Consensus over Clustered Networks using Output Feedback*

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Consensus over Clustered Networks Using Output Feedback and Asynchronous Inter-Cluster Communication by Cristian F. Nino, Federico Zegers, Sean Phillips, Warren E. Dixon

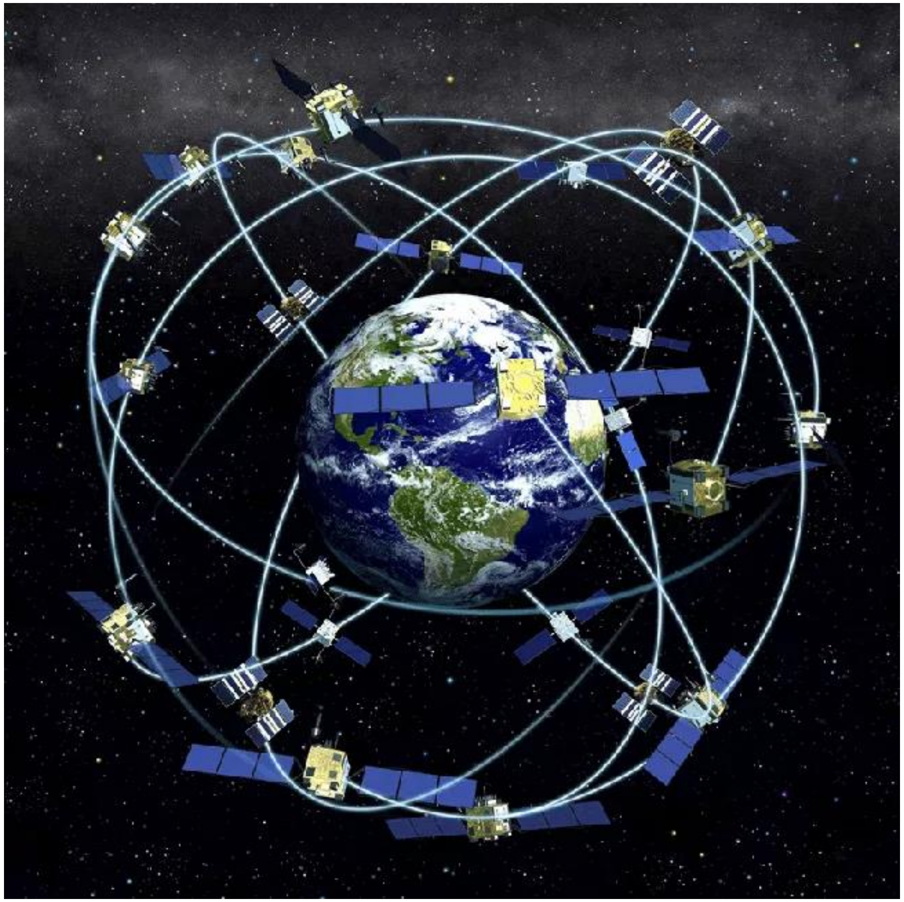
Motivation

Consensus/Agreement Problem

$$\{x_i\}_{i\in\mathcal{V}} \qquad \dot{x}_i = f_i(x_i, u_i)$$

$$\limsup_{t\to\infty} \|x_i(t) - x_k(t)\| = 0 \quad \forall i,k \in \mathcal{V}$$

- Scalability: centralized methods require at least one agent to access and process global information
- Efficiency: Networks may not be capable of supporting continuous communication for high agent populations



