Recent Advances in Safety, Optimization, and Control

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Department Electrical and Computer Engineering
University of California

CoE Review @ Duke University - December 7, 2023

Outline of Recent Results

1. **Safety**
   - Safety Certificates
     ACC23a, CDC23a, CDC23b,
     TAC (provisionally accepted) w/ Warren Dixon
   - Applications of Safety to Security
     CCTA22a and ACC23b
2. **Optimization**
   - Dynamical systems approach
     ACC23c, Optimization journal (almost ready)
     Automatica 2023, ACC23d w/ Matt Hale
   - Optimization with Computational Constraints
     CPSWeek-IoT 23 Workshop
3. **Motion Planning for Hybrid Systems**
   - RRT for feasibility and optimality
     CDC22, CCTA22b, CDC23c, ADHS24 (work in progress)

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     CPSWeek-IoT 23 Workshop
3. **Motion Planning for Hybrid Systems**

New MS student (Ryan Rodriguez) and postdoc (Himadri Basu)

Visited S. Phillips at AFRL/RV

Released new version of Hybrid Equations Toolbox (v3.0) for Matlab

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An Observer-based Switching Algorithm for Safety under Sensor Denial-of-Service Attacks

Santiago J. Leudo, Kunal Garg*, Ricardo G. Sanfelice, and Alvaro A. Cardenas

University of California, Santa Cruz, CA
*Massachusetts Institute of Technology, Cambridge, MA

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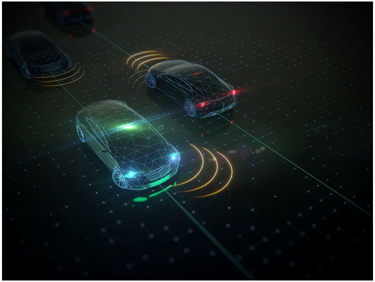
HYBRID SYSTEMS LABORATORY

Security of a CPS from a Control-Theoretic Perspective



- Attacks: Sensor data.

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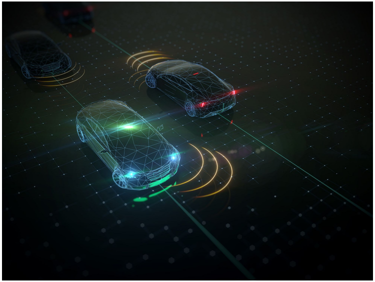

- Attacks: Sensor data.
- Attackers can disable the transmission of signals between devices: a Denial of Service (DoS) attack.

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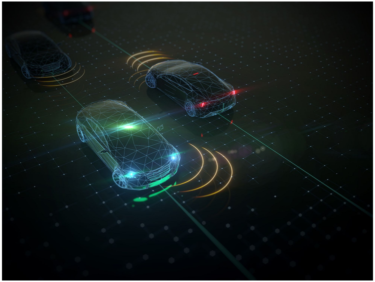
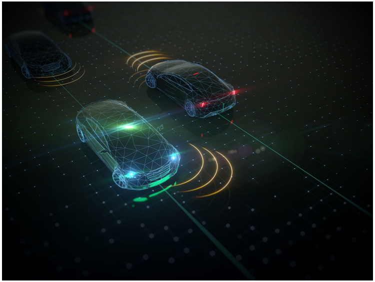


- Attacks: Sensor data.
- Attackers can disable the transmission of signals between devices: a Denial of Service (DoS) attack.
- Potential violation of safety requirements.

Security of a CPS from a Control-Theoretic Perspective



Goal: Keep a system's trajectories in a safe set even under DoS attacks.

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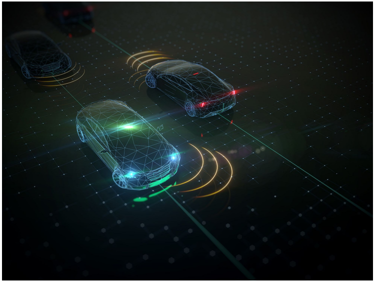

Goal: Keep a system's trajectories in a safe set even under DoS attacks.
Approach: Control scheme to bound the impact of an attack based on the **information available** to guarantee **safety**.

