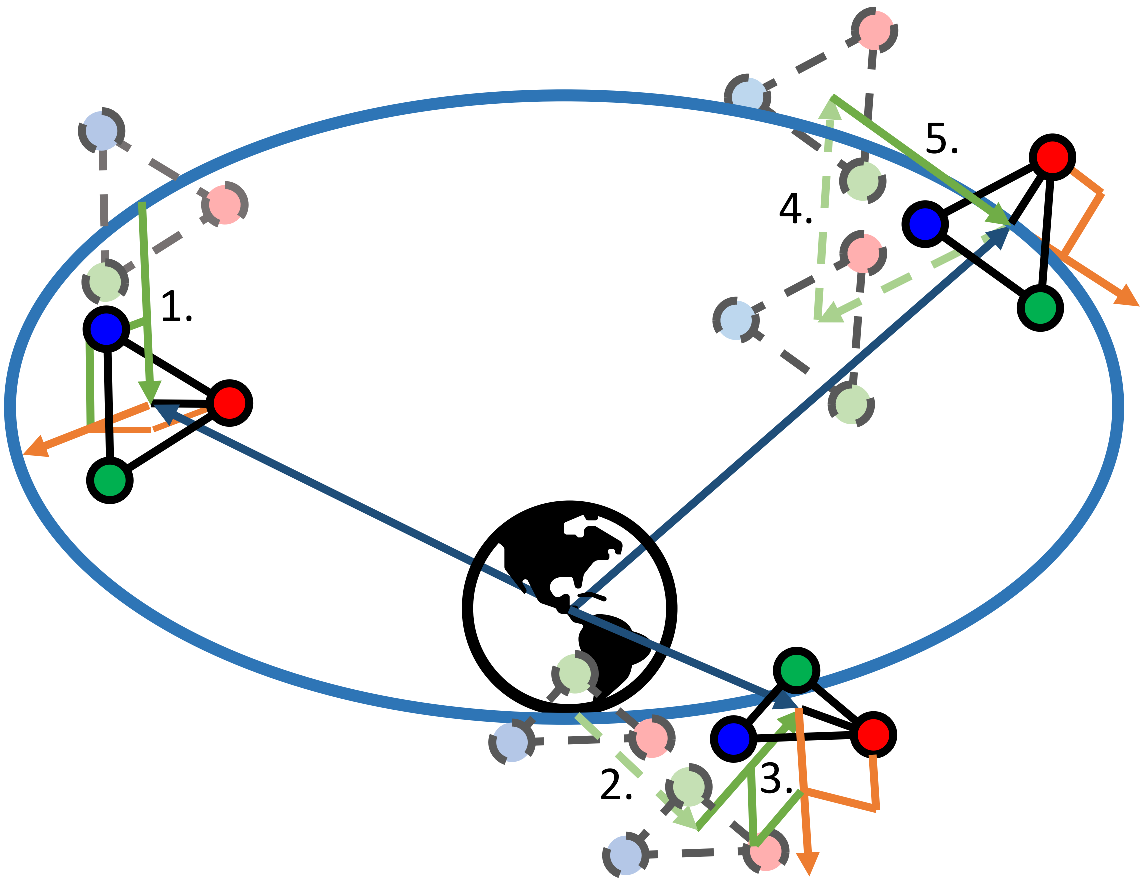
Swarm-Preserving Operations

- Translation by globally fixed displacement vector.



Swarm-Preserving Operations

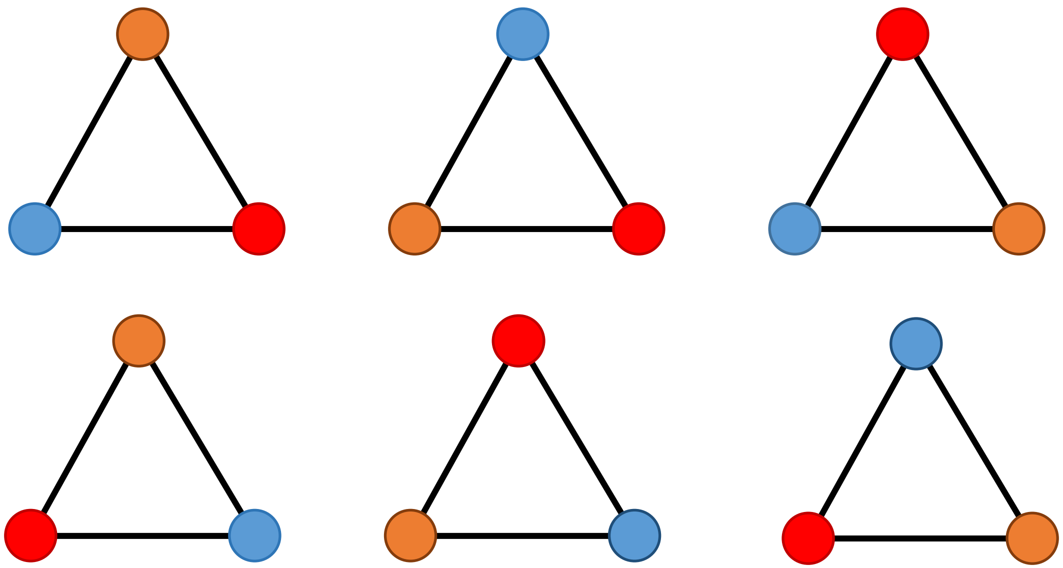
- Translation by swarm fixed displacement vector.



1. Perform rotation and shift along displacement vector.
2. Parallel transport vector from step 1.
3. Perform rotation and shift along displacement vector.
4. Parallel transport vectors from steps 1 and 3.
5. Perform rotation and shift along displacement vector.

Swarm-Preserving Operations

- Invariance of swarm optimality under transposition, or re-labeling of any two satellites.

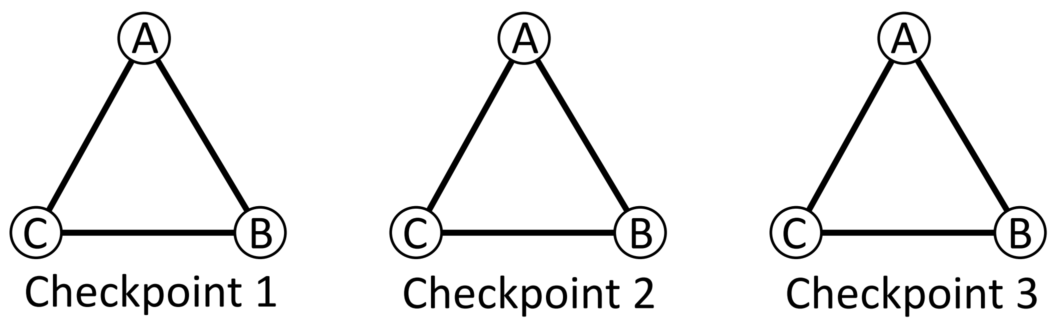


Index Chains

Index Chains

- A suitable combination of rotation, translation, and re-indexing produces geometry-based formation chains.
- We define **index chains** to determine the path followed by a satellite.
- Index chains separate the notions of satellites and positions in the swarm.
- Consider three checkpoints for a three-satellite formation, shown below:

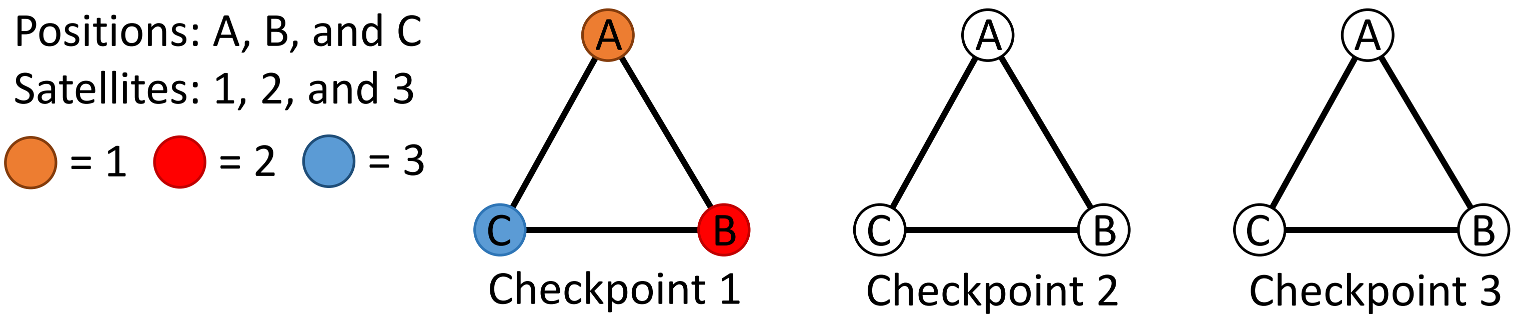
Positions: A, B, and C
Satellites: 1, 2, and 3



- For any checkpoint, position A, B, or C may contain satellite 1, 2, or 3.
  - Assume indexing for checkpoint 1 is arbitrary, so we may reduce all six cases down to one: Satellite 1 at position A; 2 at B; and 3 at C.

Index Chains

Positions: A, B, and C
Satellites: 1, 2, and 3



- We see that for each satellite / position at checkpoint 1, there are three possibilities at checkpoint 2.
- After one satellite has been placed, two possibilities remain for the next.
- After two satellites have been placed, only one possibility remains.
- There are six configurations for three satellites.
    - n! configurations for a general n satellites.

Index Chains

- It can be shown that, under these requirements, the full index chain for each satellite is fully defined by the first two checkpoints.
  - Results in n^2 index chains for n satellites.
  - Valid assignments must have unique row and column indices.

Index Chain for 3 Satellites		Moves to Position Index		
		A	B	C
Satellite Index	1 (at A)	A, A, A	A, B, C	A, C, B
	2 (at B)	B, A, C	B, B, B	B, C, A
	3 (at C)	C, A, B	C, B, A	C, C, C

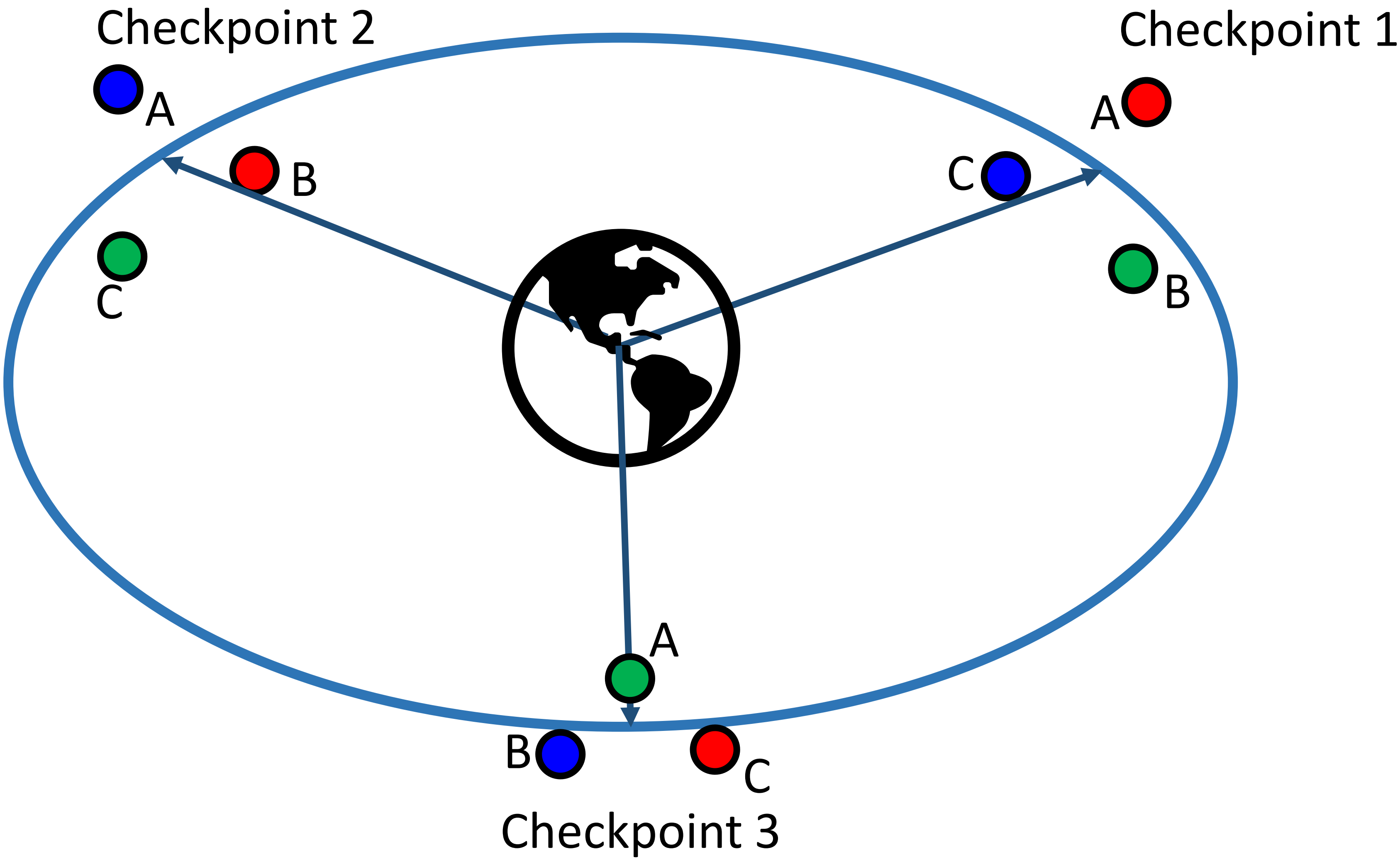
Index Chains

- Replacing the index chains on the previous table with scalar cost function values, we produce a classic 1-to-1 assignment problem.
  - The Munkres (or Hungarian) Algorithm may be used to identify the highlighted permutations with a suitable choice of cost function $\mathcal{K}$.

