Towards a Compositional Framework for Hybrid Differential Inclusions

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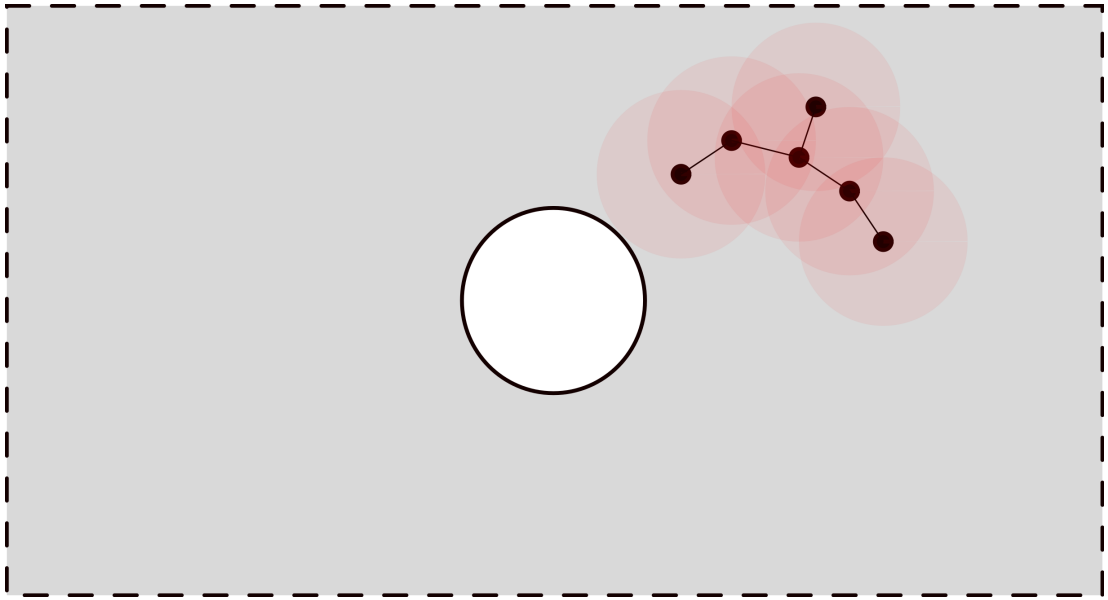
April 14, 2020

Motivation

Challenge: Autonomous generation of complex distributed cooperative behaviors requires reasoning over very large combinatorial structures.



For example, in networks where comms are constrained by distance,

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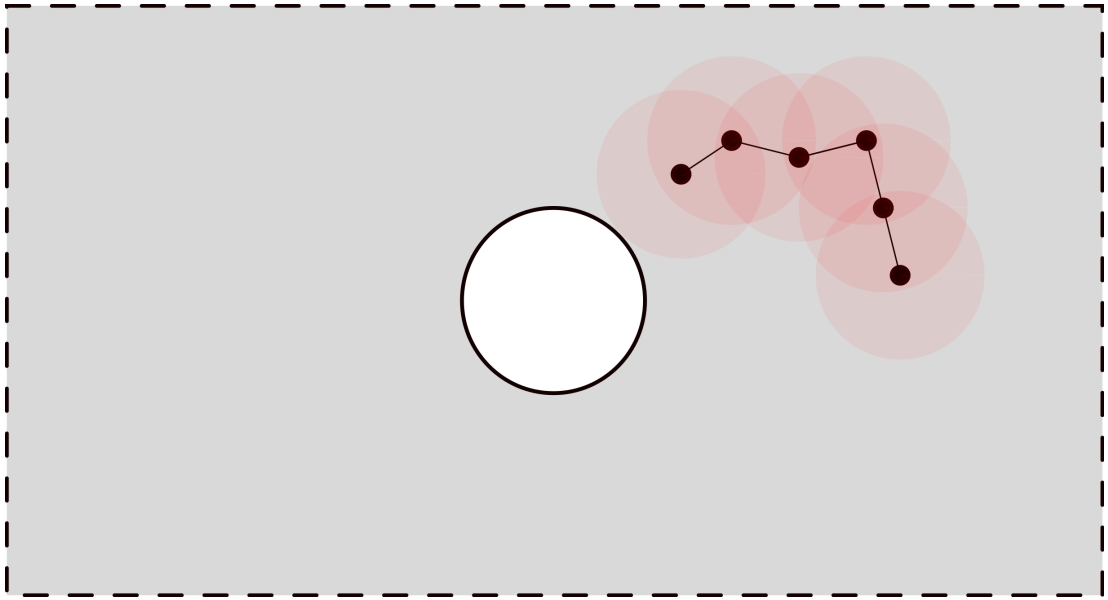

Switching between comms structures (e.g. spanning trees) is useful.