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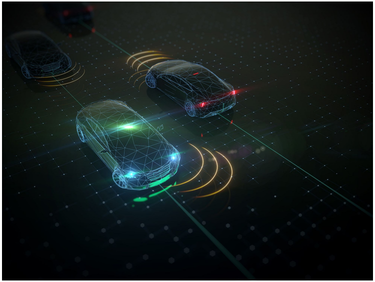

Goal: Keep a system's trajectories in a safe set even under DoS attacks.
Approach: Control scheme to bound the impact of an attack based on the **information available** to guarantee **safety**.
Assumption: **Finite duration** attacks, succeeded by intervals without attacks.

Safety Definitions

Consider the nonlinear system with state x and output y:

$$\mathcal{F}_n : \begin{cases} \dot{x} & = & F(t,x) \\ y & = & H(t,x) \end{cases}$$

- F is the *flow map*
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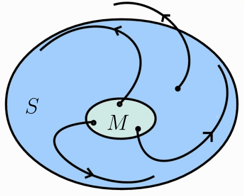

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Definition: (Conditional invariance)

A closed set $S \subset \mathbb{R}^n$ is said to be *conditionally invariant* for $\mathcal{F}_n$ with respect to $M \subset S$ if, for each $x_0 \in M$, any solution to $\mathcal{F}_n$ from x_0 remains in S.

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Definition: (Safety)

The system $\mathcal{F}_n$ is said to be *safe* with respect to (X_0, X_u), with $X_0 \subset \mathbb{R}^n \setminus X_u$, if for each $x_0 \in X_0$, any solution to $\mathcal{F}_n$ from x_0 remains in $\mathbb{R}^n \setminus X_u$.

Safety Definitions

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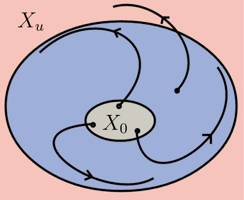

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System Model

Consider the LTI system with state x, input u and output y:

$$\mathcal{F} : \begin{cases} \dot{z} &= Ax + Bu \\ y &= Cx \end{cases}$$

where $C = \begin{bmatrix} \tilde{C} \\ \bar{C} \end{bmatrix}$.

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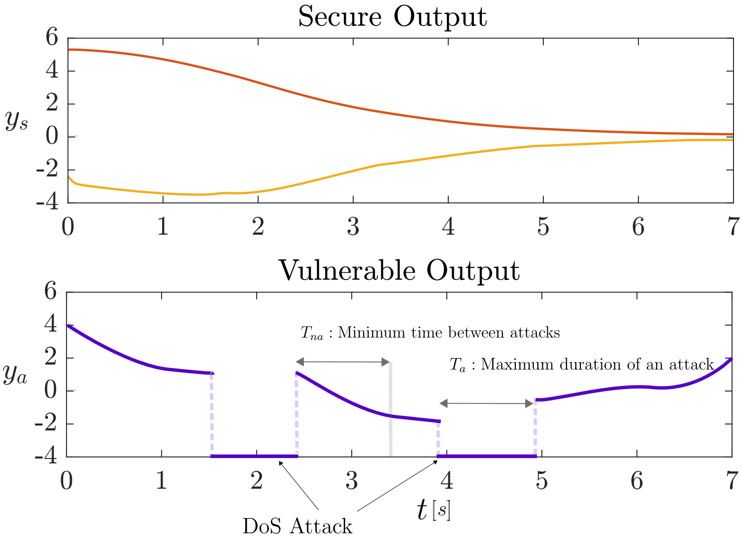
Attack Model (Denial-of-Service (DoS))

The *measured* output: $\bar{y} = \begin{bmatrix} y_s \\ y_a \end{bmatrix}$, where $y_s = \tilde{C}x$, and, along a solution $t \mapsto x(t)$,

$$y_a(t) = \begin{cases} \bar{C}x(t) & \text{if } t \notin \mathcal{T}_a, \\ Y(t, x(t)) & \text{if } t \in \mathcal{T}_a \end{cases}$$

- $\mathcal{T}_a$: the set of times of attack (known)


Secure Output
y_s
6
4
2
0
-2
-4
0
1
2
3
4
5
6
7
Vulnerable Output
y_a
6
4
2
0
-2
-4
T_{na} : Minimum time between attacks
T_a : Maximum duration of an attack
0
1
2
3
4
5
6
7
$t\,[s]$
DoS Attack

Design an algorithm to render the set S conditionally invariant for the system $\mathcal{F}$ with respect to the set X_0 using output measurements only.