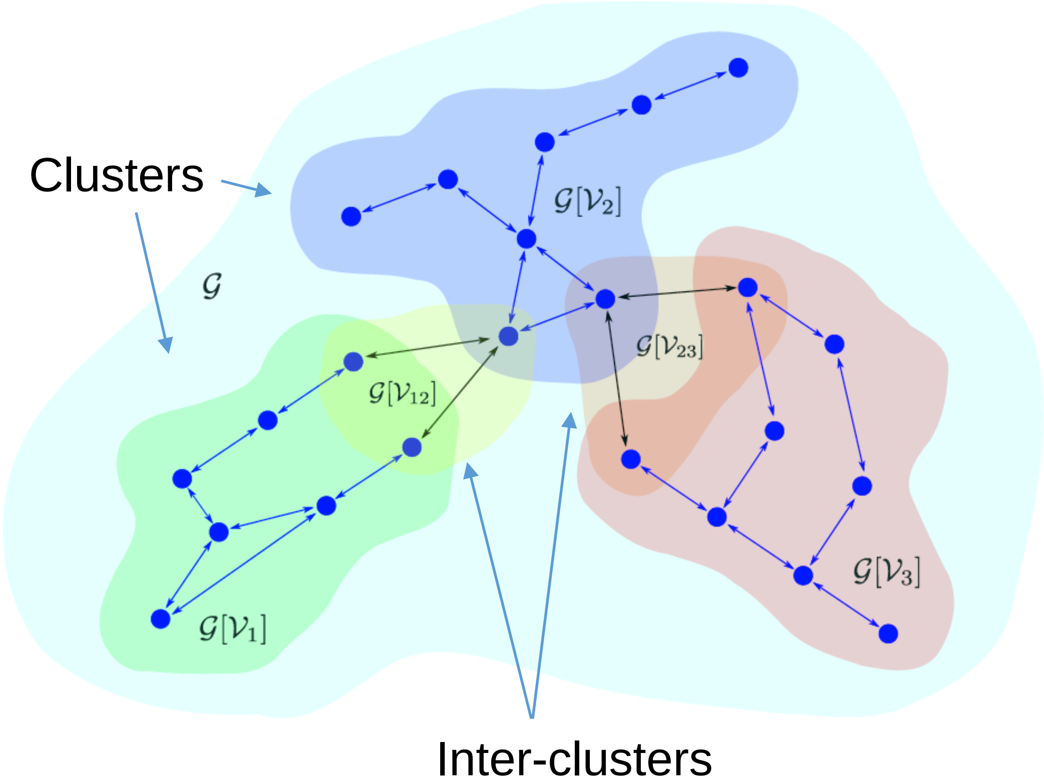
Clustered Networks and Inter-Clusters

$$\mathcal{C} \triangleq \{\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_M\}$$

$$\mathcal{V}_p \subset \mathcal{V} \quad \forall p \in [M],$$

$$\mathcal{V}_p \cap \mathcal{V}_q = \varnothing \quad \forall p, q \in [M]$$

$$\bigcup_{p \in [M]} \mathcal{V}_p = \mathcal{V}$$



F. Zegers, "Lyapunov-Based Control of Distributed Multi-Agent Systems With Intermittent Communication," University of Florida, 2021.

Problem Formulation

Given a C-MAS with:

- N agents
- Undirected Communication Graph
  - $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$
- Vertex Set
  - $\mathcal{V} \triangleq \{1, 2, ..., N\}$
- Edge Set
  - $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$
- Cluster Set
  - $\mathcal{C} \triangleq \{\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_M\}$
- Agent Model
  - $\dot{x}_i = Ax_i + Bu_i$
  - $y_i = Cx_i$

  where:
  $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times d}, C \in \mathbb{R}^{m \times n}$
  $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^d, y_i \in \mathbb{R}^m$
- Only y_i is measurable by agent i

Objective:
Design a distributed controller and observer that enables consensus in the states:

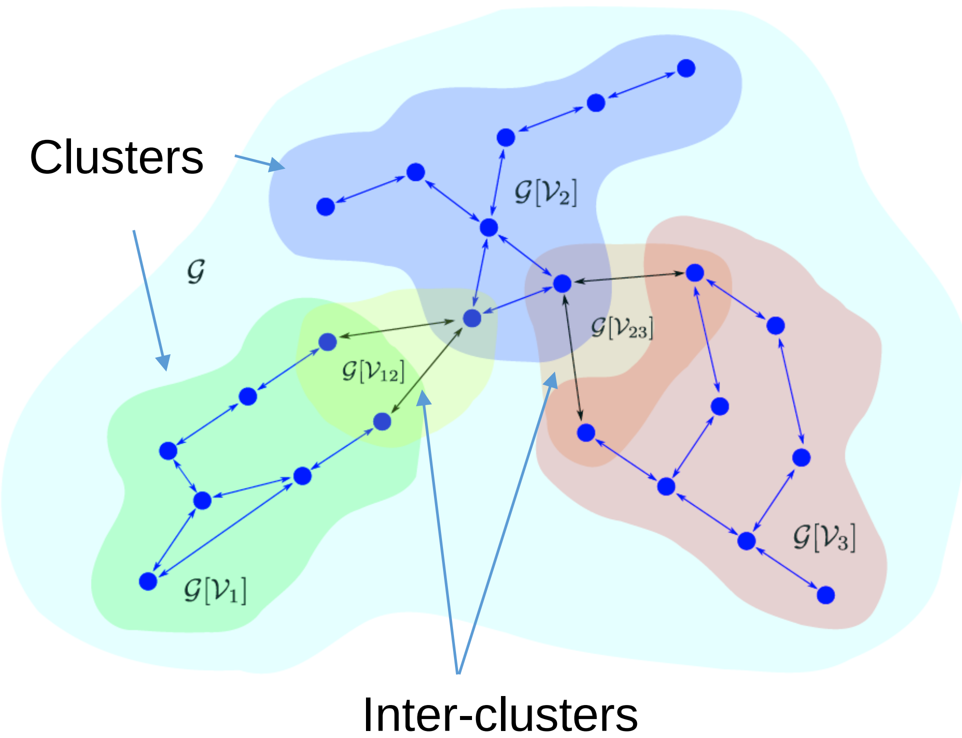
$$e_i \triangleq x_i - \frac{1}{N} \sum_{\ell \in \mathcal{V}} x_\ell, \quad \lim_{t \to \infty} \|e_i(t)\| = 0 \quad \forall i \in \mathcal{V}$$

by using output feedback and asynchronous inter-cluster communication (continuous communication in clusters).

Assumptions

1. The static graph $\mathcal{G}$ is connected.
2. At most one inter-cluster between each cluster pair.

Intermittent Communication



The sequence of output measurement times for all agents is $\{\mathtt{t}_\mathtt{s}\}_{\mathtt{s}=0}^{\infty}$, which is generated by

$$\dot{\rho} = -1, \qquad \rho_i \in [0, T_4]$$
$$\rho^+ \in [T_3, T_4], \quad \rho = 0.$$

The sequence of communication events between within inter-clusters is is $\{t_s^r\}_{s=0}^{\infty}$, which is generated by

$$\dot{\tau}_r = -1, \qquad \tau_r \in [0, T_2^r]$$
$$\tau_r^+ \in [T_1^r, T_2^r], \quad \tau_r = 0$$

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State Observer of Agent i

The state estimate evolves according to:

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i + \beta_i$$
$$\hat{y}_i = C\hat{x}_i,$$

where the variable β_i evolves according to

$$\begin{aligned} \dot{\beta}_i &= 0_n, & \rho \in [0, T_4], \\ \beta_i^+ &= K_2 \left(y_i - \hat{y}_i \right), & \rho = 0, \end{aligned}$$