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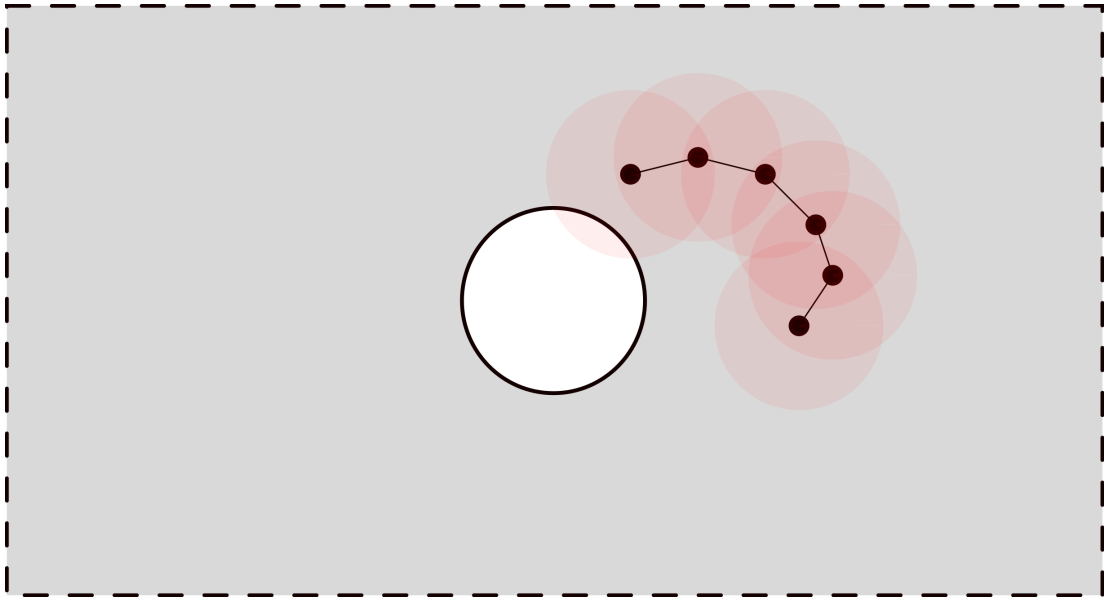

Coordinated motion under a fixed conroller. . .

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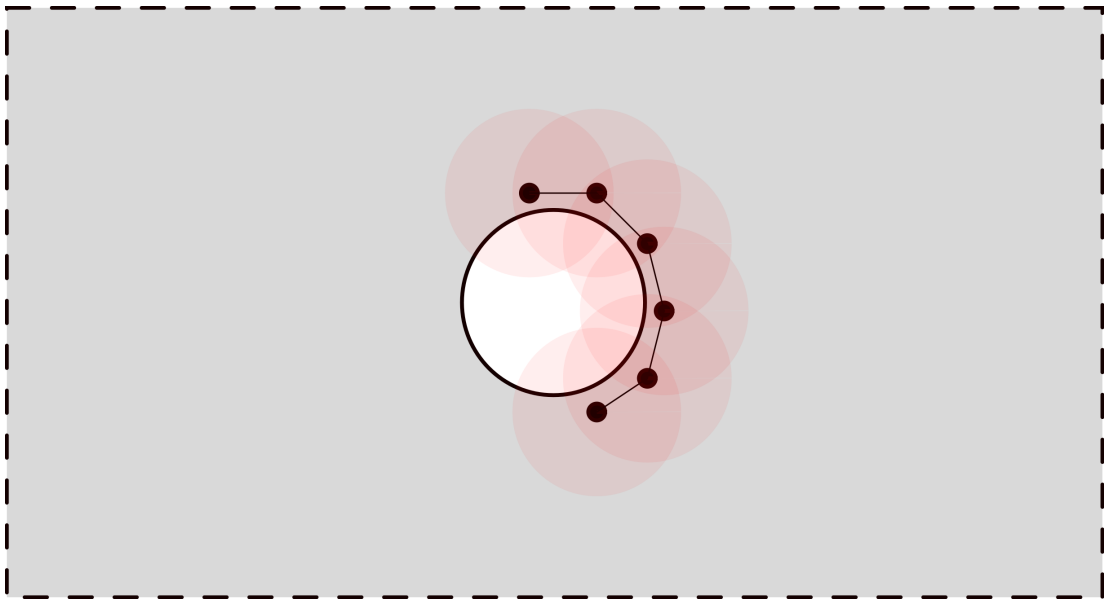