Safety Problem

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Solution Approach:

Design an observer-based feedback law that induces conditional invariance of S with respect to X_0 using the measured output $\bar{y}$.

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- System output **under attack** $\rightarrow$ the observer uses the **non-attacked output** components.

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- System output **under attack** $\rightarrow$ the observer uses the **non-attacked output** components.
- System output **not attacked** $\rightarrow$ the observer uses the **complete output** vector.

Solution Approach

1. Design functions g_1 and g_2 for the observer

$$\dot{\hat{x}} = \begin{cases} A\hat{x} + Bu + g_1(Cx, C\hat{x}) & \text{if} \quad t \notin \mathcal{T}_a, \\ A\hat{x} + Bu + g_2(\tilde{C}x, C\hat{x}) & \text{if} \quad t \in \mathcal{T}_a, \end{cases}$$

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- If $x(0) \in X_0, \hat{x}(0) \in \hat{X}_0(x_0) \Rightarrow x(t) \in S$ during attacks.

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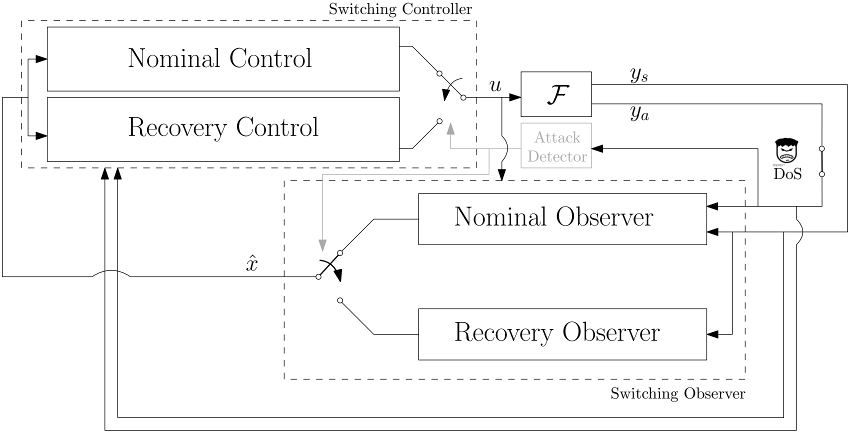
- If $x(0) \in S, \hat{x}(0) \in \tilde{X} \Rightarrow x(t) \in S$ when no attacks and belongs to X_0 at the beginning of the next attack.

Solution Approach



Attack detector [Phillips et al - CDC 17]

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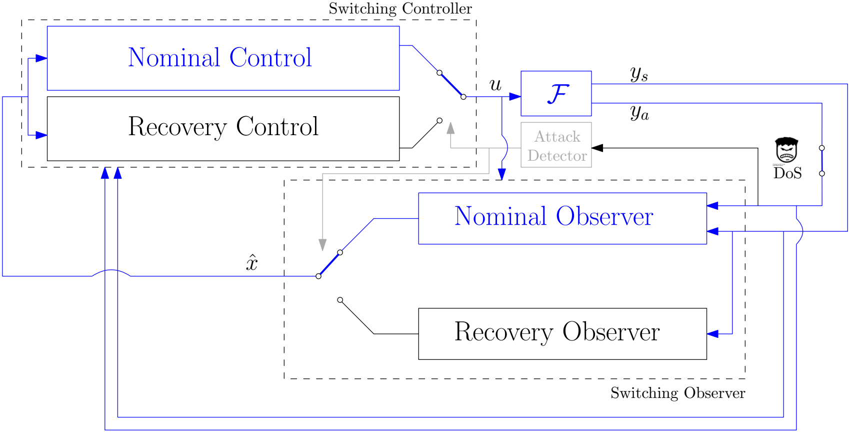