Chain Cost for 3 Satellites		Moves to Position Index		
		A	B	C
Satellite Index	1 (at A)	$\mathcal{K}_{1\to A}$	$\mathcal{K}_{1\to B}$	$\mathcal{K}_{1\to C}$
	2 (at B)	$\mathcal{K}_{2\to A}$	$\mathcal{K}_{2\to B}$	$\mathcal{K}_{2\to C}$
	3 (at C)	$\mathcal{K}_{3\to A}$	$\mathcal{K}_{3\to B}$	$\mathcal{K}_{3\to C}$

Assessing Chain Cost

Assessing Chain Cost



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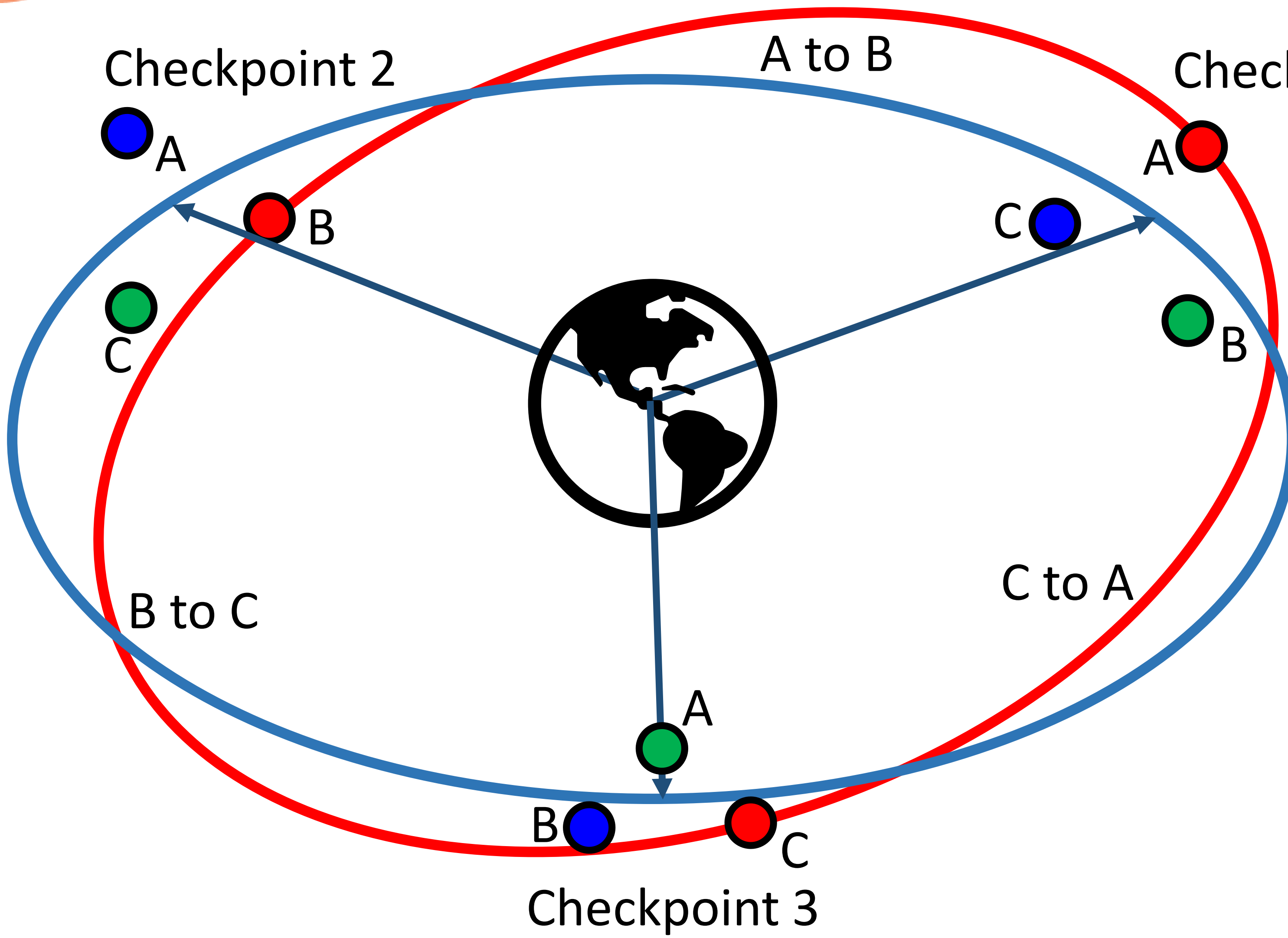

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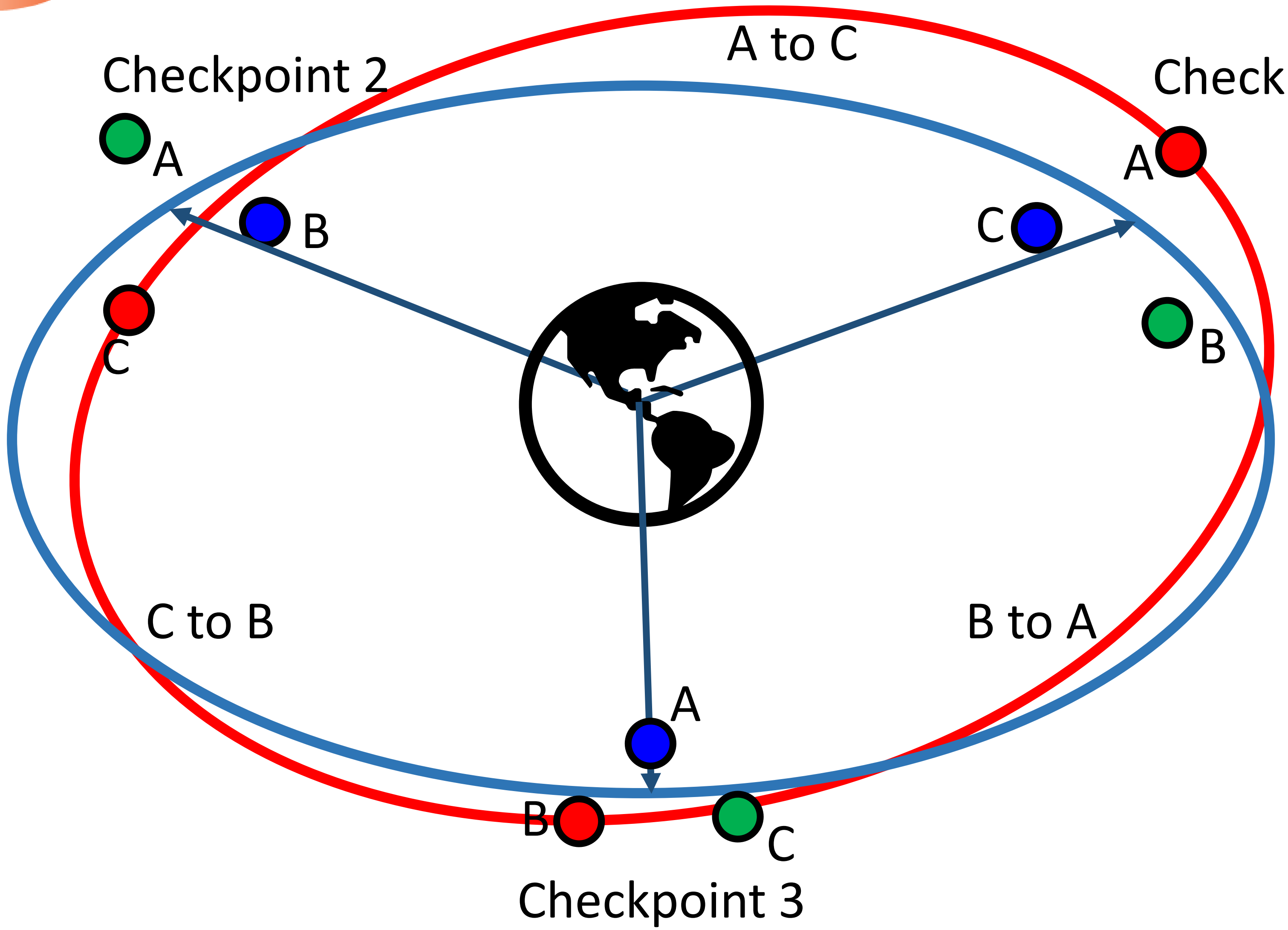

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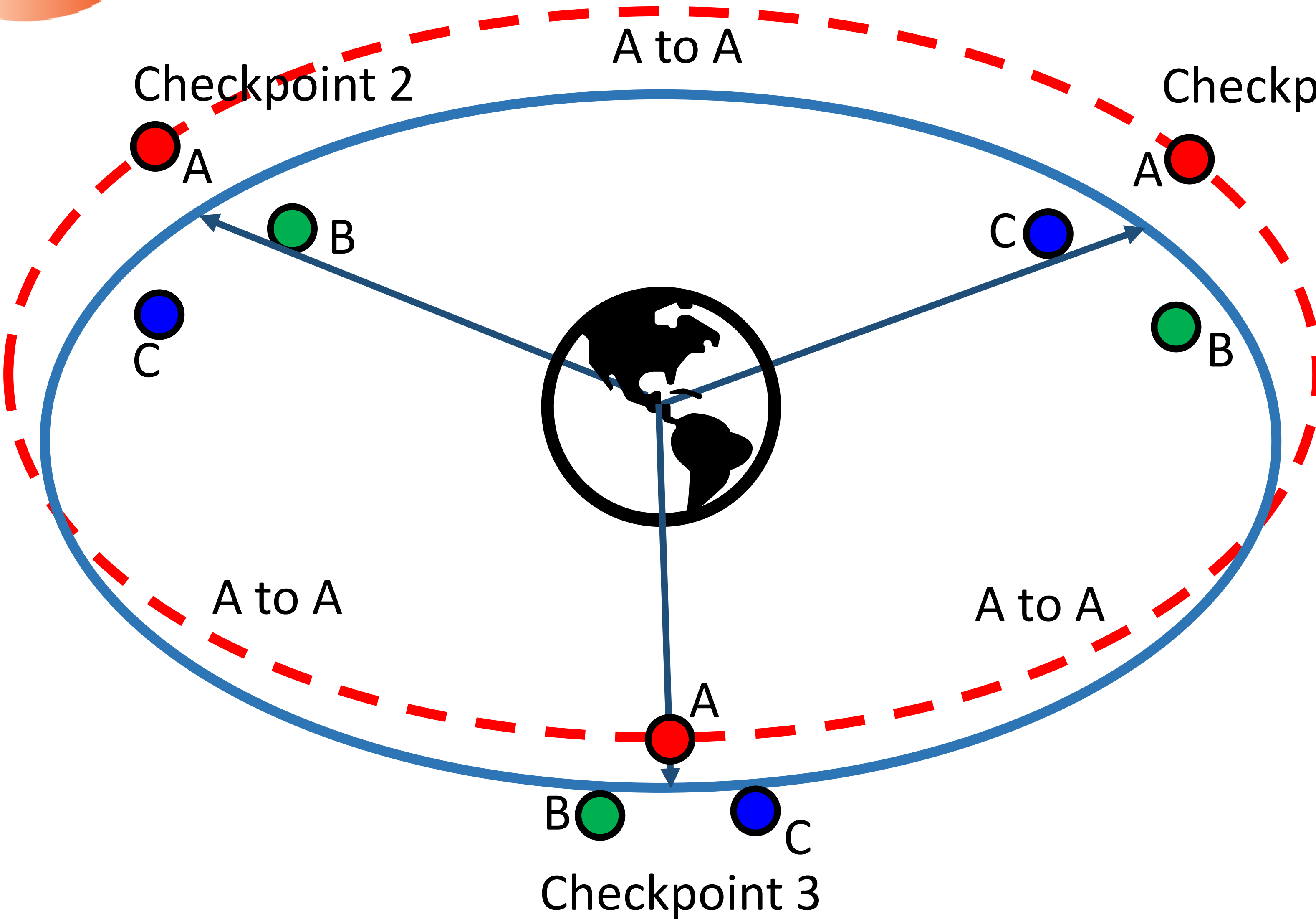


Assessing Chain Cost

- Fixed parameters of the chain cost function:

Description	Symbolic Representation	Degrees of Freedom
Normalized positions (relative to swarm centroid)	$\hat{\rho}_i \;\forall\; i \in \{1, 2, \dots, n\}$	$2n$
Radius of swarm envelope	ρ	1
Radius of swarm trajectory	a_0	1
Offset angle of swarm (polygonal swarm shapes)	ϕ	1
Earth-centered coordinate basis of swarm trajectory	$\hat{X}, \hat{Y}, \hat{Z}$	3