. . . may run into obstacles. . .

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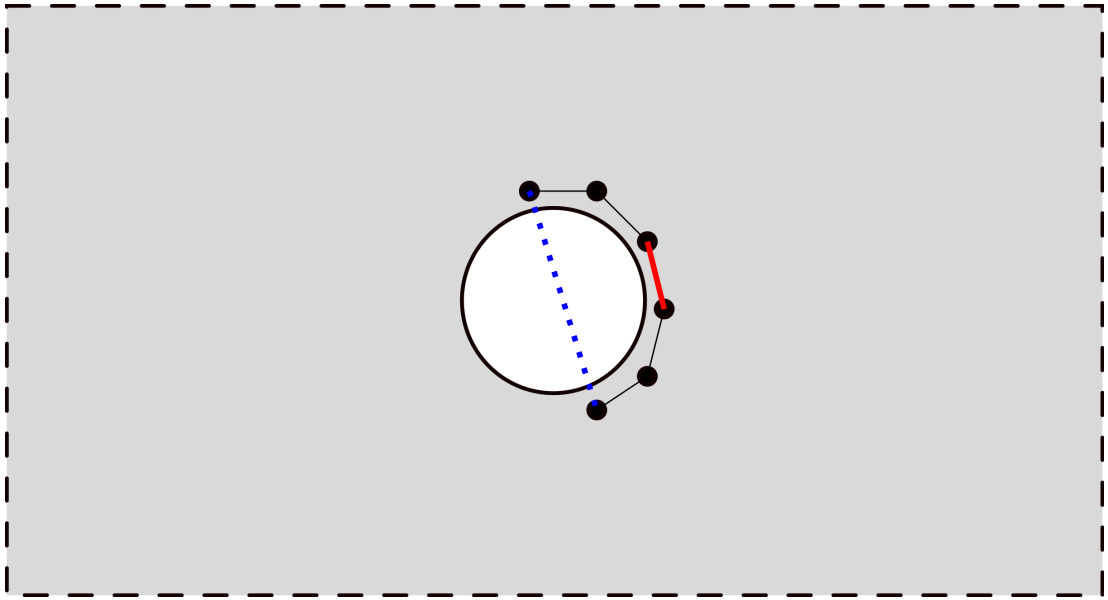

. . . suggesting a reassessment of the comms structure. . .

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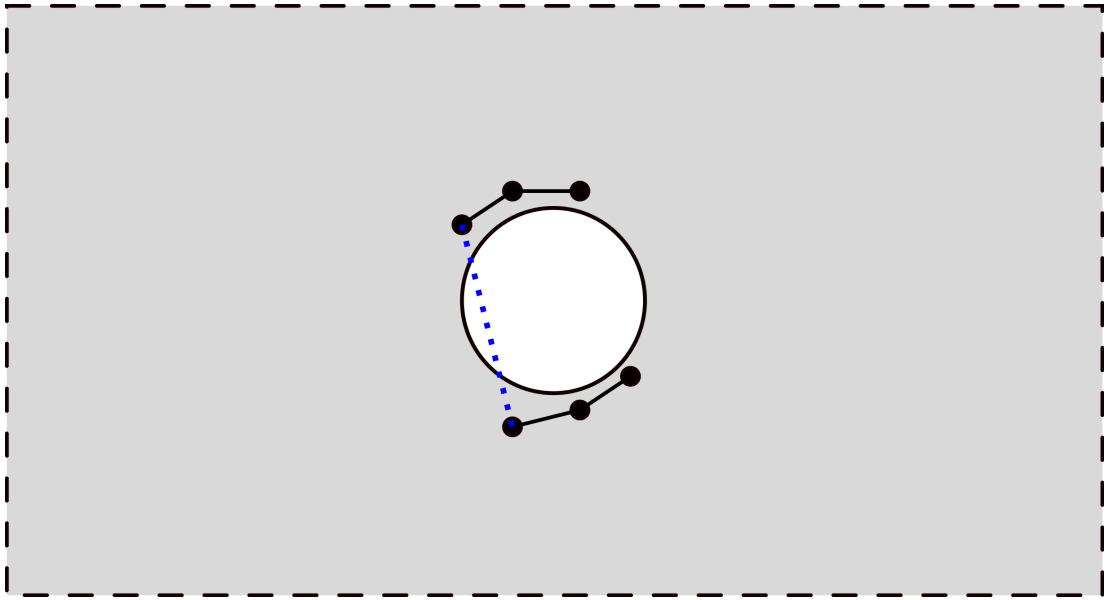