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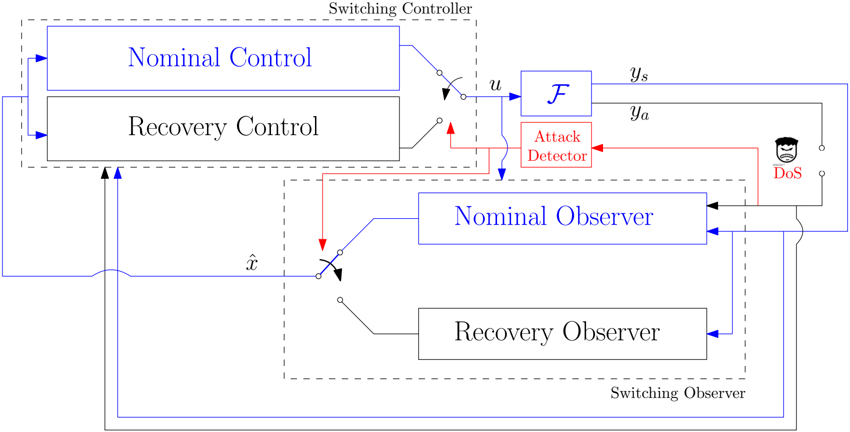

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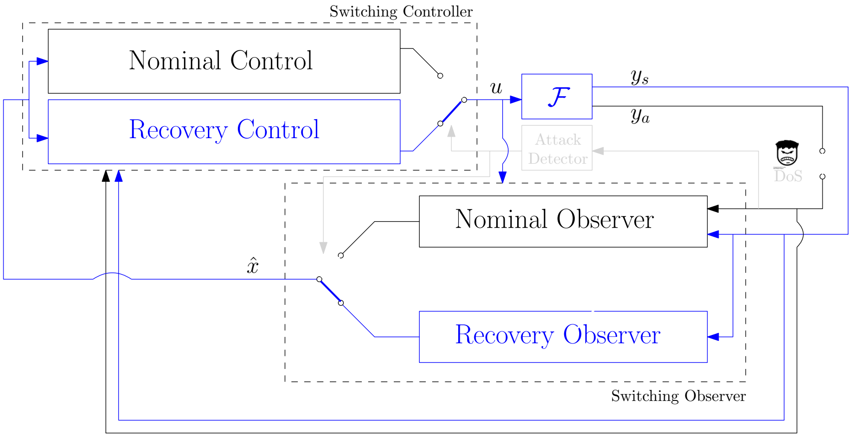

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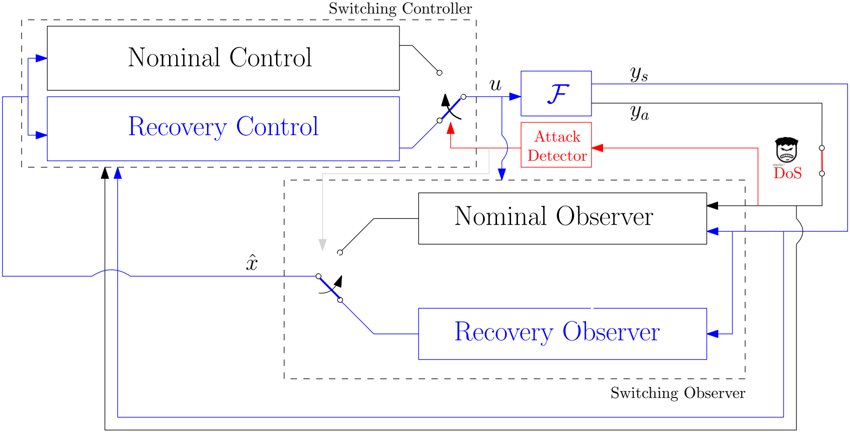

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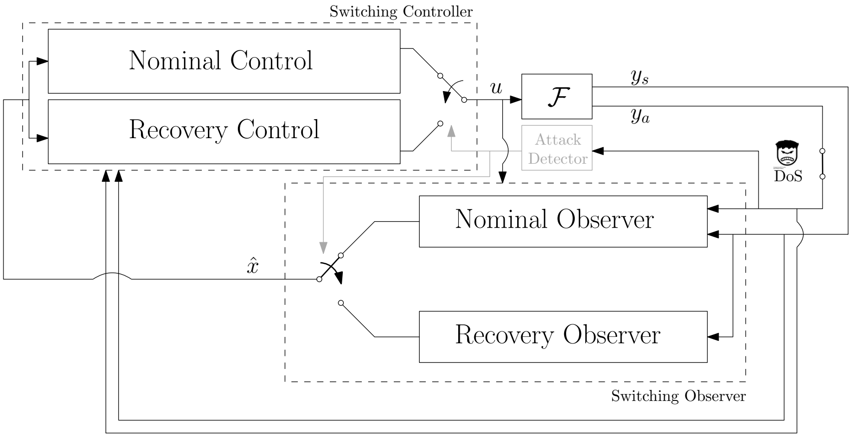