Assessing Chain Cost

- Geometric variables incorporated in the chain cost function:

Description	Symbolic Representation	Degrees of Freedom
Axis of initial rotation	$\hat{u}$	2
Angle of initial rotation	θ	1
Initial displacement	$\vec{s}$	3
Axis of incremental rotation	$\hat{w}$	2
Interrotational displacement	$\Delta\vec{s}$	3

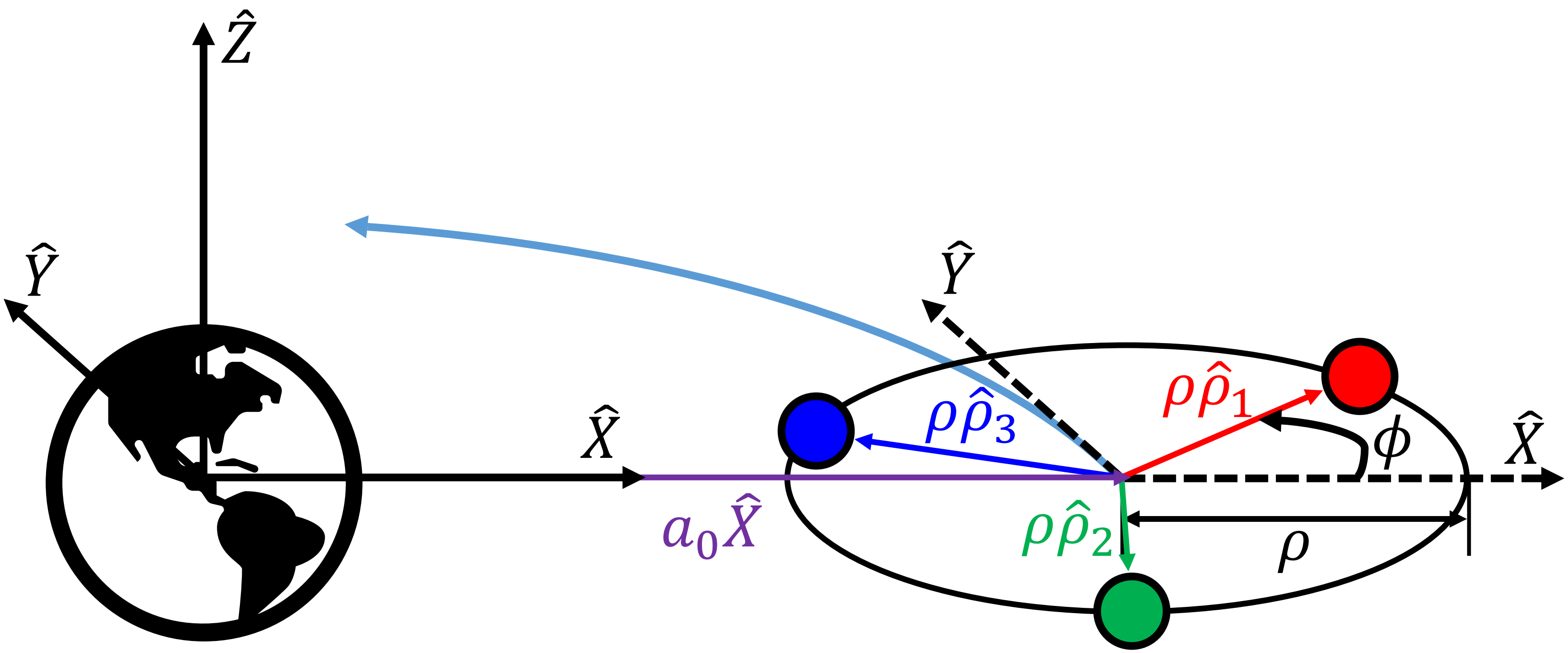
Assessing Chain Cost

- Dynamical variables incorporated in the chain cost function:

Description	Symbolic Representation	Degrees of Freedom
Semi-major axis	a	1
Orbital eccentricities for each satellite	$e_i \ \forall \ i \in \{1, 2, \dots, n\}$	n
Initial true anomaly ($t = 0$) for each satellite	$\nu_i \ \forall \ i \in \{1, 2, \dots, n\}$	n

- Total degrees of freedom: $4n + 18$

- Fixed parameters of the chain cost function:



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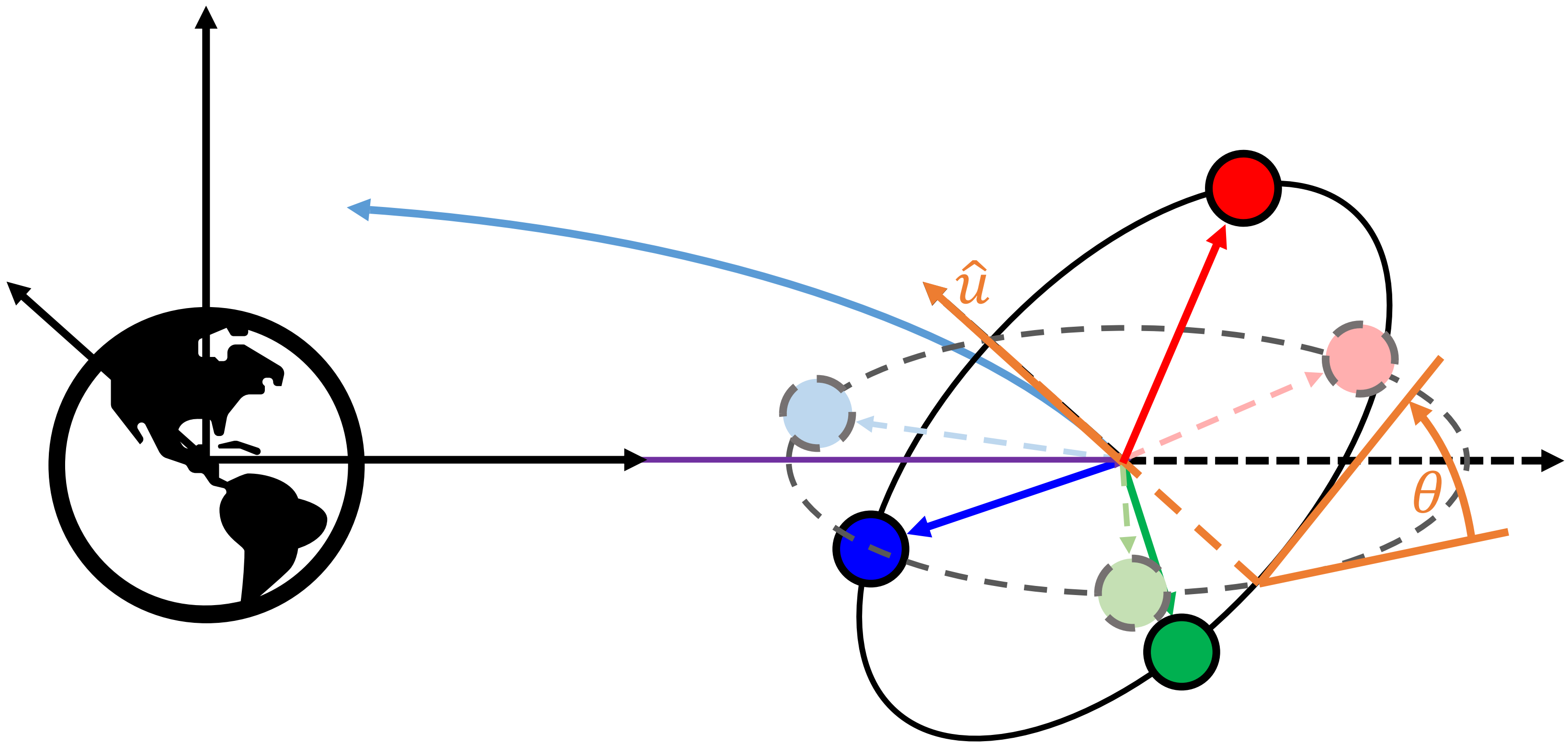

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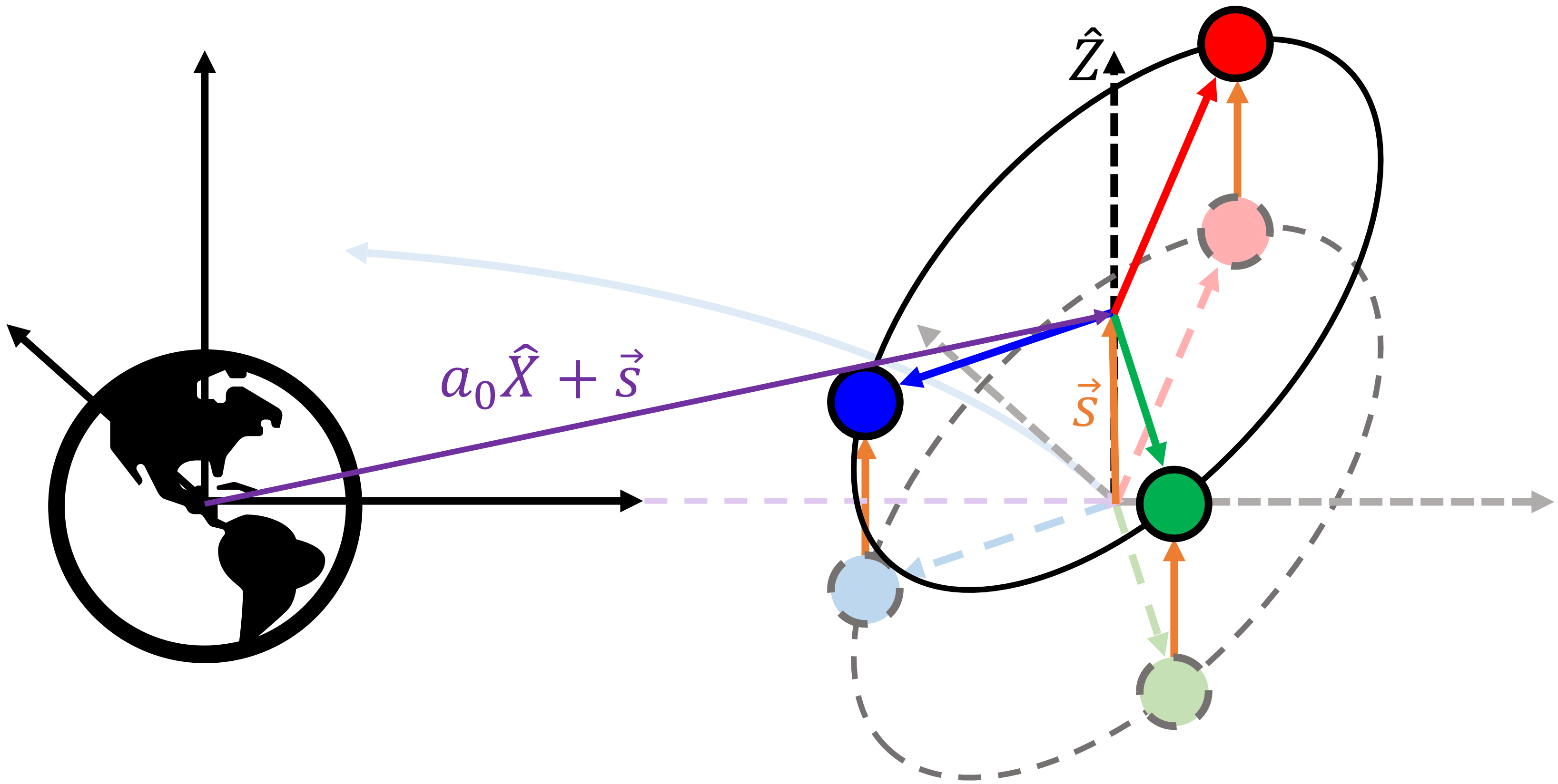