. . . including temporary disconnects with the aim of reconnecting soon thereafter. . .

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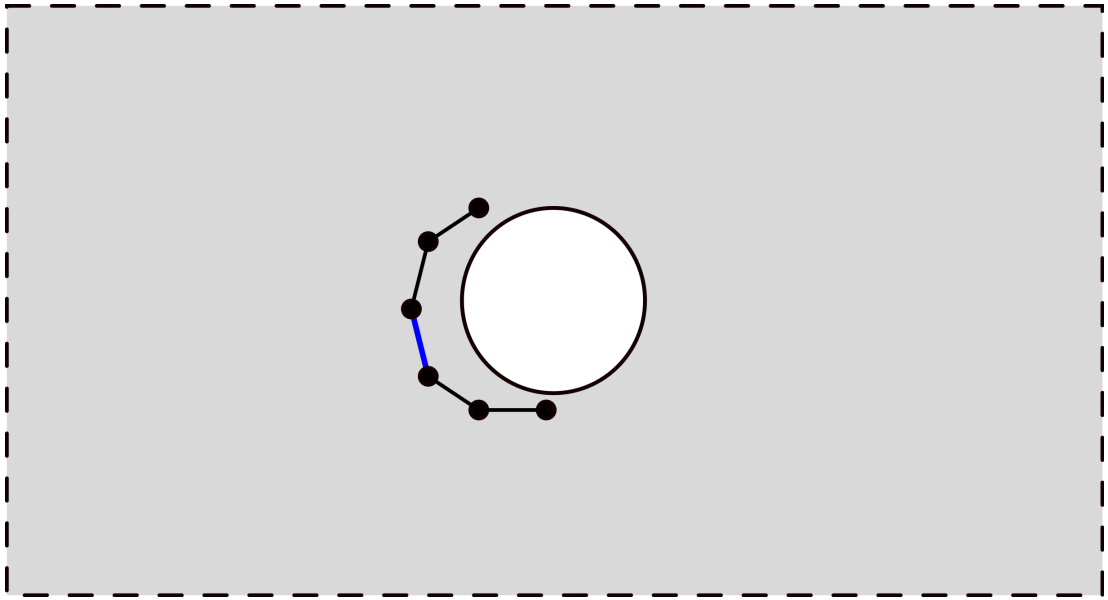

. . . using a different connectivity structure.