Switching Observer Design

Reconstruction of the state with potential unobservable modes when under attack for feedback control design. The switching observer

$$\dot{\hat{x}} = \begin{cases} A\hat{x} + Bu + L(Cx - C\hat{x}) & \text{if} \quad t \notin \mathcal{T}_a, \\ A\hat{x} + Bu + \tilde{L}(\tilde{C}x - \tilde{C}\hat{x}) & \text{if} \quad t \in \mathcal{T}_a, \end{cases}$$

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Basic Assumptions

- The pair (A, B) is controllable and the pair (C, A) is detectable.
- We design L so that $A - LC$ has all its eigenvalues in the open left-half plane.
- We design $\tilde{L}$ so that $A - \tilde{L}\tilde{C}$ has as many eigenvalues (but not necessarily all of them) as possible in the open left-half plane.

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Define the estimation error as $e = x - \hat{x}$ with

$$\dot{e} = \begin{cases} (A - LC)e & \text{if} \quad t \notin \mathcal{T}_a, \\ (A - \tilde{L}\tilde{C})e & \text{if} \quad t \in \mathcal{T}_a. \end{cases}$$

Estimation Error Bounds

Lemma 1. Under No Attacks

For given $T_{na}, \bar{e}_0 > 0$, if at the end of an attack the norm of **the estimation error** $e = x - \hat{x}$ is bounded by $\bar{e}_0$, then

$$|e(t)| \leq \gamma_1(t)\bar{e}_0 \qquad \forall t \in [0, T_{na}]$$

where $\gamma_1(t) := c_1 \exp\left(-\bar{\lambda}_1 t\right)$ with $c_1, \bar{\lambda}_1 > 0$.

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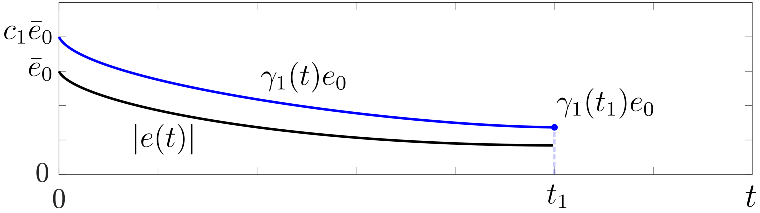

Lemma 2. Under Attacks

For given $T_a, \bar{e}_0 > 0$, if at the beginning of an attack the norm of **the estimation error** e is bounded by $\bar{e}_0$, then

$$|e(t)| \leq \gamma_2(T_a)\bar{e}_0 \qquad \forall t \in [0, T_a]$$

where $\gamma_2(T_a) := \max_{t \in [0, T_a]} \hat{c}_1 \exp\left(-\hat{\lambda}_1 t\right) + \hat{c}_2 \exp\left(\hat{\lambda}_2 t\right)$

with $\hat{c}_1, \hat{\lambda}_1, \hat{c}_2, \hat{\lambda}_2 > 0$.

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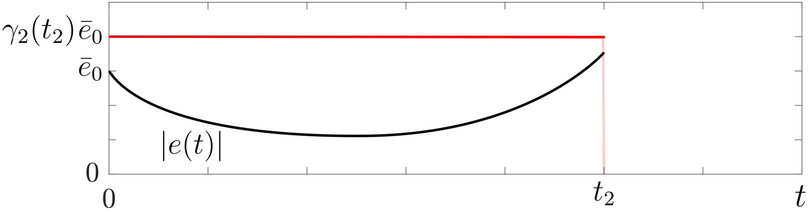

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Global Bound on Estimation Error

Theorem 1.