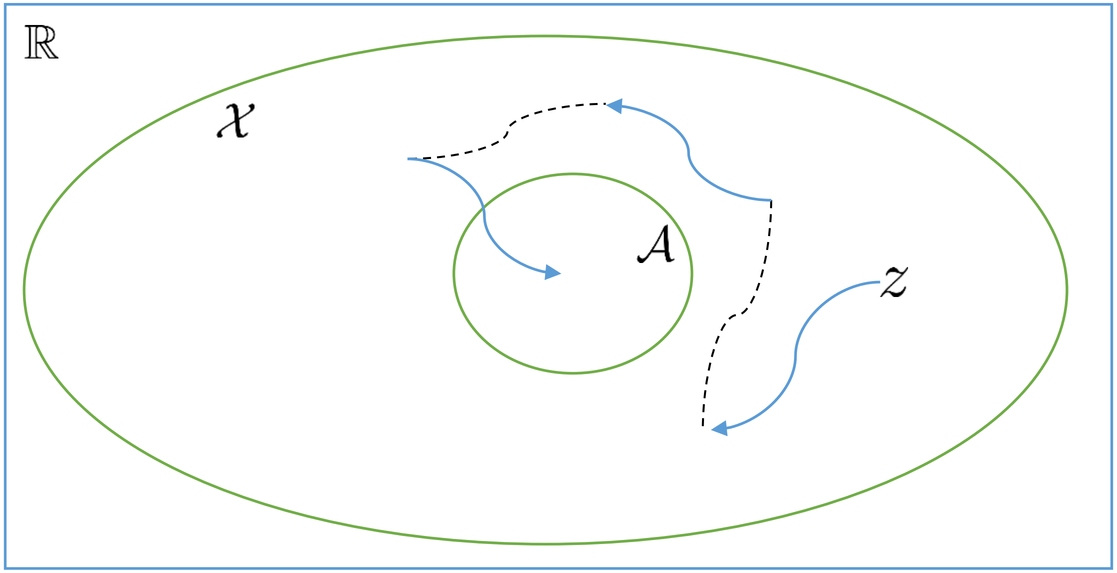
and $K_2 \in \mathbb{R}^{n \times m}$ is a user-defined matrix.

Analysis Overview

flow map — system state — flow set

$$\mathcal{H}: \begin{cases} \dot{z} = f(z), & z \in C, \\ z^{+} \in G(z), & z \in D, \end{cases}$$

jump map — jump set



Hybrid System Development

Agent Controller

- $u_i \triangleq K_1 \left(\eta_i + \sum_{r \in [M^*]} \eta_{i,r}\right), K \in \mathbb{R}^{d \times n}$

- η_i contains state estimate information that is continuously available
- $\eta_{i,r}$ contains state estimate information that is intermittently available

Same cluster component:

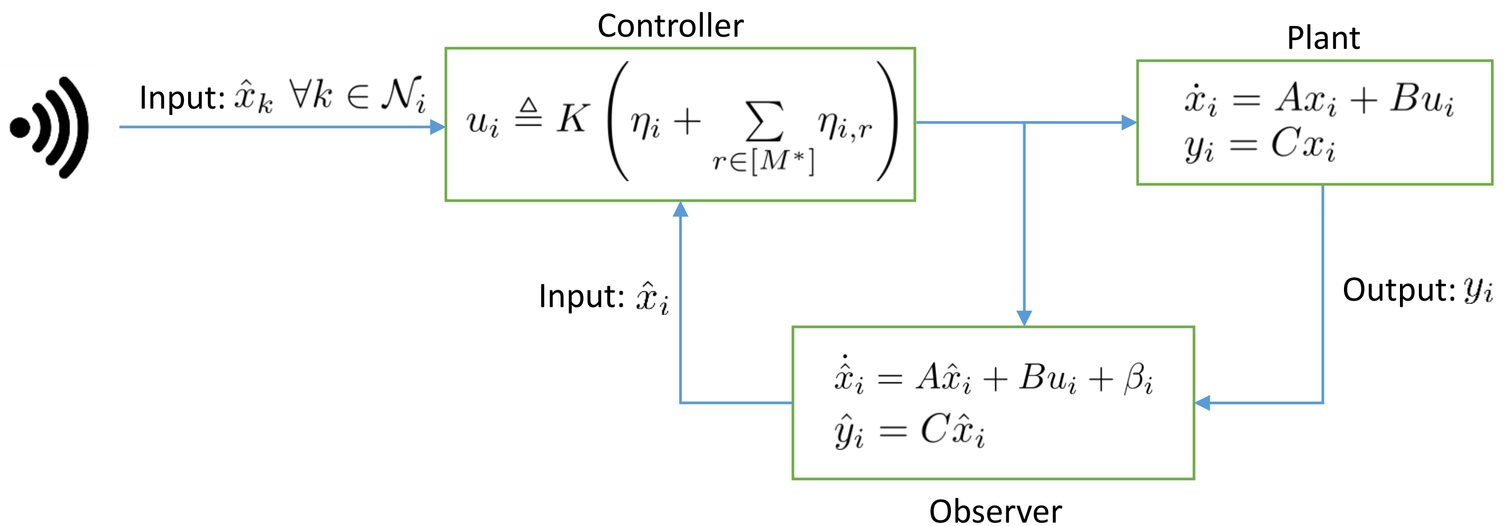
$$\eta_i \triangleq \sum_{k \in \mathcal{N}_i^0} (\hat{x}_k - \hat{x}_i)$$