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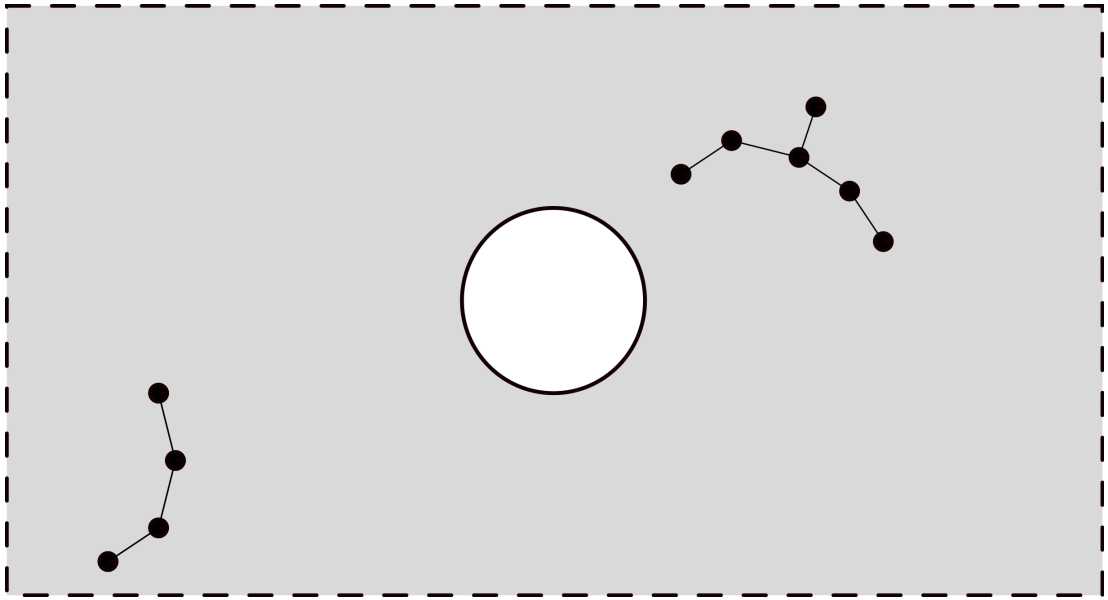

In the presence of additional resources. . .

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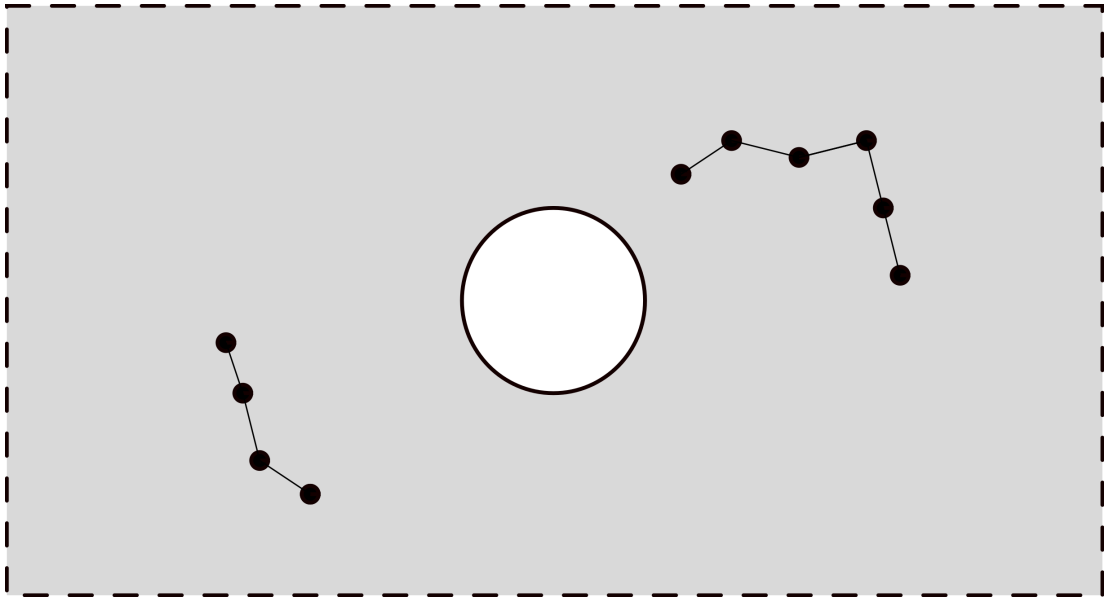

. . . a reactive control paradigm may provide alternative solutions. . .