Given $\bar{E}, T_{na}, T_a > 0$, if the initial estimation error $e(0)$ satisfies $|e(0)| \leq \bar{E}$ and $\gamma_1(T_{na})\gamma_2(T_a) \leq 1$, then

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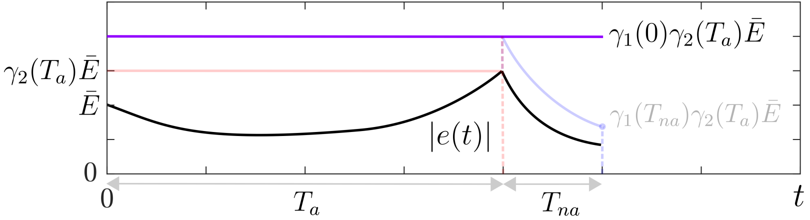

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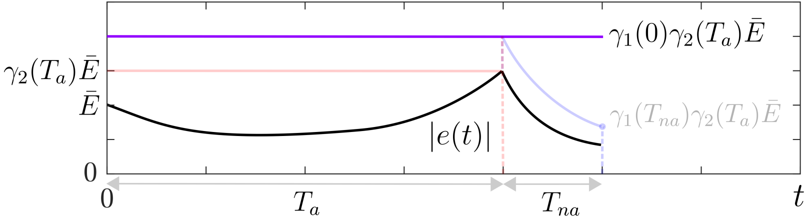
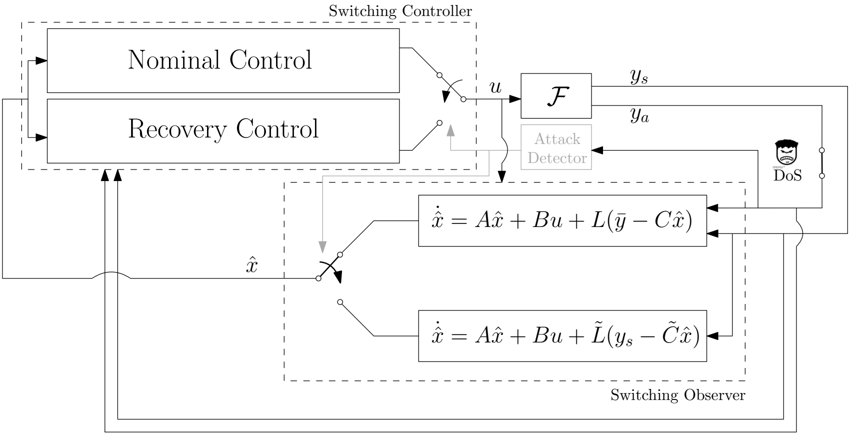
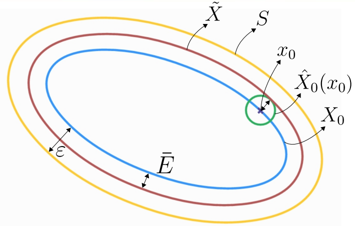


Using the proposed observer, **the norm of the error always remains bounded**.

Switched Observer

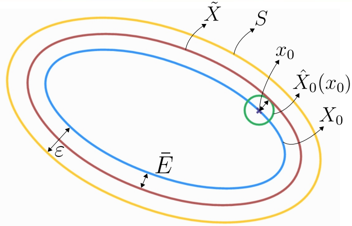




Construction of Sets of Initial Conditions

Pick $\varepsilon > (1 + \gamma_1(0)\gamma_2(T_a))\bar{E}$

- Set of initial states: $X_0 := S \setminus (\partial S + \varepsilon\mathbb{B})$
- Set of initial estimates: $\hat{X}_0(x_0) := x_0 + \bar{E}$
- Allowed initial estimates: $\tilde{X} := X_0 + \bar{E}\mathbb{B}$



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With **bounded estimation error** and $x(0) \in X_0, \hat{x}(0) \in \hat{X}_0(x_0)$:

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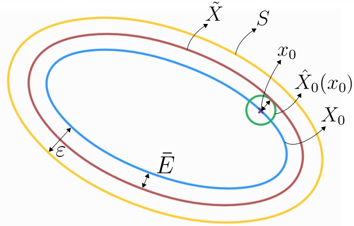

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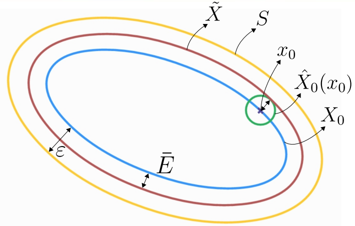

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- The sets X_0 and $\hat{X}_0$ are defined such that the $|e(0)| \leq \bar{E}$.
- Forward invariance of $\tilde{X}$ for the observer implies conditional invariance of the set S for $\mathcal{F}$ with respect to X_0.