Inter-cluster r component:

$$\begin{aligned} \dot{\eta}_{i,r} &= 0_n, & \tau_r &\in [0, T_2^r] \\ \eta_{i,r}^+ &= \sum_{k \in \mathcal{N}_i^r} (\hat{x}_k - \hat{x}_i), & \tau_r &= 0. \end{aligned}$$

Control Strategy for Agent i

State Estimation via Output Feedback



Hybrid System

- State Vector: $\xi \triangleq \left[e^\top, \hat{e}^\top, \zeta^\top, \widetilde{\Theta}^\top, \tau^\top, \rho^\top\right]^\top$
- Letting $z = [e^\top, \hat{e}^\top, \zeta^\top, \widetilde{\Theta}^\top]^\top$, the hybrid system is given by

$$\begin{aligned} \dot{\xi} &= f(\xi), \quad && \xi \in C, \\ \xi^+ &\in G(\xi), \quad && \xi \in D. \end{aligned}$$

- Flow Set: $C \triangleq \mathcal{X}$, where $\mathcal{X} \triangleq \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times \mathbb{R}^{nNM^*} \times \mathcal{T} \times \mathcal{R}$
- Flow Map: $f : \mathcal{X} \to \mathcal{X}$, where

$$f(\xi) \triangleq \begin{bmatrix} \mathbf{A}z \\ -1_{M^*+1} \end{bmatrix}, \quad \mathbf{A} \triangleq \begin{bmatrix} \mathbf{A}_{11} & L \otimes BK_1 & \mathbf{0} & \mathbf{A}_{14} \\ \mathbf{0} & \mathbf{A}_{22} & -I_{nN} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{32} & I_N \otimes K_2 C & \mathbf{0} \\ \mathbf{A}_{41} & \mathbf{A}_{42} & -\bar{L}\left(1_{M^*} \otimes I_{nN}\right) & \mathbf{A}_{44} \end{bmatrix},$$

$$\begin{aligned} \mathbf{A}_{11} &\triangleq I_N \otimes A - L \otimes BK_1, \\ \mathbf{A}_{14} &\triangleq -\tfrac{1}{N}\left(L_C \otimes BK_1\right)\left(1_{M^*}^\top \otimes I_{nN}\right), \\ \mathbf{A}_{22} &\triangleq I_N \otimes \left(A - K_2 C\right), \\ \mathbf{A}_{32} &\triangleq -\left(I_N \otimes K_2 C\left(A - K_2 C\right)\right), \\ \mathbf{A}_{41} &\triangleq -\bar{L}\left(1_{M^*} \otimes \left(I_N \otimes A - L \otimes BK_1\right)\right), \\ \mathbf{A}_{42} &\triangleq \bar{L}\left(1_{M^*} \otimes \left(\left(I_N \otimes \left(A - K_2 C\right)\right) - \left(L \otimes BK_1\right)\right)\right), \\ \mathbf{A}_{44} &\triangleq \bar{L}\left(1_{M^*} \otimes \left(I_N \otimes BK_1\right)\left(1_{M^*}^\top \otimes I_{nN}\right)\right). \end{aligned}$$

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Hybrid System

- Jump Set: $D \triangleq \left(\bigcup_{r\in[M^*]} D_r\right) \cup \mathtt{D}$,

  where $D_r \triangleq \{\xi \in \mathcal{X} : \tau_r = 0\}$, $\mathtt{D} \triangleq \{\xi \in \mathcal{X} : \rho = \mathtt{0}\}$.