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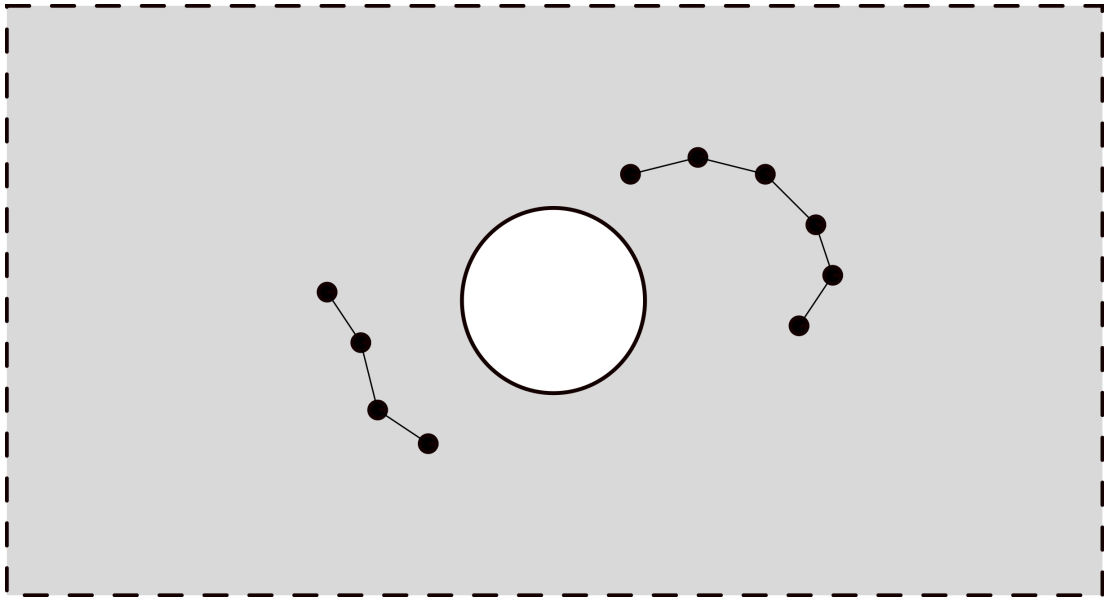

[agents move according to original plans]

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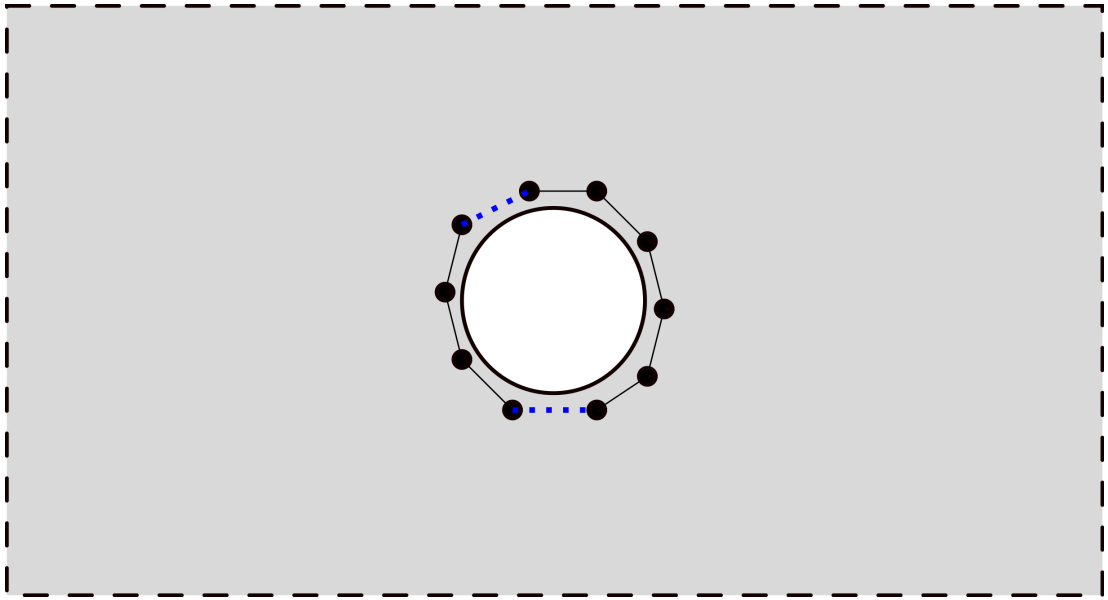

[rendevous generates new comms connections]