QP-based Feedback Law Synthesis

Control barrier function (CBF)-based approach for control design.

Control Objective

Enforce the estimate $\hat{x}$ in the set $\tilde{X}$ to guarantee safety of S.

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Sufficient: Design an observer-based feedback law κ such that for each $\hat{x}(0) \in \bar{X}$, the estimate $\hat{x}(t) \in \bar{X} \subset \tilde{X}$, for all $t \geq 0$.

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CBF Conditions

Under no attacks:

$$\frac{\partial}{\partial \hat{x}} h(\hat{x}(t)) \left(A\hat{x}(t) + B\kappa_1(\hat{x}(t), \bar{y}(t)) + L(\bar{y}(t) - C\hat{x}(t))\right) \leq \alpha_1(-h(\hat{x}(t))),$$

Under attack:

$$\frac{\partial}{\partial \hat{x}} h(\hat{x}(t)) \left(A\hat{x}(t) + B\kappa_2(\hat{x}(t), \bar{y}(t)) + \tilde{L}(\bar{y}_s(t) - \tilde{C}\hat{x}(t))\right) \leq \alpha_2(-h(\hat{x}(t))).$$

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Quadratic Programming (QP) Formulation to Compute Input u

Synthesize the control input via solving:

- For each $\hat{x} \in \bar{X}$ and $\bar{y}$ such that $x \in S$ when there is no attack:

$$\min_{(v,\eta)} \quad \frac{1}{2}|v - K\hat{x}|^2 + \frac{1}{2}\eta^2$$

$$\text{s.t.} \qquad \frac{\partial}{\partial \hat{x}} h(\hat{x}) \left(A\hat{x} + Bv + L(\bar{y} - C\hat{x})\right) \leq -\eta h(\hat{x}),$$

- For each $\hat{x} \in \bar{X}$ and $y_s = \tilde{C}x$ such that $x \in S$ when under attack:

$$\min_{(v_s,\zeta)} \quad \frac{1}{2}|v_s - K\hat{x}|^2 + \frac{1}{2}\zeta^2$$

$$\text{s.t.} \qquad \frac{\partial}{\partial \hat{x}} h(\hat{x}) \left(A\hat{x} + Bv_s + \tilde{L}(y_s - \tilde{C}\hat{x})\right) \leq -\zeta h(\hat{x}).$$

where K is the optimal LQR gain for the pair (A, B).

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Theorem 2. Main Result

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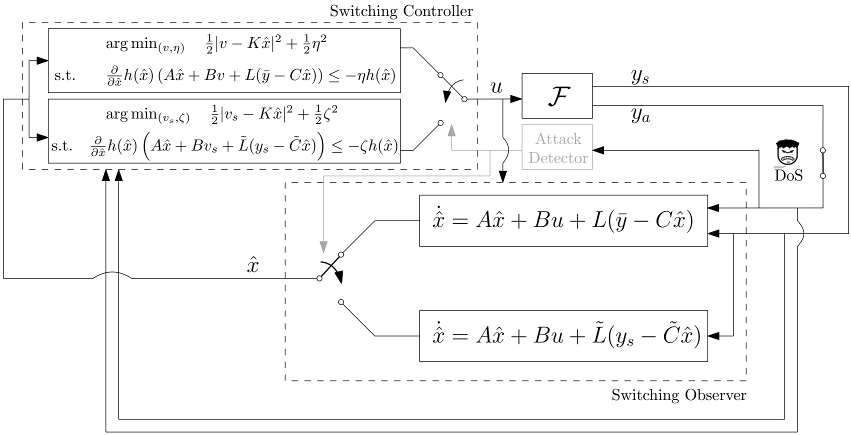
- Feasibility of the QPs with proper state initialization renders the estimate $\hat{x}(t) \in \bar{X}$ at all times.
- Then **the state x remains in S at all times.**