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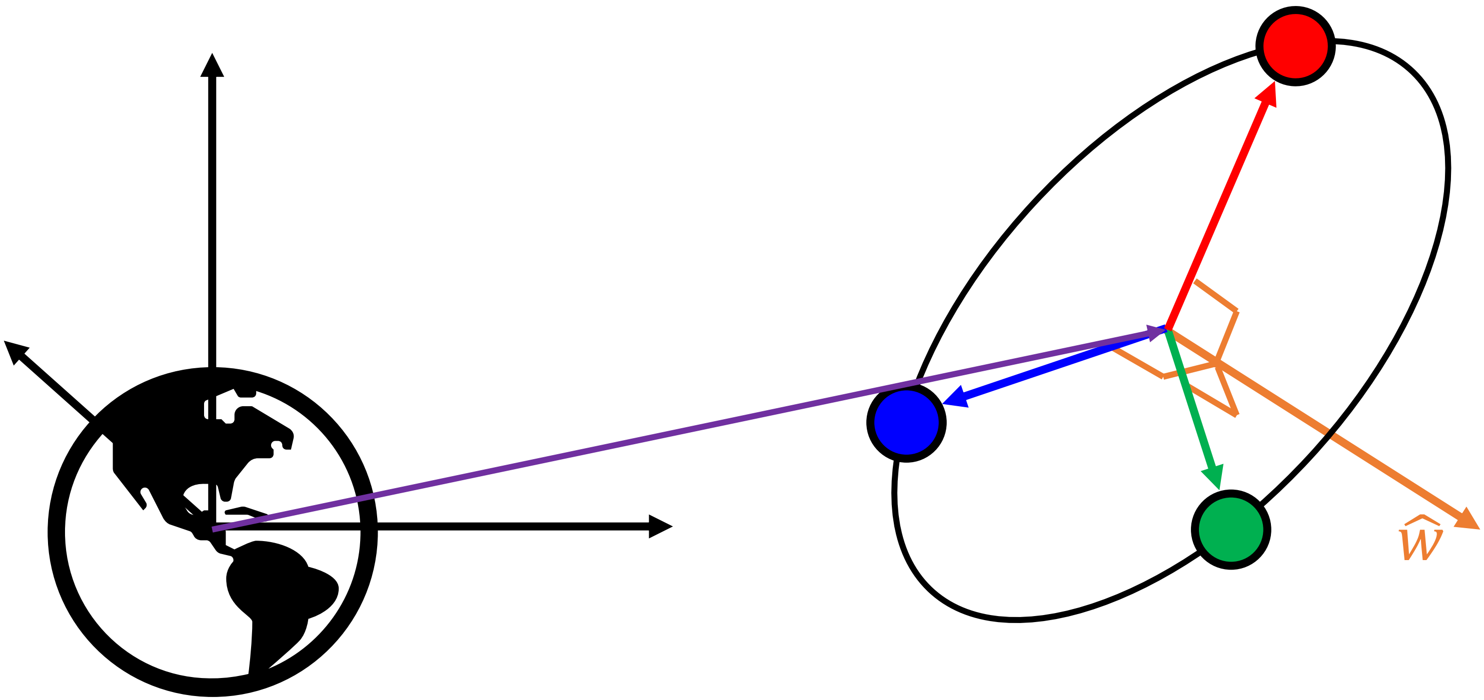
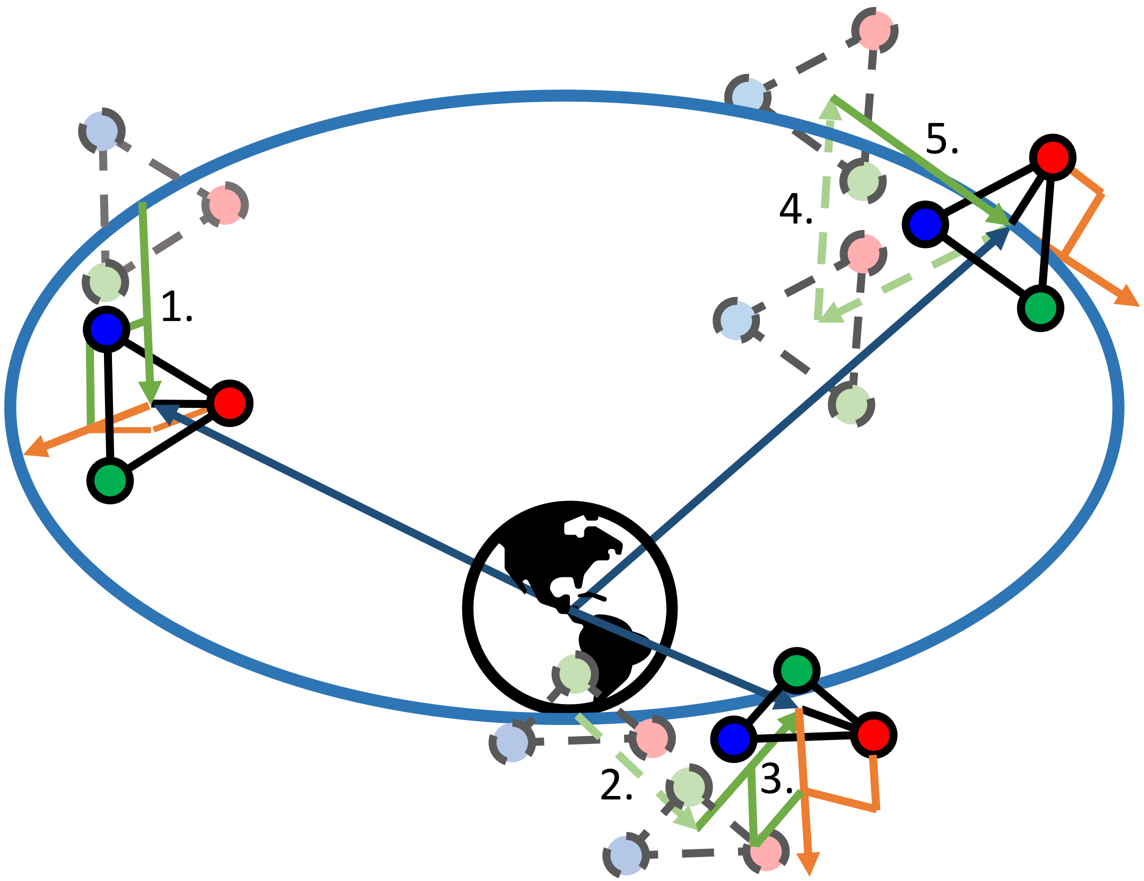


- Translation by swarm fixed displacement vector.



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Assessing Chain Cost

Chain Cost for 3 Satellites		Moves to Position Index		
		A (index 1)	B (index 2)	C (index 3)
Satellite Index	1 (at A)	$\mathcal{K}_{1\to1}$	$\mathcal{K}_{1\to2}$	$\mathcal{K}_{1\to3}$
	2 (at B)	$\mathcal{K}_{2\to1}$	$\mathcal{K}_{2\to2}$	$\mathcal{K}_{2\to3}$
	3 (at C)	$\mathcal{K}_{3\to1}$	$\mathcal{K}_{3\to2}$	$\mathcal{K}_{3\to3}$

- We may now define the chain cost function for each element of this matrix:

$$\mathcal{K}_{i_1\to i_2} = \frac{1}{n}\sum_{j=0}^{n-1}\left\| \vec{r}_{j\,|\,i_1,i_2}^{\,\text{geometry}} - \vec{r}_{j\,|\,i_1,i_2}^{\,\text{dynamics}} \right\|^2$$

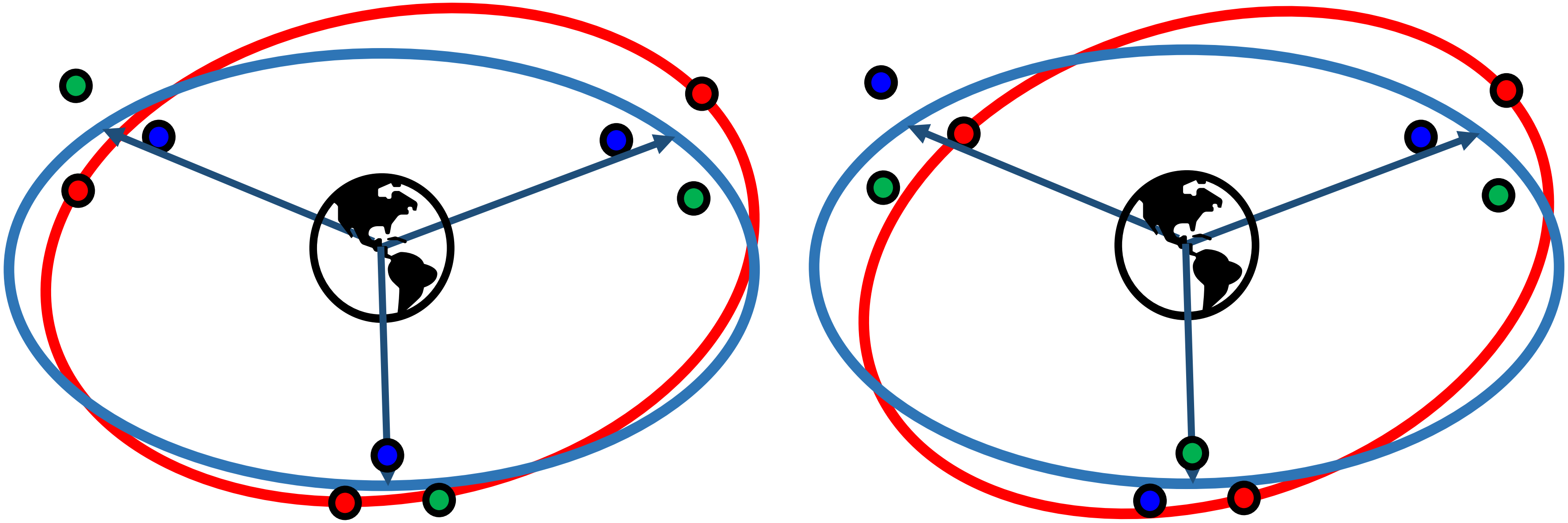
Reproducing LISA

Reproducing LISA

- Process is intended for swarms with any number of satellites.
- Process must produce results of LISA swarm to be considered viable.
- Consider a LISA-like configuration with the following parameters.

Parameter / Variable	Value
$\hat{\rho}_i$	$\cos\big((i-1)\cdot 120°\big)\,\hat{X} - \sin\big((i-1)\cdot 120°\big)\,\hat{Y}$
ρ	1,000 km
a_0	10,000 km
ϕ	0°
$\hat{u}$	$\hat{Y}$
θ	$-60°$
$\hat{w}$	$\sin(120°)\,\hat{X} + \cos(120°)\,\hat{Z}$

- Dynamic variables can be defined for the three-satellite case by specifying motion as either **right-handed or left-handed** with respect to $\hat{w}$.
  - Right-handed motion means that each satellite moves clockwise from one checkpoint to the next (i.e., satellite 1 follows path A, B, C).
  - Left-handed motion moves counter-clockwise (i.e., path A, C, B).



Left-handed motion **Right-handed motion**

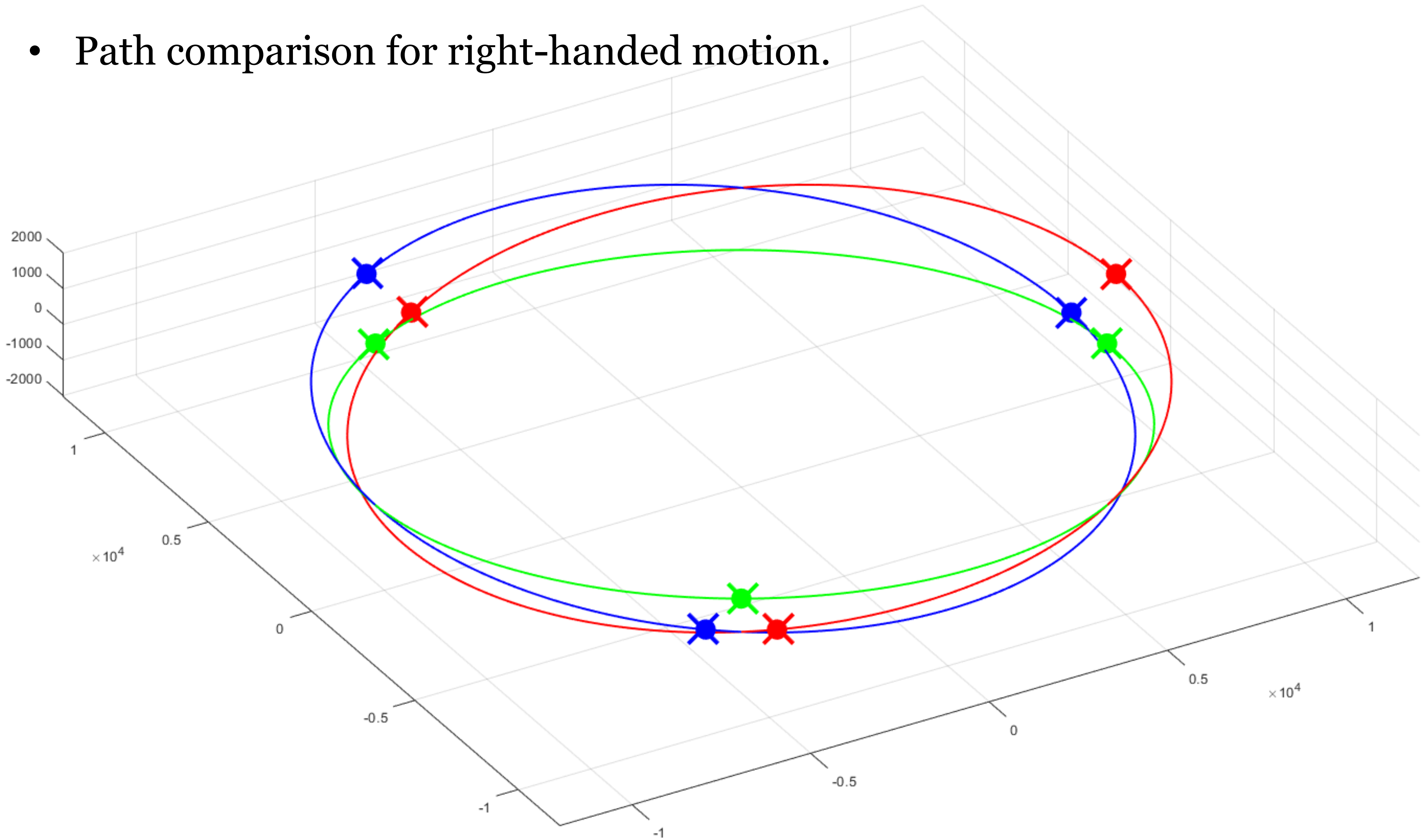
Reproducing LISA

- Left-handed motion $\Longrightarrow$ path A, C, B; right-handed motion $\Longrightarrow$ path A, B, C.