- Jump Map: $G : \mathcal{X} \rightrightarrows \mathcal{X}$ defined by

$$G(\xi) \triangleq \left\{G_r(\xi) : \xi \in D_r \text{ for } r \in [M^*]\right\} \cup \left\{\mathtt{G}(\xi) : \xi \in \mathtt{D}\right\},$$

$$\mathtt{G}(\xi) \triangleq \left[e^\top, \hat{e}^\top, 0_{nN}^\top, \widetilde{\Theta}^\top, \tau^\top, [T_3, T_4]\right]^\top,$$

$$G_r(\xi) \triangleq \begin{bmatrix} e \\ \hat{e} \\ \zeta \\ \left[\tilde{\theta}_1^\top, \ldots, \tilde{\theta}_{r-1}^\top, 0_{nN}^\top, \tilde{\theta}_{r+1}^\top, \ldots, \tilde{\theta}_{M^*}^\top\right]^\top \\ \left[\tau_1, \ldots, \tau_{r-1}, [T_1^r, T_2^r], \tau_{r+1}, \ldots, \tau_{M^*}\right]^\top \\ \rho \end{bmatrix}.$$

- Attractor:

$$\mathcal{A} \triangleq \left\{\xi \in \mathcal{X} : \ \forall_{i,k\in\mathcal{V}}\ e_i = e_k,\ \forall_{i,k\in\mathcal{V}}\ \hat{e}_i = \hat{e}_k,\ \forall_{i,k\in\mathcal{V}}\ \zeta_i = \zeta_k,\ \forall_{p,q\in[M^*]}\ \tilde{\theta}_p = \tilde{\theta}_q\right\}.$$

Result

Theorem 1: Solutions to our Hybrid System Exponentially Converge to our Attractor

Suppose Assumptions 1 and 2 are satisfied. Given $0 < T_1^i \leq T_2^i$ for all $i \in \mathcal{V}$ and $0 < T_3 \leq T_4$ for all $r \in [M^*]$, the set $\mathcal{A}$ is globally exponentially stable for the hybrid system $\mathcal{H}$ with data (C, f, D, G) if there exists a $\sigma > 0$, gain matrices $K_1 \in \mathbb{R}^{d \times n}$, $K_2 \in \mathbb{R}^{n \times m}$, and symmetric positive definite matrices $P \in \mathbb{R}^{n(N-1) \times n(N-1)}$, $H \in \mathbb{R}^{n(N-1) \times n(N-1)}$, $W \in \mathbb{R}^{n(N-1) \times n(N-1)}$, $Q_r \in \mathbb{R}^{n(N-1) \times n(N-1)}$ for each $r \in [M^*]$ such that

$$\widetilde{\Psi}^\top \mathbf{A}^\top \widetilde{\Psi} R(\rho, \tau) + R(\rho, \tau) \widetilde{\Psi}^\top \mathbf{A} \widetilde{\Psi} + \dot{R}(\rho, \tau) \leq \mathbf{0} \quad \forall (\tau, \rho) \in \mathcal{T} \times \mathcal{R},$$

where $R(\rho, \tau) \triangleq \operatorname{diag}(P, H, W e^{\sigma \rho}, Q_1 e^{\sigma \tau_1}, Q_2 e^{\sigma \tau_2}, \ldots, Q_{M^*} e^{\sigma \tau_{M^*}})$ and $\widetilde{\Psi} = I_{3+M^*} \otimes \Psi \otimes I_n$.

