Probabilistic Connectivity of Random Graphs and Their Unions

Matthew Hale

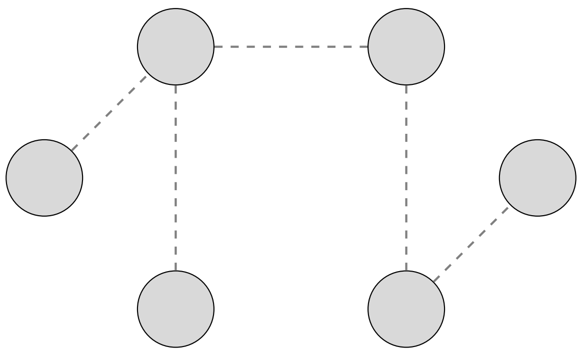
Department of Mechanical and Aerospace Engineering
University of Florida

Joint work with Beth Bjorkman, Thomas Lamkin, Ben Robinson, and Craig Thompson

AFOSR Center of Excellence Review
April 14, 2020

Why Study Random Graphs?

1. Adversaries can jam communications, which makes them unpredictable
2. Congested bandwidth makes information flow intermittent
3. Randomness encodes the lack of control over these events

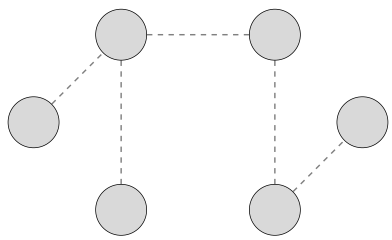


Fundamental Goal

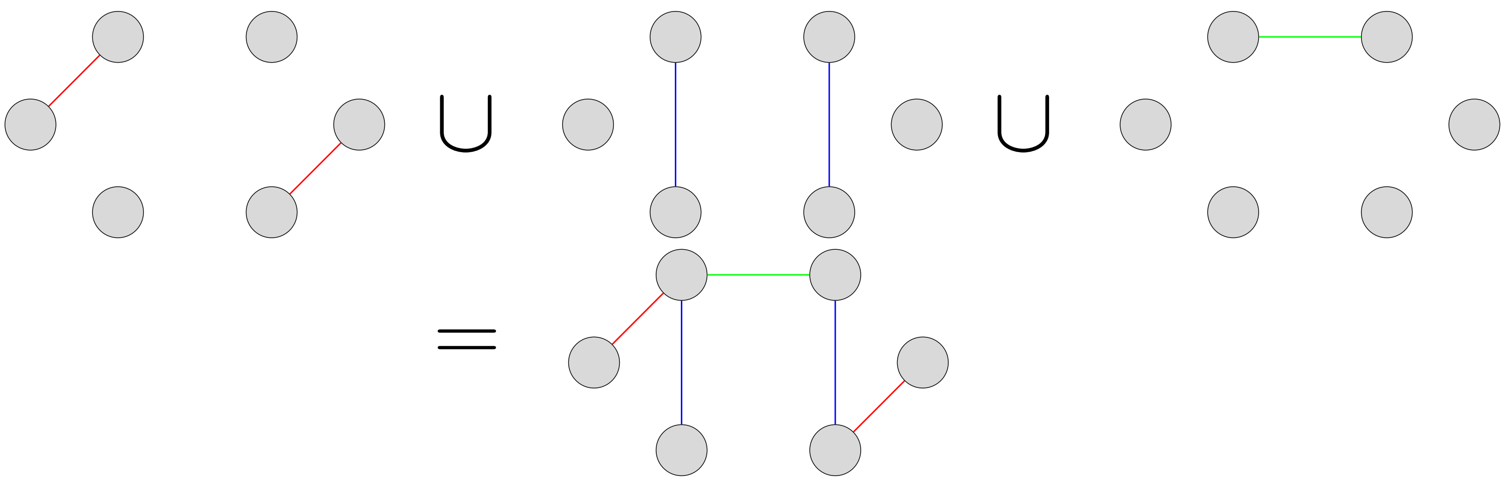
Understand when "enough" information flows for a team of autonomous agents to collaborate.

Time-Varying Graphs are Common

- With autonomous agents, communication graphs may look like



- No single graph is connected, but their union is



How to Make Connected Unions of Graphs?

Common Assumption

There exists an N such that $\bigcup_{k=t+1}^{t+N} G(k)$ is connected for any t.