Parameter / Variable	Left-Handed Motion	Right-Handed Motion
$\vec{s}$	$-(20.14\text{ km})\,\hat{Z}$	$(66.62\text{ km})\,\hat{Z}$
$\Delta\vec{s}$	$\vec{0}$ km	$\vec{0}$ km
a	10,076 km	10,032 km
e_1	0.0454	0.0508
e_2	0.0454	0.0508
e_3	0.0454	0.0508
ν_1	180°	180°
ν_2	305°	65°
ν_3	55°	295°

Reproducing LISA

- Path comparison for right-handed motion.



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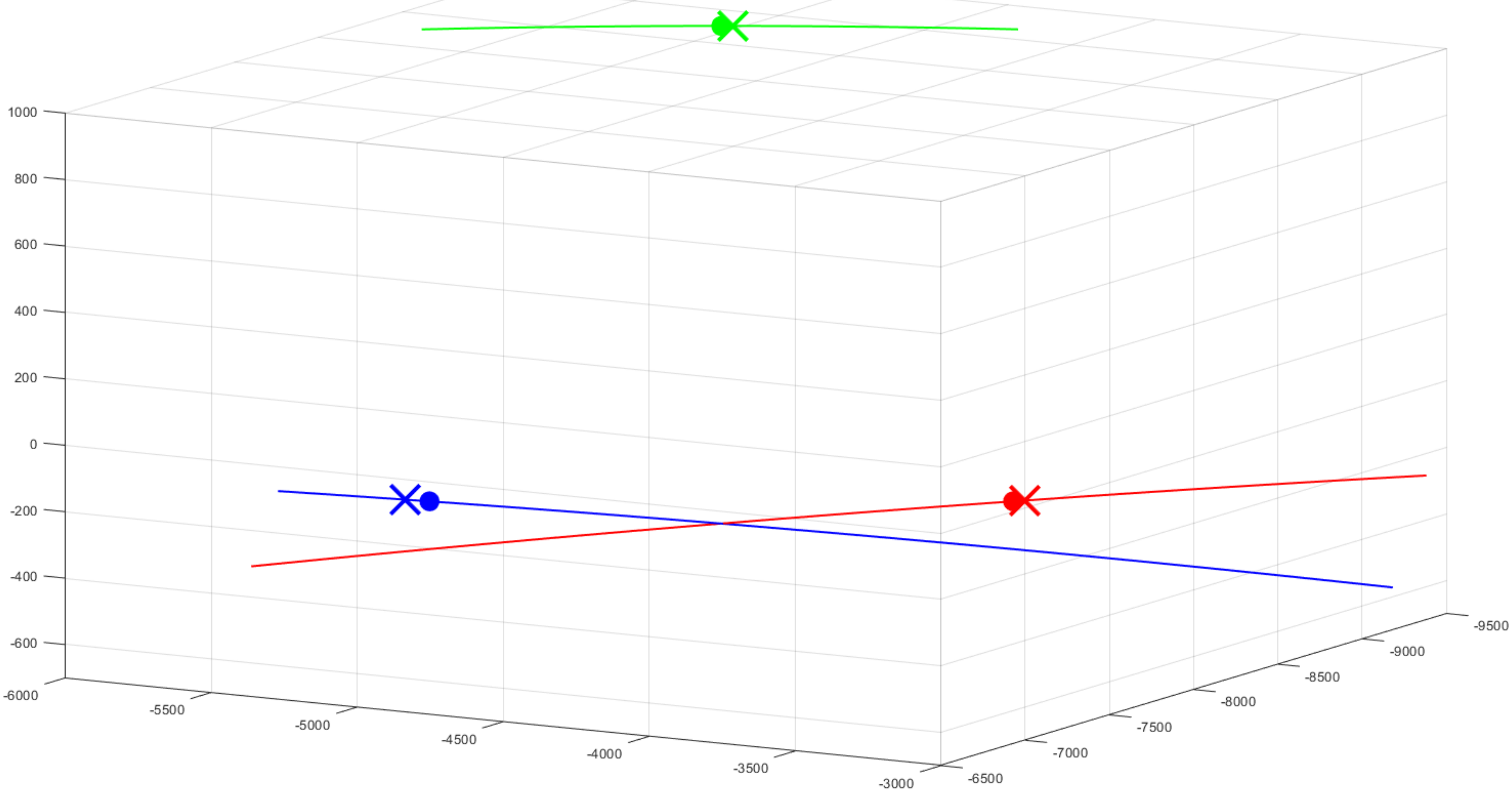

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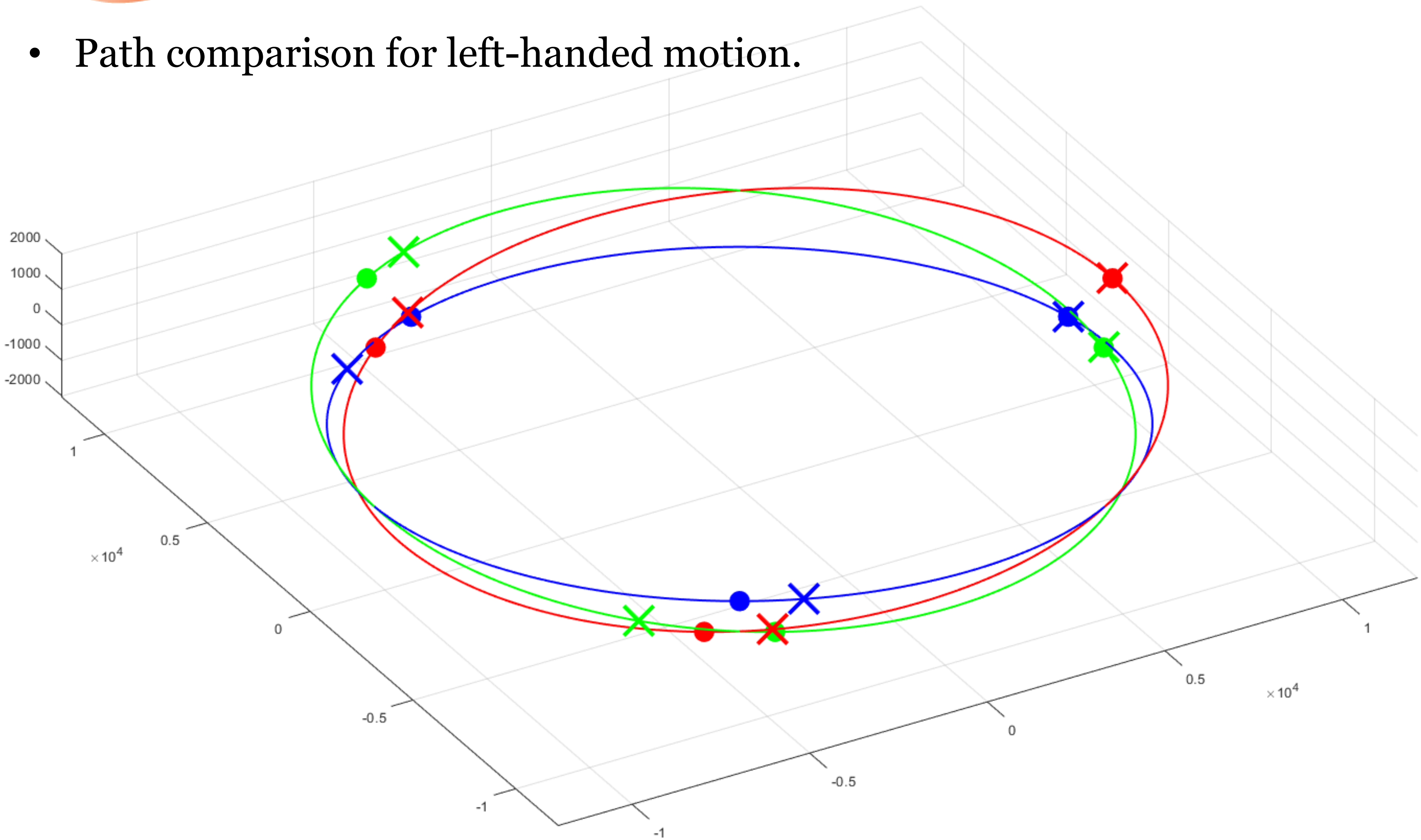

Conclusions

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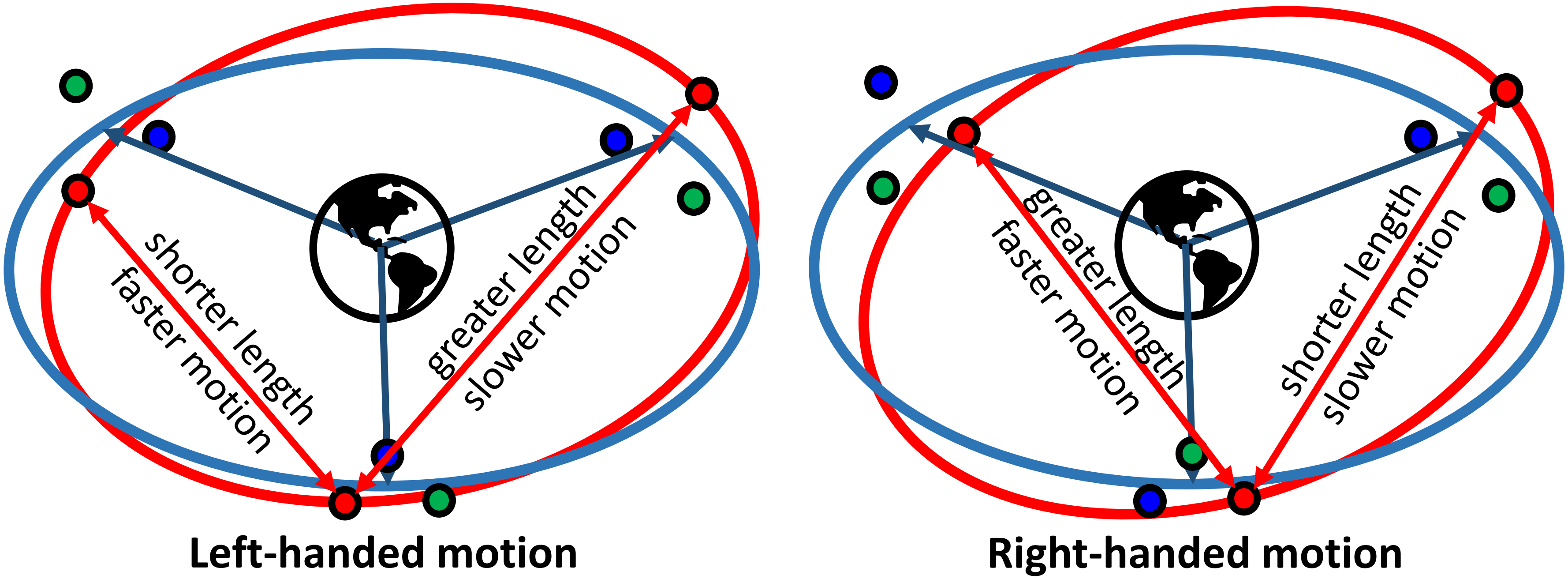
- The difference in timing between left- and right-handed motion is the result of uneven spacing between checkpoints 2 and 3.
  - Left-handed motion squeezes together the positions near periapse.
  - Comparing the periapse leg of the orbit to the apoapse legs, we see that traversal in the same amount of time is physically impossible.