Recall:

$$\mathcal{A} \triangleq \left\{ \xi \in \mathcal{X} : \ \forall_{i,k \in \mathcal{V}} \ e_i = e_k, \ \forall_{i,k \in \mathcal{V}} \ \hat{e}_i = \hat{e}_k, \ \forall_{i,k \in \mathcal{V}} \ \zeta_i = \zeta_k, \ \forall_{p,q \in [M^*]} \ \tilde{\theta}_p = \tilde{\theta}_q \right\}.$$

- State estimation errors are brought into agreement
- Regulation of our solutions into the attractor does not imply our state estimation errors converge to zero for each agent

Theorem 2: State reconstruction is achieved

Simulation

- Consider a C-MAS with 15 agents
- LTI dynamics given by:

$$A = \begin{bmatrix} 0 & 1 \\ -0.65 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0.88 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

- Corresponds to the SS representation of a 1-DOF harmonic oscillator
- Agents in the C-MAS are coupled via a chain with Adjacency matrix

$$A = [a_{ik}] \in \mathbb{R}^{15\times 15} : a_{i+1,i} = a_{i,i+1} = 1 \text{ for all } i \in \mathcal{V} \text{ and } a_{ik} = 0 \text{ otherwise}$$

- The node set is partitioned into three clusters:

$$\mathcal{V} = \{1, 2, ..., 15\}, \mathcal{V}_1 = \{1, 2, 3, 4, 5\}, \mathcal{V}_2 = \{6, 7, 8, 9, 10\}, \mathcal{V}_3 = \{11, 12, 13, 14, 15\}$$

- Inter-Cluster Sets:

$$\mathcal{V}^1 = \{5, 6\}, \mathcal{V}^2 = \{10, 11\}$$

- Inter-Cluster Timer Bounds:

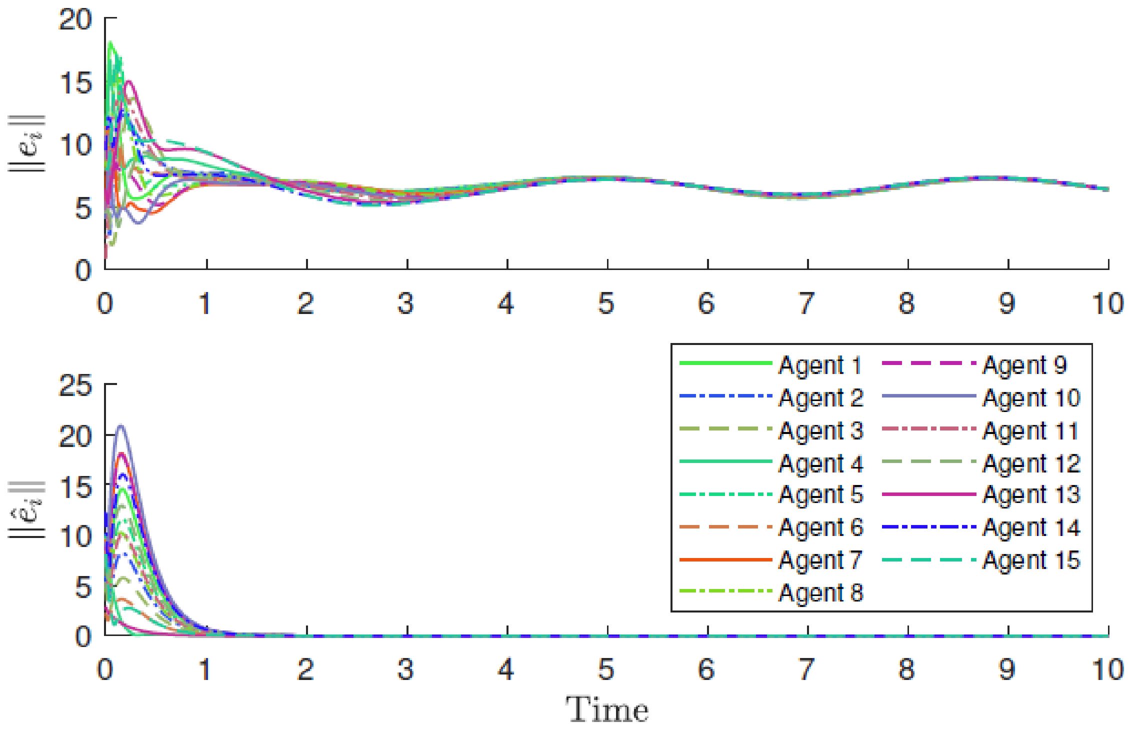
$$T_1^1 = 0.035, T_2^1 = 0.104, T_1^2 = 0.011, T_2^2 = 0.042$$