Solution Scheme



Integrator with Nondetectable Modes under Attack

System $\mathcal{F}$ with state $x = (x_1, x_2) \in \mathbb{R}^2$, input $u \in \mathbb{R}$, and dynamics

$$\begin{aligned} \dot{x}_1 &= \frac{x_1}{2} + x_2 \\ \dot{x}_2 &= u \\ y &= (x_1, x_2) \end{aligned}$$

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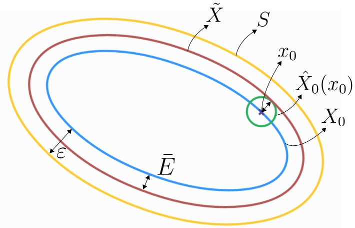
- $y_a = x_1$ is only available when there are no attacks.
- Attacks of max. $T_a = 1.6$s. No attacks for min. $T_{na} = 0.05$s.



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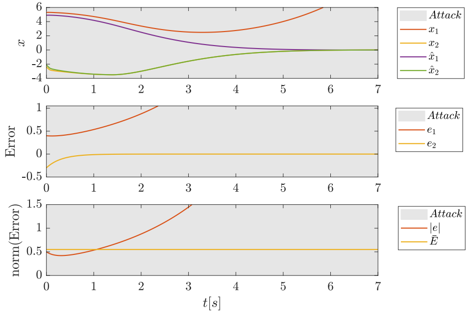

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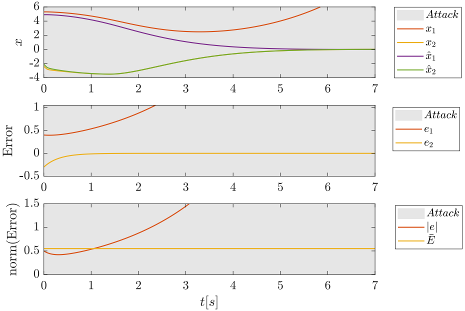

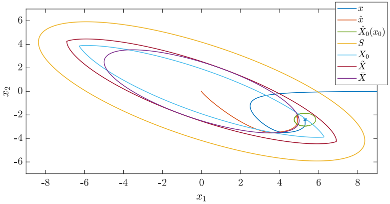

Numerical Example

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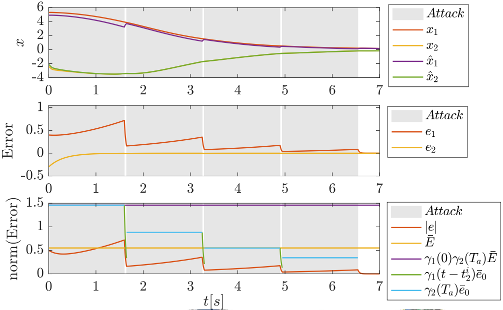

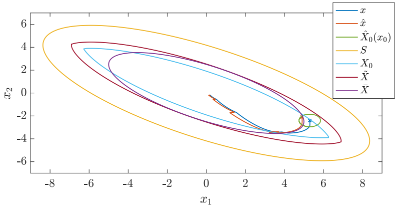

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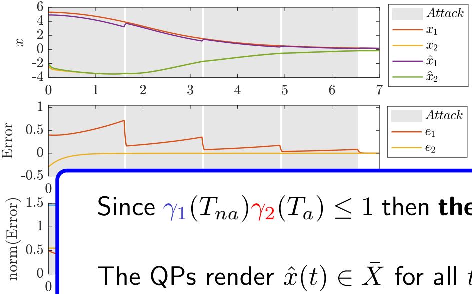

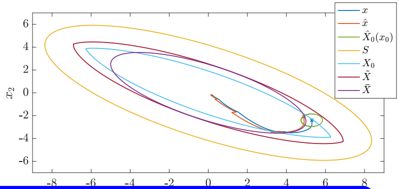

Since $\gamma_1(T_{na})\gamma_2(T_a) \leq 1$ then **the error is bounded at all times.**

The QPs render $\hat{x}(t) \in \bar{X}$ for all t, then $x(t) \in S$ for all t.

Conclusion

- **Switched controller** design with a **switched observer** that ensures a LTI system to recover *safely* from finite-time DoS attacks in some of the system outputs.
- Conditional invariance of a set with respect to a subset of initial conditions by employing a **barrier function** approach and **bounding the estimation error at all times**.

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Future Work

- Finite-time observer and tighter bound to relax the conservatism.

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