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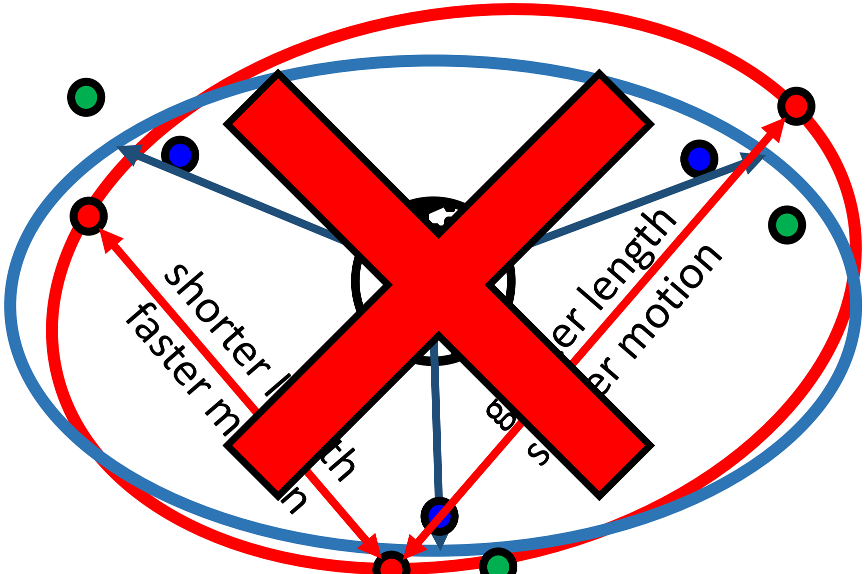
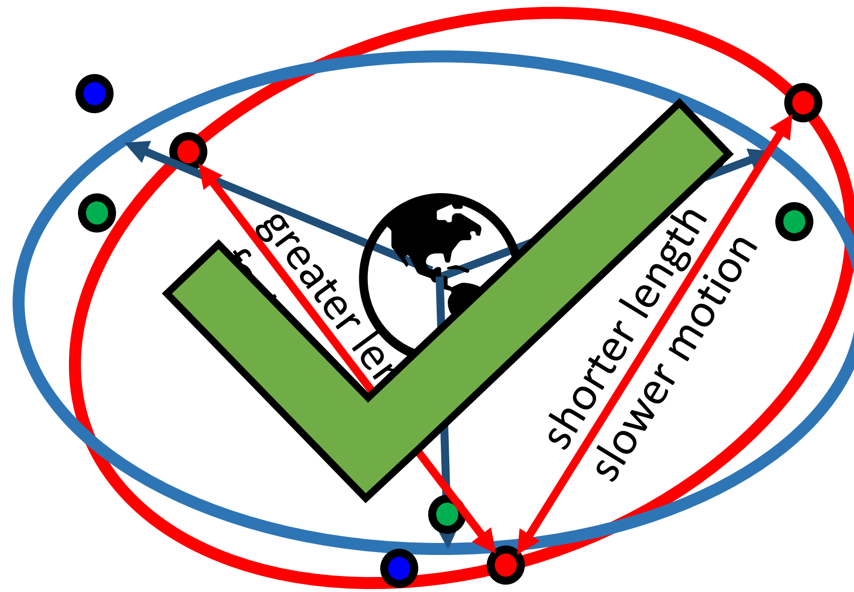


Left-handed motion



Right-handed motion

Conclusions

- Substituting parameter and variable values into the chain cost function, we apply the Munkres algorithm to the resulting cost matrix.

Chain Cost for 3 Satellites (root mean-squared error)		Moves to Position Index		
		A	B	C
Satellite Index	1 (at A)	1,328 km	30 km	1,443 km
	2 (at B)	969 km	1,286 km	46 km
	3 (at C)	46 km	969 km	1,286 km

- Average cost of right-handed motion: 41 kilometers.

Conclusions

- Summary:
  - Demonstrated that ‘formation chain’ framework produces accurate results for known swarm configurations.
  - Provided a rigorous, mathematical basis for swarm initialization.
  - Gained insight into relationship between dynamics and geometry.
- Next Steps:
  - Publish formation chain framework as a solution to the swarm initialization problem.
  - Apply formation chain framework to different swarm configurations.
  - Explore additional methods to refine swarm optimality in general.

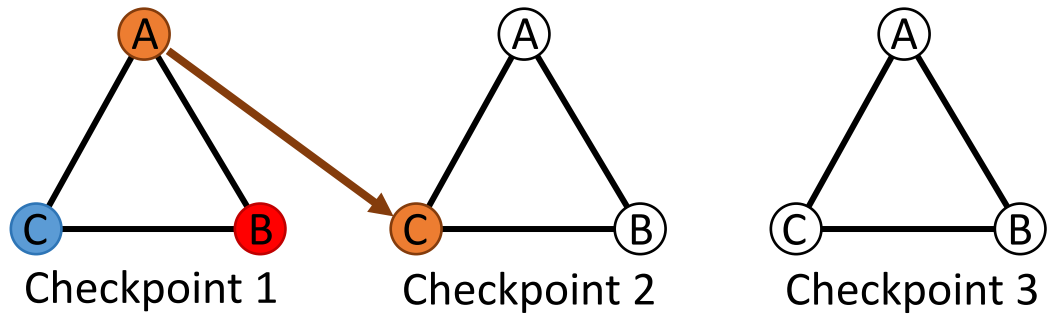
Questions

Index Chains

Bonus (Index Chains)

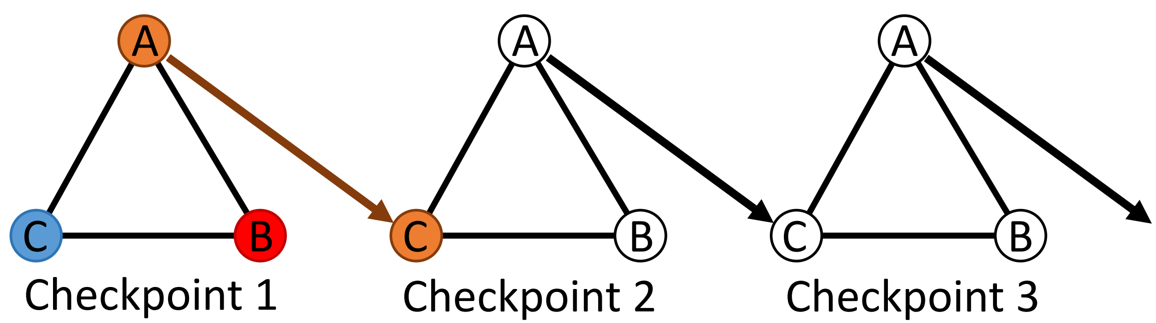
- Consider the case in which satellite 1 moves from position A to position C:

Positions: A, B, and C
Satellites: 1, 2, and 3
= 1 = 2 = 3



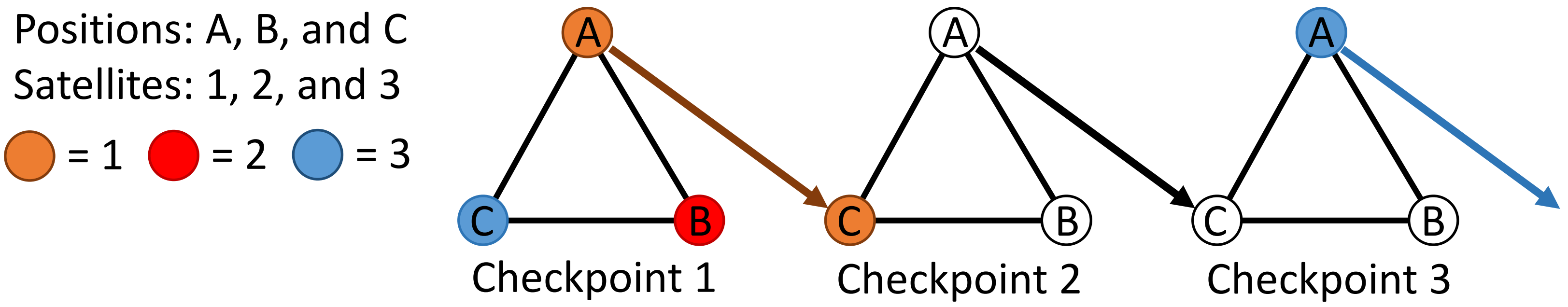
- Because no checkpoint can be 'special', the satellite at A must *always* move to C from one checkpoint to the next.

Positions: A, B, and C
Satellites: 1, 2, and 3
= 1 = 2 = 3

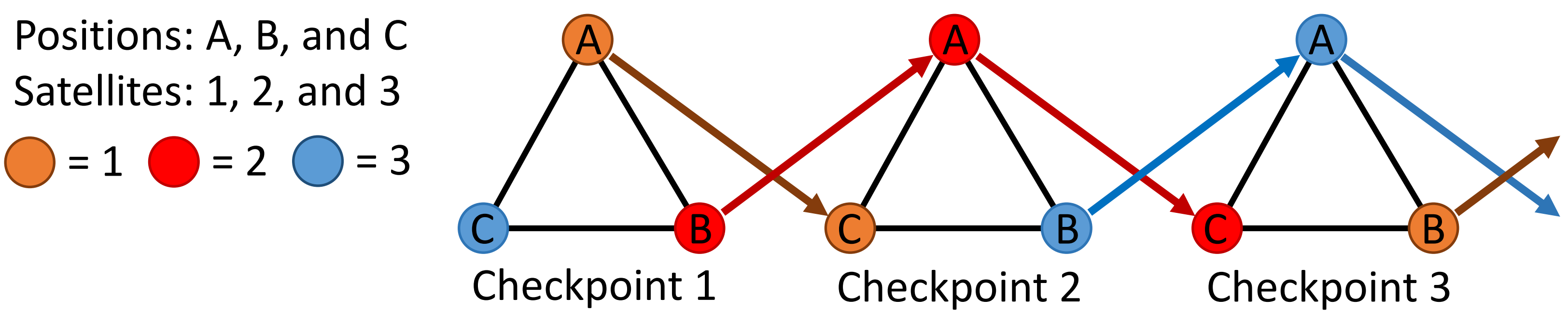


Bonus (Index Chains)

- Consequently, satellite 3 must have arrived at position C from position A:



- Next, since satellite 1 must arrive at A after three successive motions, the satellite at C cannot move to position A; the satellite at B must move to A.



Assessing Chain Cost

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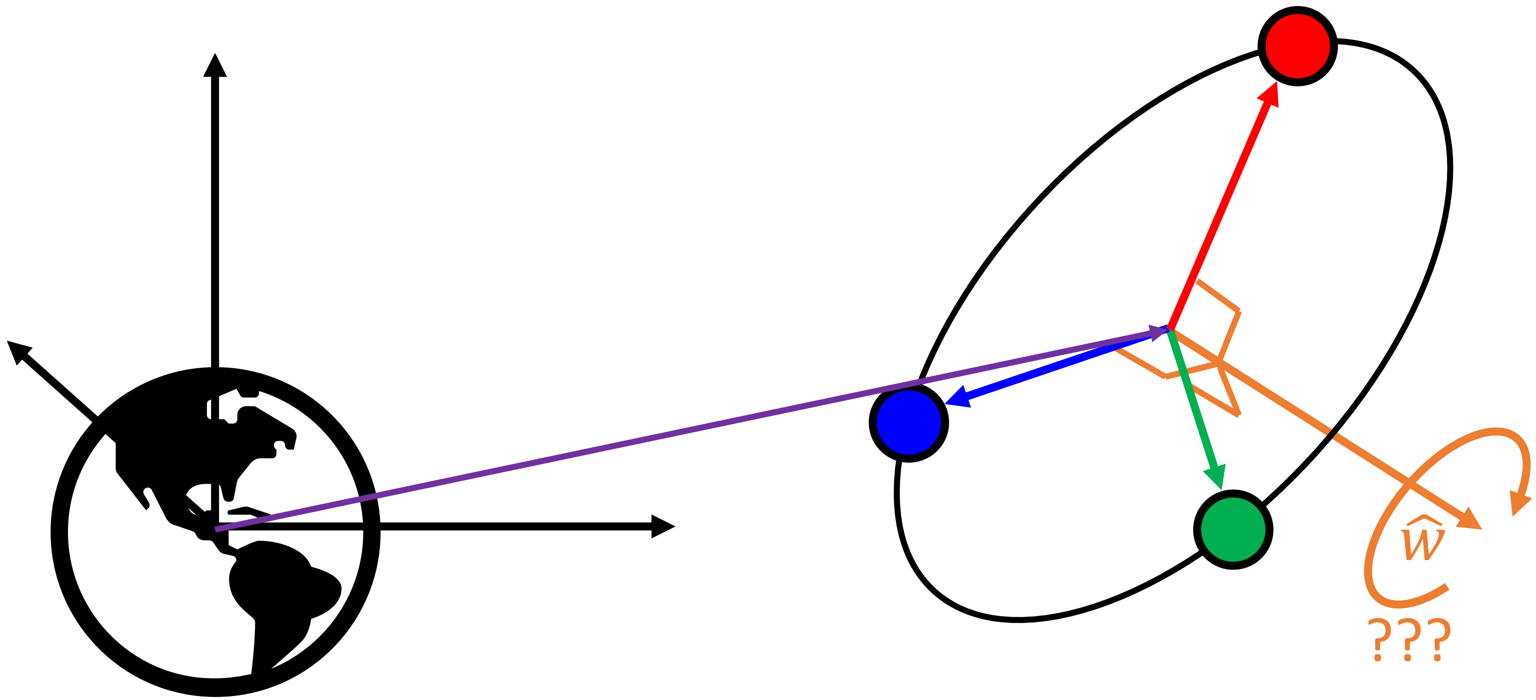
- Geometric variables incorporated in the chain cost function:



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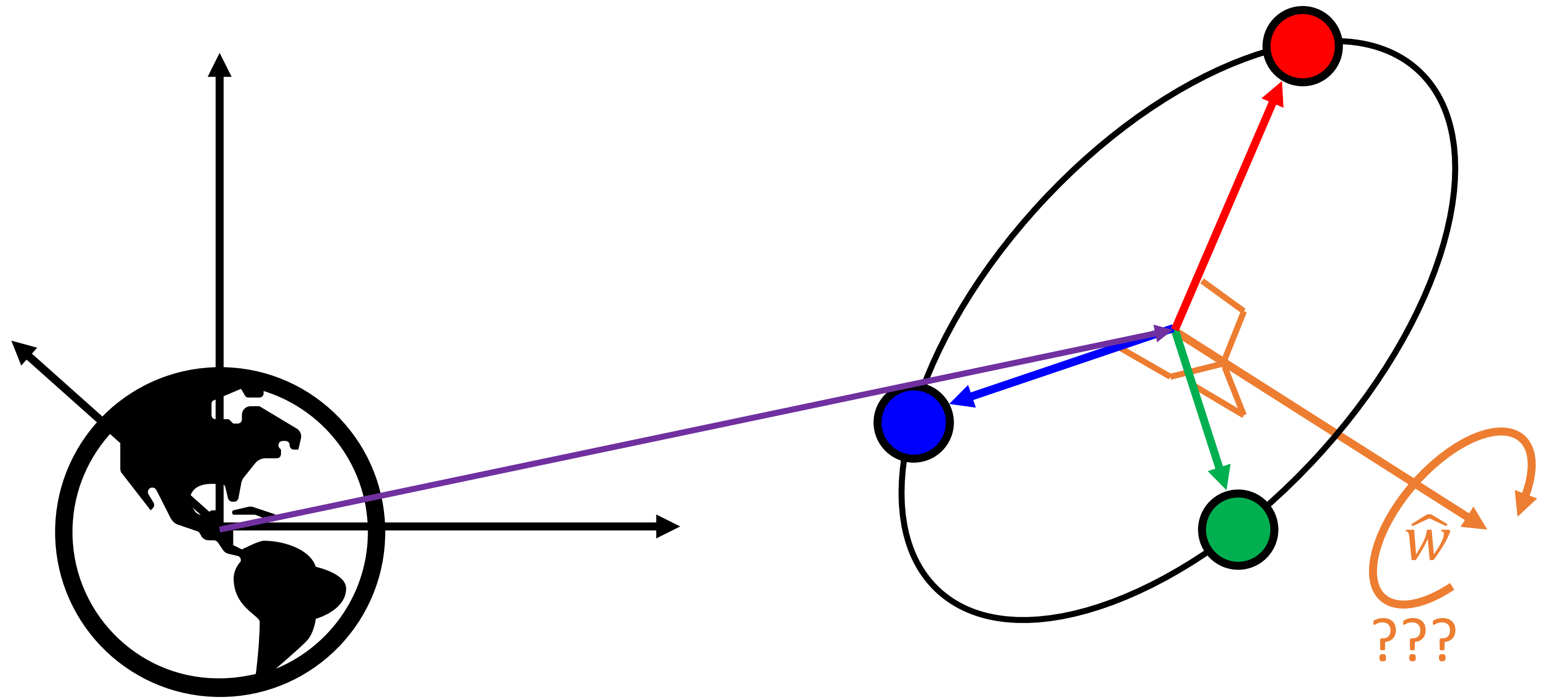

n rotations for n satellites $\Longrightarrow$ angle of rotation = $m\Delta$ where $\Delta = 2\pi/n$.

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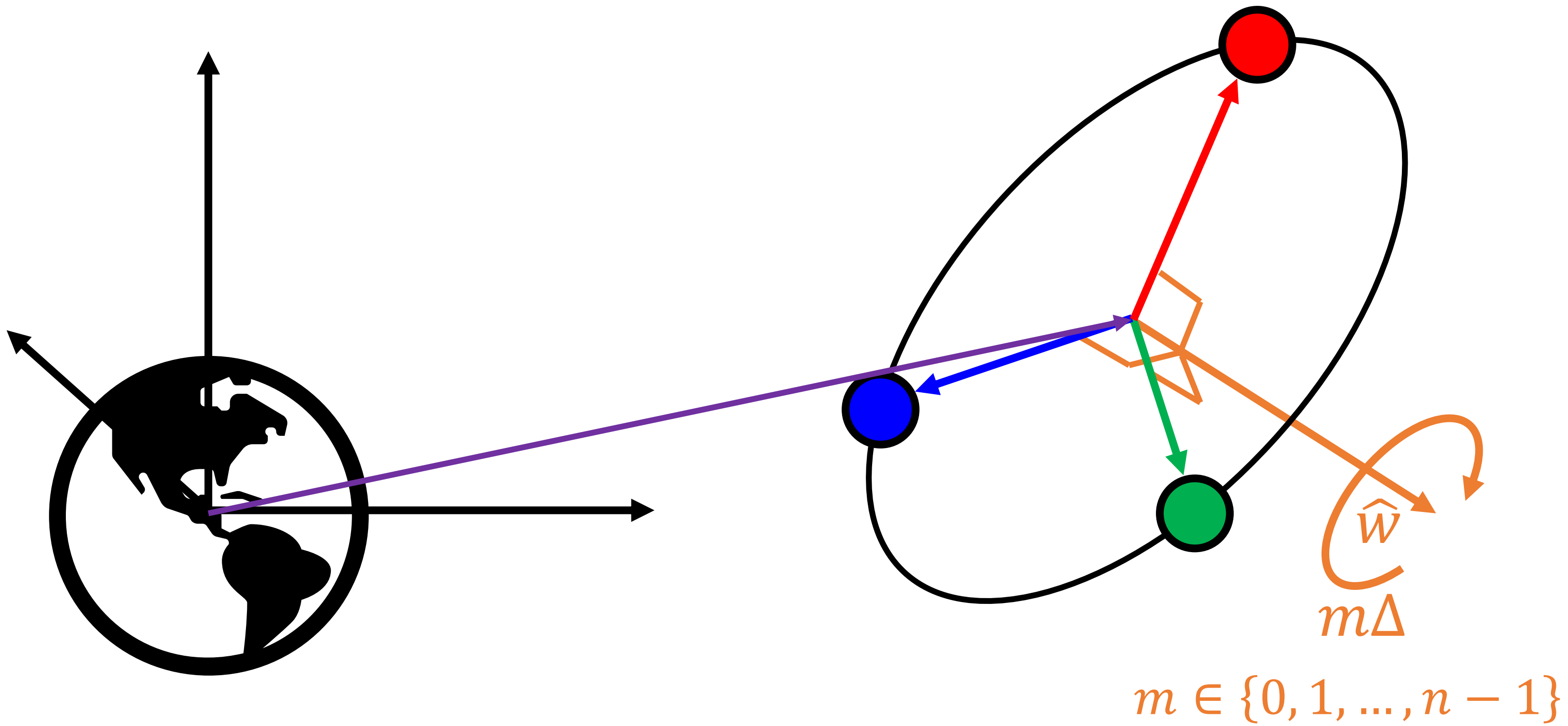

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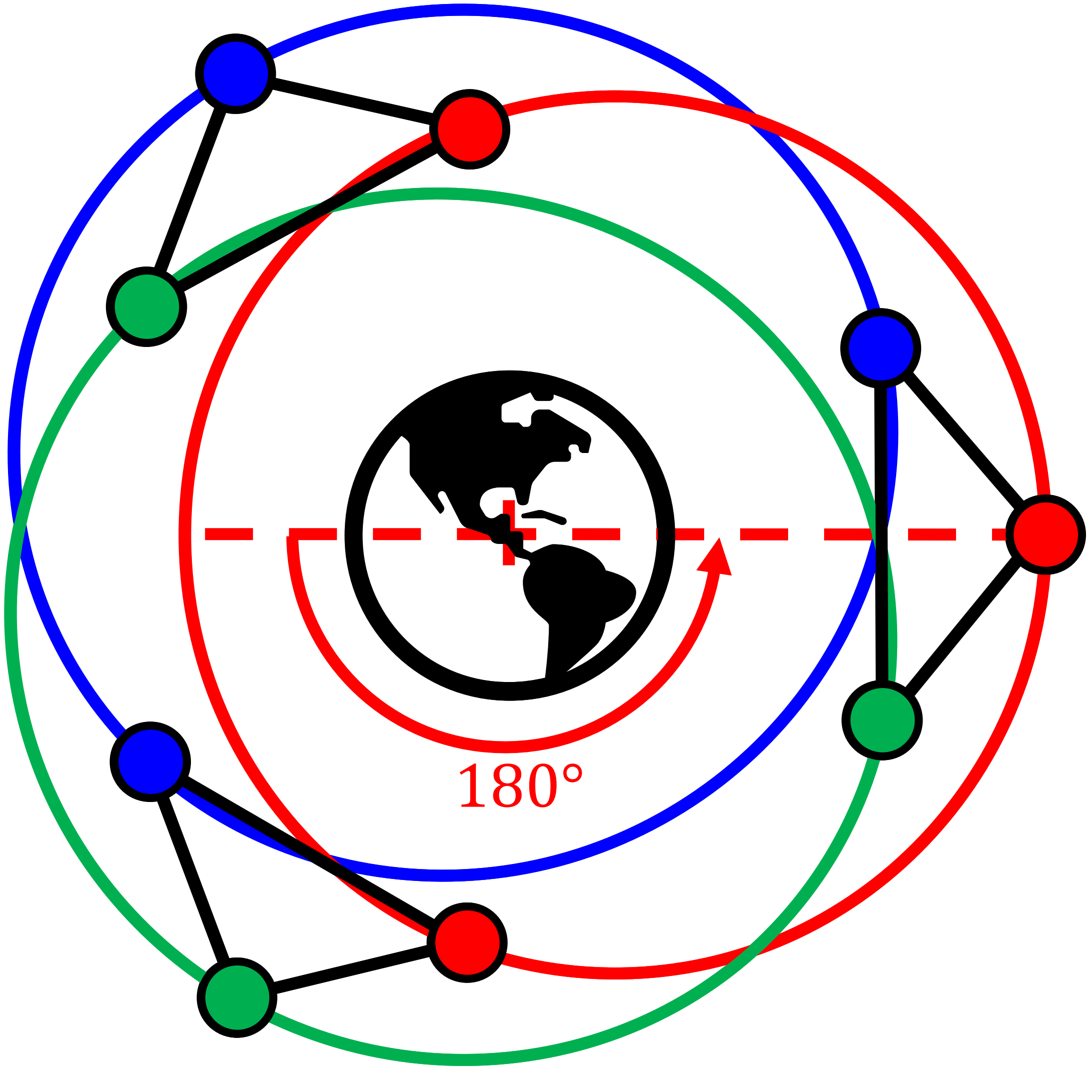
	2 (at B)	$\mathcal{K}_{2\to 1,m}$	$\mathcal{K}_{2\to 2,m}$	$\mathcal{K}_{2\to 3,m}$
	3 (at C)	$\mathcal{K}_{3\to 1,m}$	$\mathcal{K}_{3\to 2,m}$	$\mathcal{K}_{3\to 3,m}$

- Must perform assignment analysis for each m value from 0 to $n-1$.
- We may now define the chain cost function for each element of this matrix:

$$\mathcal{K}_{i_1\to i_2,m} = \frac{1}{n}\sum_{j=0}^{n-1}\left\|\vec{r}_{j\,|\,i_1,i_2,m}^{\,\text{geometry}} - \vec{r}_{j\,|\,i_1,i_2,m}^{\,\text{dynamics}}\right\|^2$$

Reproducing LISA

Bonus (Reproducing LISA)