- This lets autonomous agents
  1. Rendezvous
  2. Assemble formations
  3. Solve optimization problems
- The value of N tells us how quickly we do these, relates the flow of information to performance

Mathematical Question

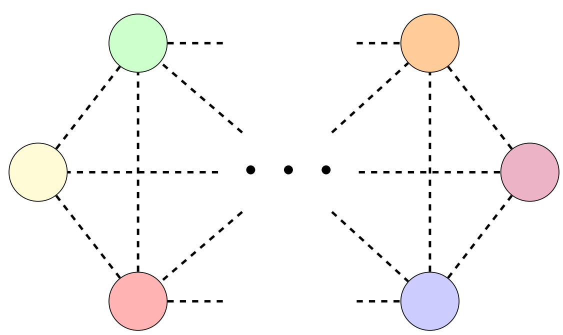
Q: What is the probability that a specific union is connected?

A: $\mathbb{P}\left[\bigcup_{k=t+1}^{t+N} G(k) \text{ connected}\right] \geq f(\text{graph parameters})$

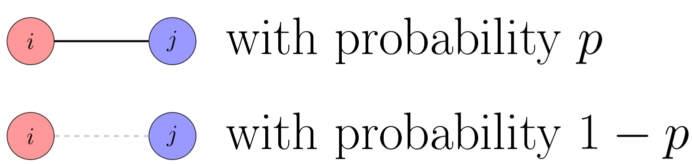
UF UNIVERSITY of FLORIDA Duke UNIVERSITY TEXAS The University of Texas at Austin UC SANTA CRUZ

Erdős-Rényi Graphs

- Defined by parameters $n \in \mathbb{N}$ and $p \in (0, 1)$
- Graphs are on n nodes:

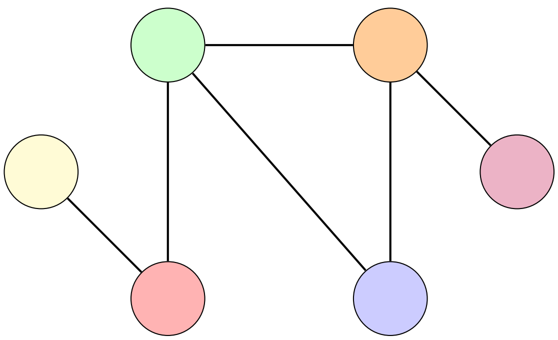


- Each edge appears with probability p:



New Analyses Required for Multi-Agent Systems

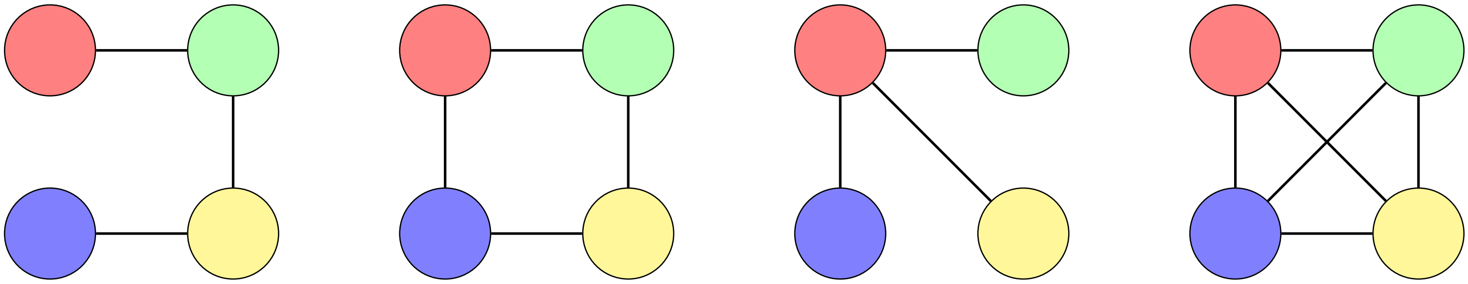
- We require innovations beyond the existing literature:
  - Common to take #nodes $\rightarrow \infty$, but we have small networks
  - The standard Erdős-Rényi model includes all edges, but we won't have that



  - We also avoid assuming $p \sim \frac{\log n}{n}$ or $p = f(n)$ because p is outside our control