- Output feedback sensing timers

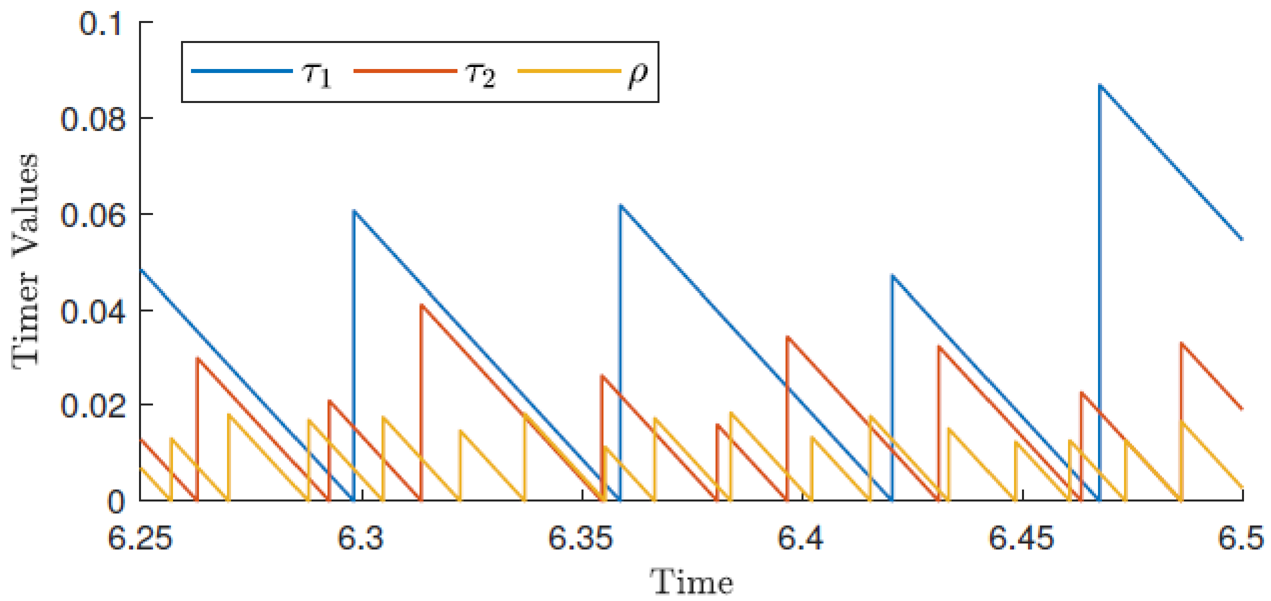
$$T_3 = 0.01, T_4 = 0.02$$

Consensus in the states is achieved

The C-MAS achieves state reconstruction



Results



0.1
0.08
0.06
0.04
0.02
0
τ_1
τ_2
ρ
Timer Values
6.25
6.3
6.35
6.4
6.45
6.5
Time

Conclusion and Future Work

Conclusion

- Achieved the consensus objective over clustered networks with output feedback
- Intermittent and Asynchronous communication with inter-clusters

Future work:

- Intermittent and Asynchronous Output feedback
- Intermittent and Asynchronous communication within each cluster
- Robustness to disturbances and model uncertainty

Questions?