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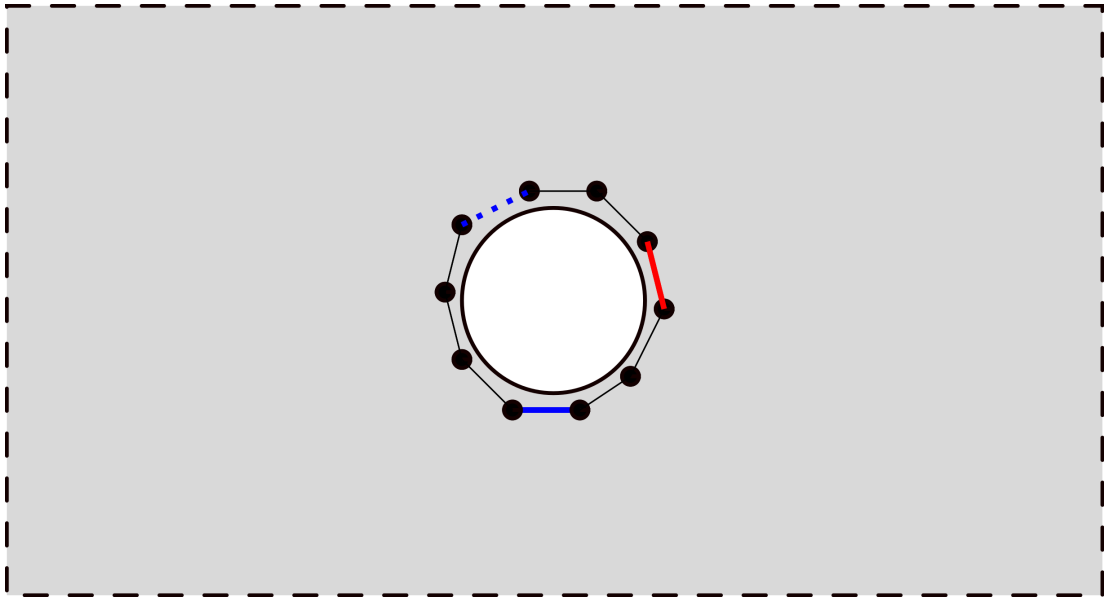

[less risky strategy becomes available]

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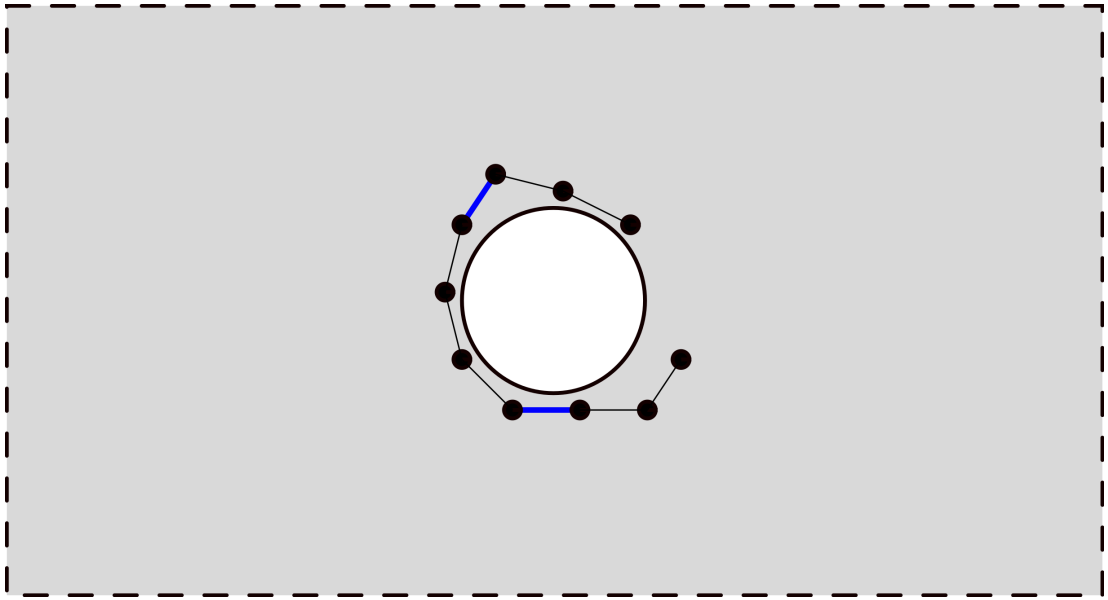

[resolution through edge-creation and edge-exchanges]