We Study Graphs using Algebraic Connectivity

- Define $L = D - A$, and denote its i^{th} eigenvalue by $\lambda_i(L)$
- A classic result says G is connected if and only if $\lambda_2(L) > 0$
- What is
$$\mathbb{P}\left[\lambda_2\left(L\left(\bigcup_{k=t+1}^{t+N} G(k)\right)\right) > 0\right]?$$
- We don't prioritize any graph

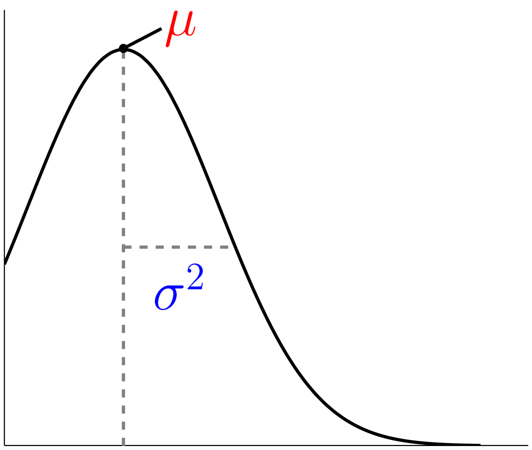


UF UNIVERSITY of FLORIDA | Duke UNIVERSITY | TEXAS The University of Texas at Austin | UC SANTA CRUZ

Working on Question 2

- Applying the Paley-Zygmund inequality, we find

$$\mathbb{P}\left[\lambda_2(L) > 0\right] \geq \frac{\mathbb{E}[\lambda_2(L)]^2}{\mathbb{E}[\lambda_2^2(L)]}$$



- Plan:
  1. Upper bound $\mathbb{E}[\lambda_2^2(L)]$
  2. Lower bound $\mathbb{E}[\lambda_2(L)]^2$

Bounding $\mathbb{E}[\lambda_2^2(L)]$

- Let ℓ be a uniformly sampled positive eigenvalue of L
- Then $\mathbb{E}[\lambda_2^2(L)] \leq \mathbb{E}[\ell^2]$
- Because of how we sample,

$$\mathbb{E}[\ell^2] = \frac{1}{n-1}\text{tr}\left(\mathbb{E}[L^2]\right)$$

Lemma: Upper bound on $\mathbb{E}[\lambda_2^2(L)]$

- For m total possible edges, edge probability p, d_i the possible degree of node i: a union of N random graphs has

$$\mathbb{E}[\lambda_2^2(L)] \leq \frac{4m\big(1-(1-p)^N\big) - 2m\big(1-(1-p)^N\big)^2 + \big(1-(1-p)^N\big)\sum_{i=1}^n d_i^2}{n-1}$$

UF UNIVERSITY of FLORIDA · Duke UNIVERSITY · TEXAS The University of Texas at Austin · UC SANTA CRUZ

Bounding $\mathbb{E}[\lambda_2(L)]^2$

- Now take n uniformly sampled eigenvalues. Let ℓ_n be the smallest
- Then
$$\mathbb{E}[\lambda_2(L)] \geq \left(1 - \left(\frac{n-2}{n-1}\right)^{n-1}\right)^{-1} \left[\mathbb{E}[\ell_n] - \frac{2m\left(1-(1-p)^N\right)}{n-2}\left(\frac{n-2}{n-1}\right)^{n}\right]$$

Lemma: Lower bound on $\mathbb{E}[\lambda_2(L)]$

- For n nodes, edge probability p, m total possible edges, agent i with possible degree d_i: a union of N random graphs has $\hat{p}(N) = 1-(1-p)^N$ and

$$\mathbb{E}[\lambda_2] \geq \max\Bigg\{0, \left(1 - \left(\frac{n-2}{n-1}\right)^{n-1}\right)^{-1} \Bigg[\frac{2m\hat{p}(N)}{n-1}\left(1 - \left(\frac{n-2}{n-1}\right)^{n-1}\right) - \frac{1}{n-1}\left(2m\hat{p}(N)(n-1)(2-\hat{p}(N)) + \hat{p}(N)^2(n-1)\sum_{i=1}^{n} d_i^2 - 4m^2\hat{p}(N)^2\right)^{1/2}\Bigg]\Bigg\}$$