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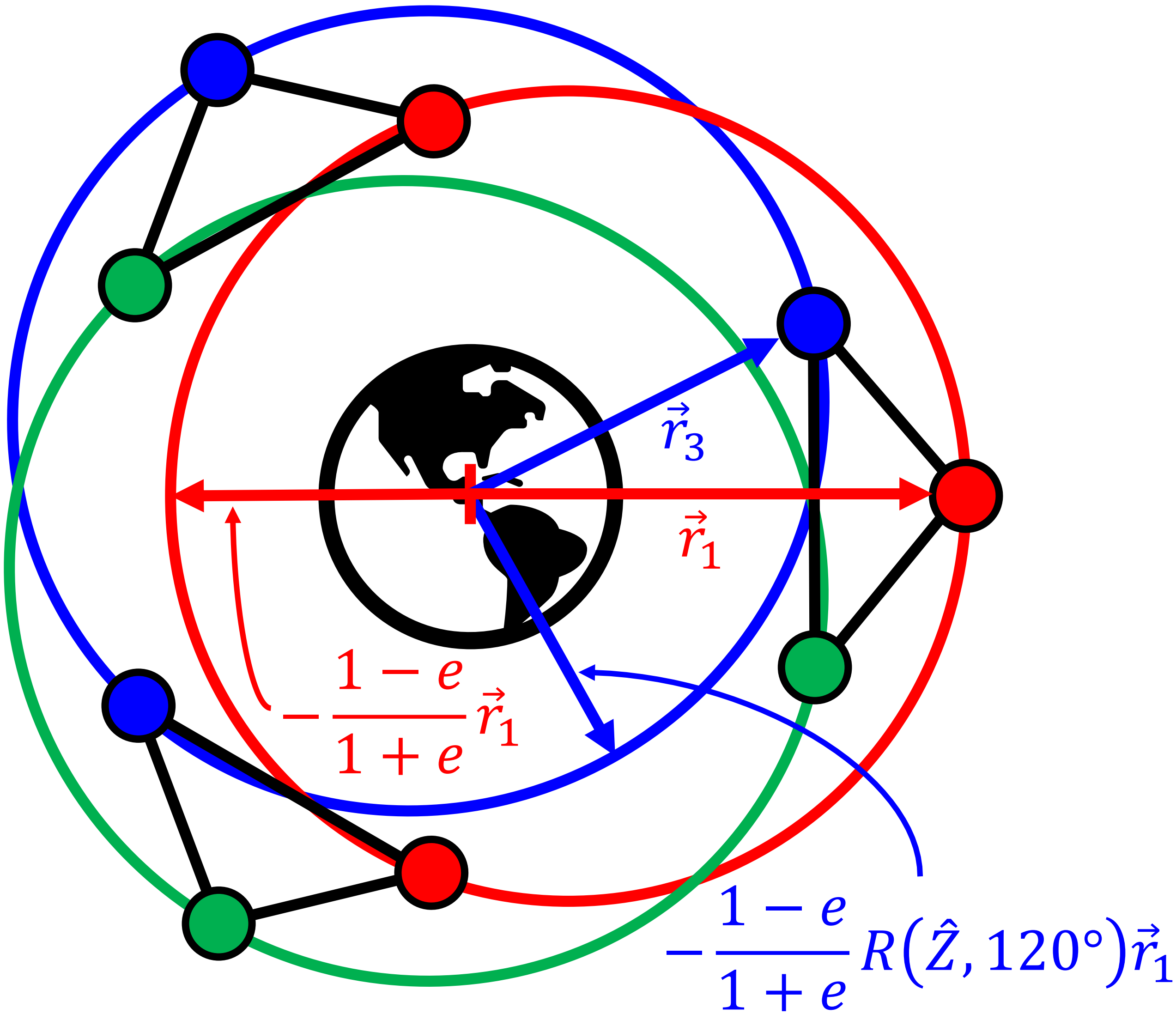

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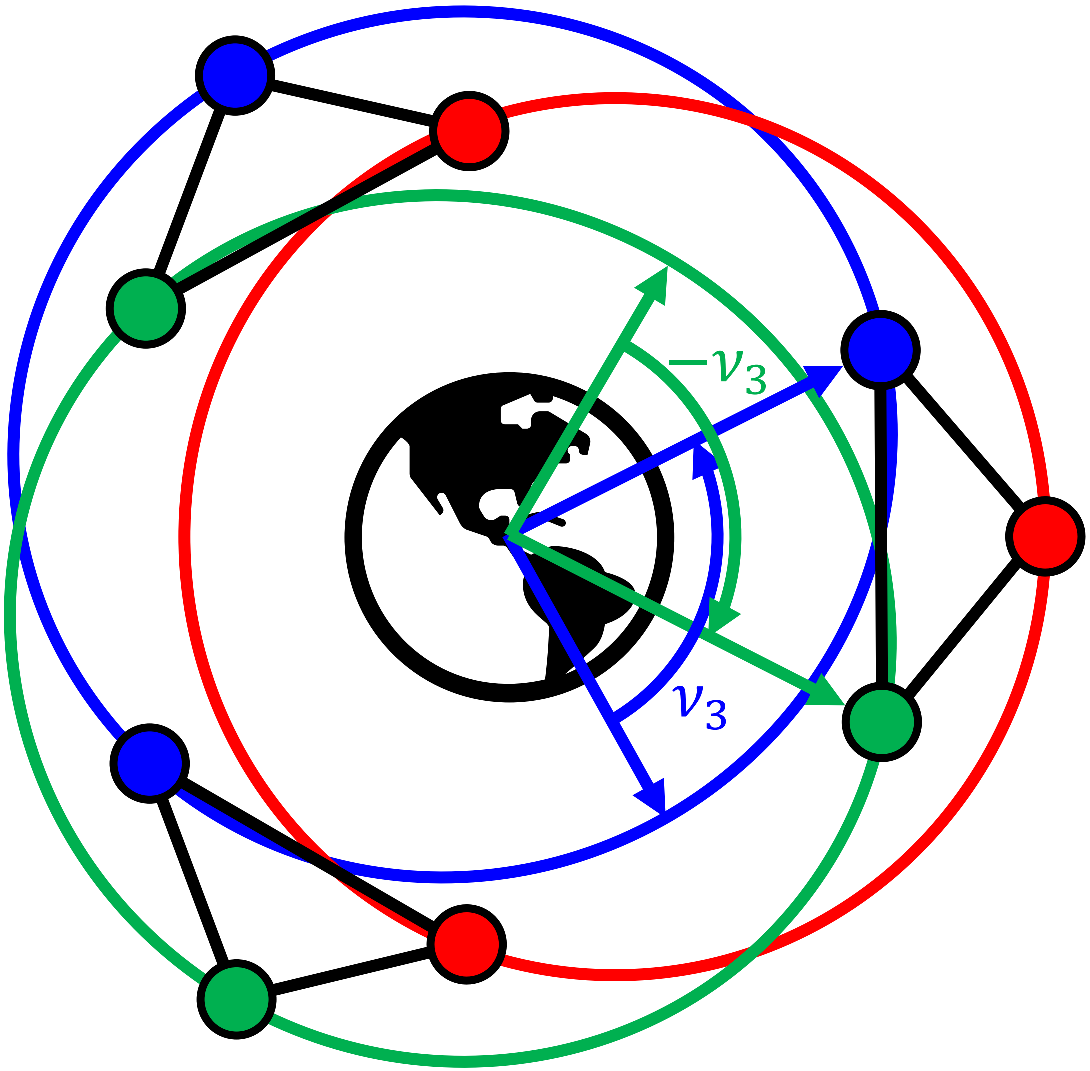
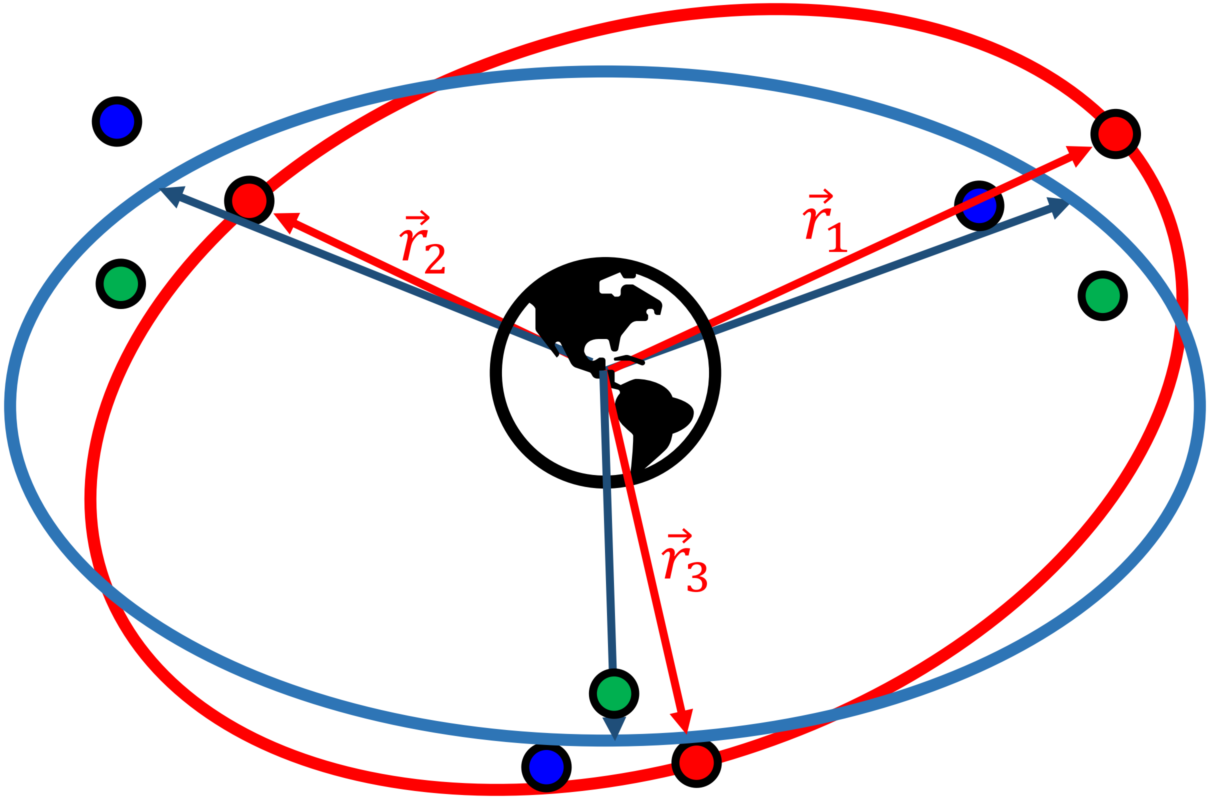


Bonus (Reproducing LISA)

- The exact value of the dynamical variables in the three-satellite case can be determined by defining the following vectors:



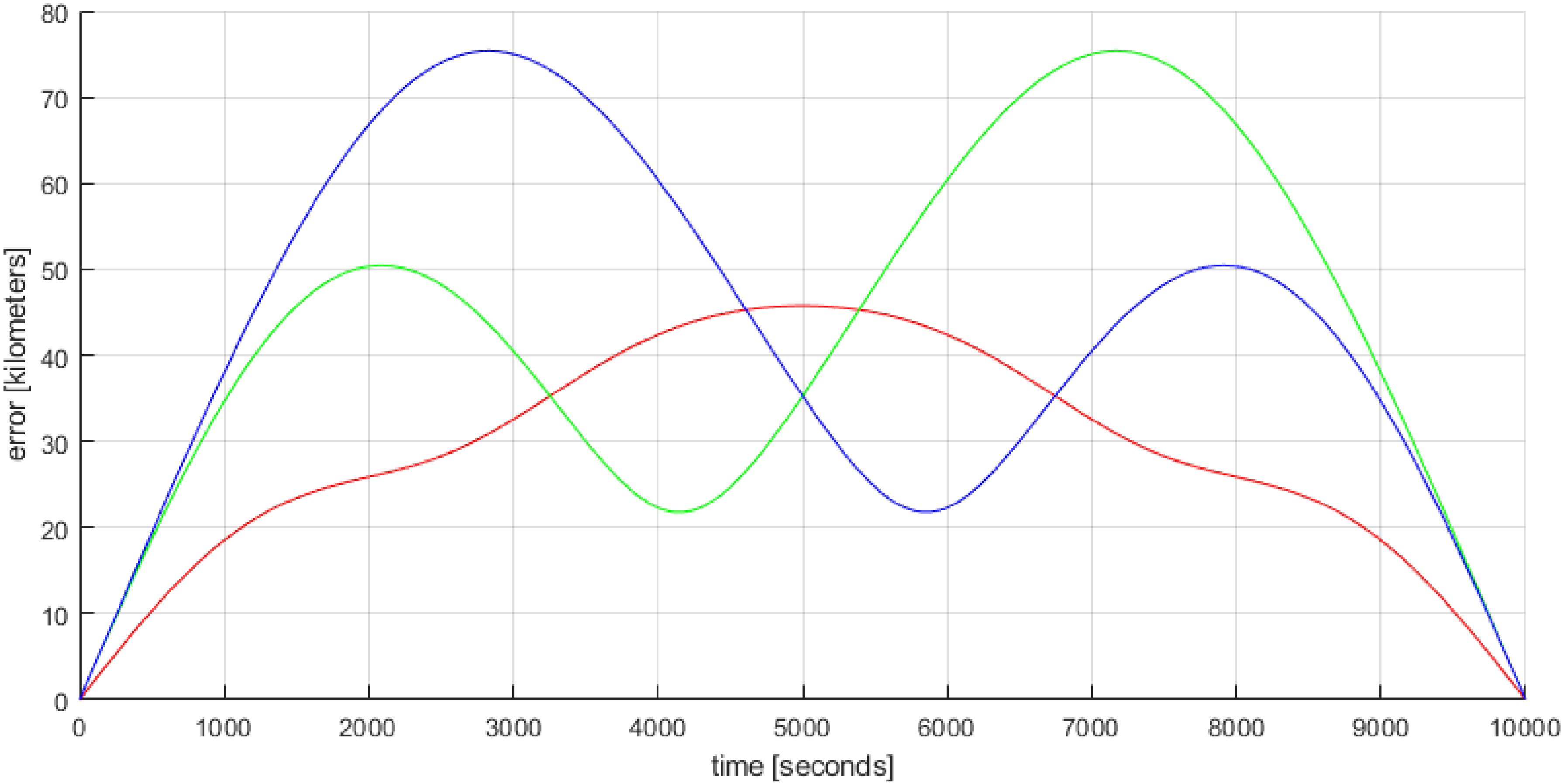
$\vec{r}_1$ = position of satellite 1 at checkpoint 1.

$\vec{r}_2$ = position of satellite 1 at checkpoint 2.

$\vec{r}_3$ = position of satellite 1 at checkpoint 3.

Bonus (Reproducing LISA)

- Path comparison for right-handed motion.



Bonus (Reproducing LISA)

- Path comparison for left-handed motion.