Main Results: Connected Random Graphs

Theorem: Random Graphs

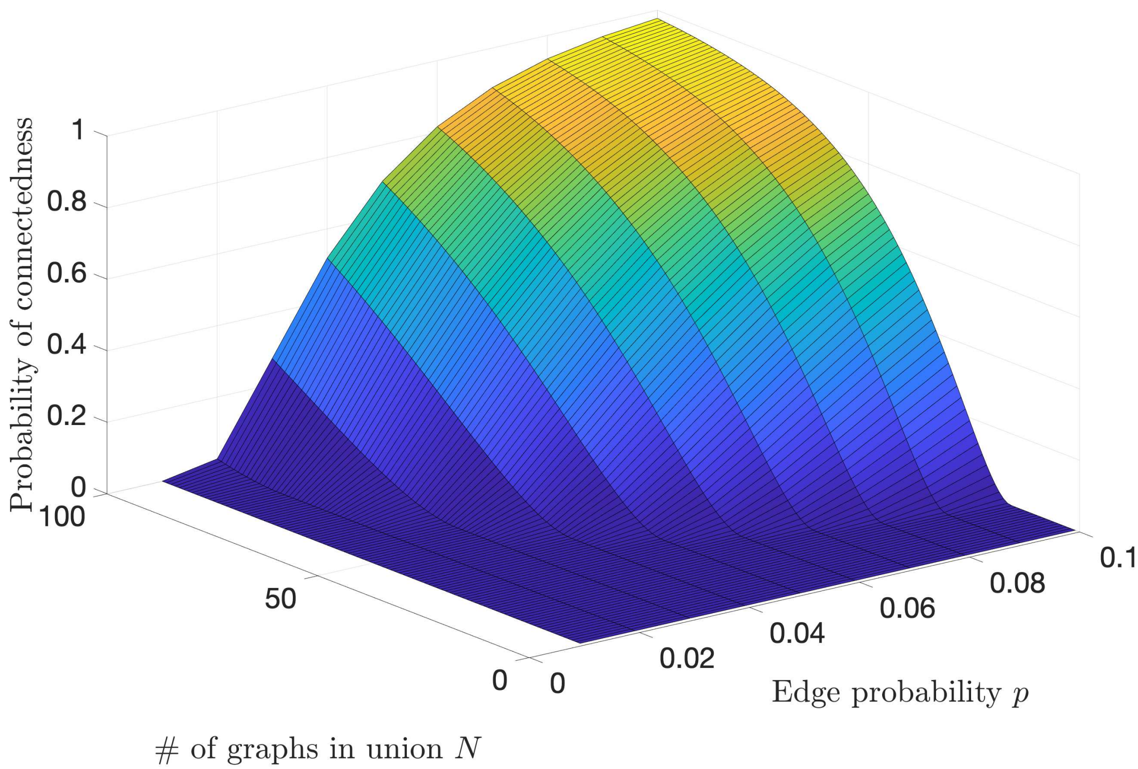
For n nodes, m possible edges, probability p, agent i's degree d_i: a union of N random graphs has $\hat{p}(N) = 1 - (1-p)^N$ and

$$\mathbb{P}[\lambda_2(L) > 0] \geq \frac{\left(\max\left\{0, 2m\hat{p}(N)\left(1 - \left(\frac{n-2}{n-1}\right)^{n-1}\right) - S\big(m, n, \hat{p}(N), \{d_i\}_{i\in[n]}\big)\sqrt{n-1}\right\}\right)^2}{(n-1)\left(1 - \left(\frac{n-2}{n-1}\right)^{n-1}\right)^2 \left(4m\hat{p}(N) - 2m\hat{p}(N)^2 + \hat{p}(N)^2 \sum_{i=1}^n d_i^2\right)},$$

$$S\big(m, n, \hat{p}(N), \{d_i\}_{i\in[n]}\big) = \sqrt{2m\hat{p}(N)(n-1)(2-\hat{p}(N)) + \hat{p}(N)^2(n-1)\sum_{i=1}^n d_i^2 - 4m^2\hat{p}(N)^2}$$

Numerical Results

- Consider $n = 20$ agents, vary p and N



Next Steps

- Formalize tradeoffs between locomotion, communication and performance:
  - When is it worth it to move to improve comms?
  - What balance of communication and motion best uses limited energy?
- Assess vulnerability to attacks on comms:
  - How bad can comms get before performance is unacceptable?
  - How close are we to failing to complete our task?

Thank you