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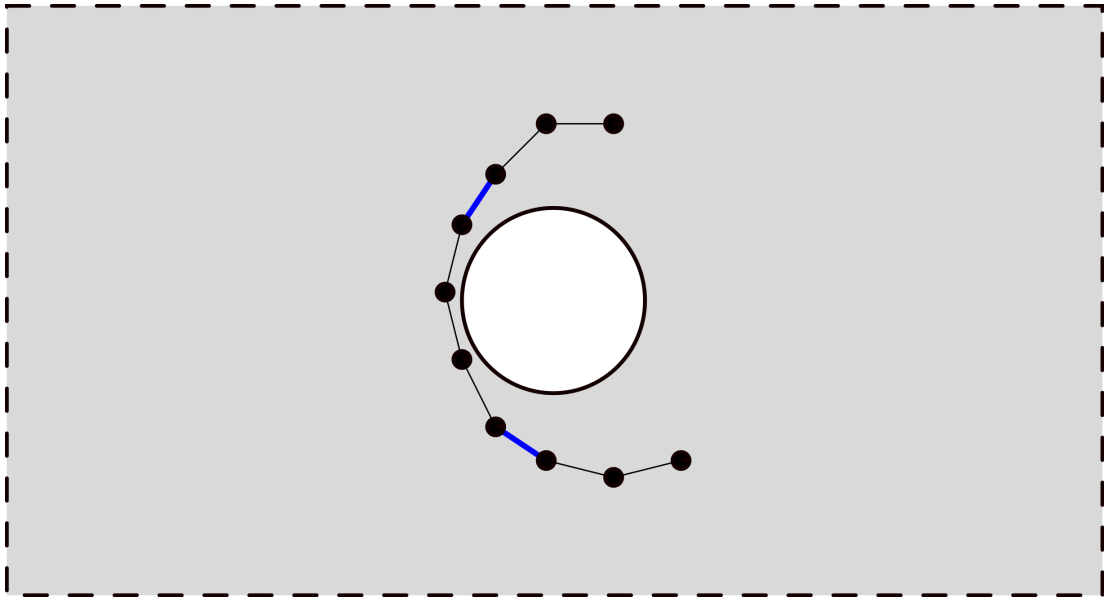

[continued motion as a group]

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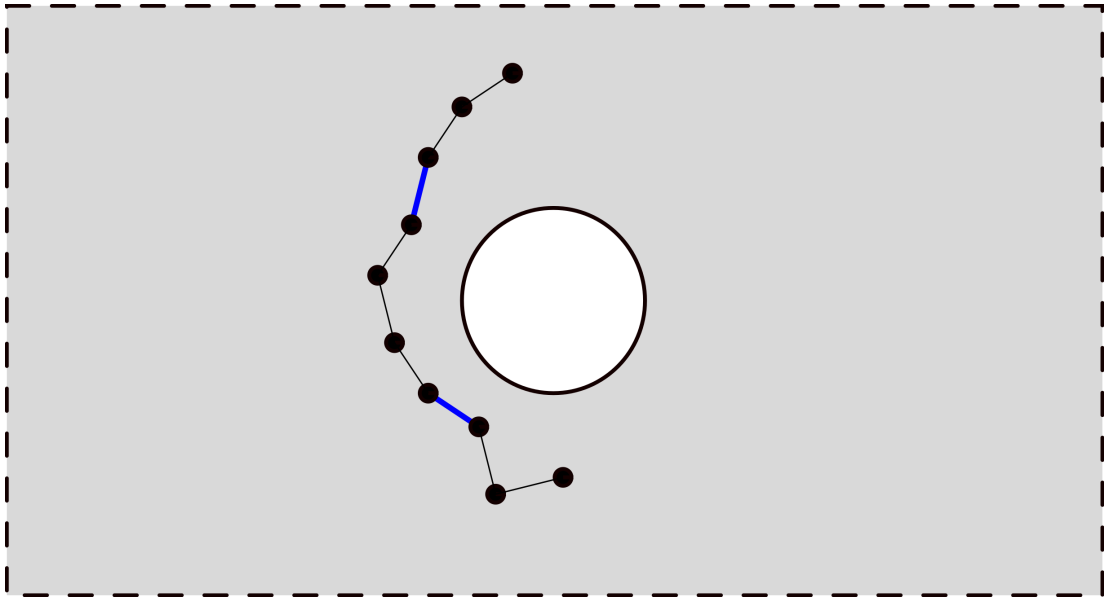

[they live happily ever after]

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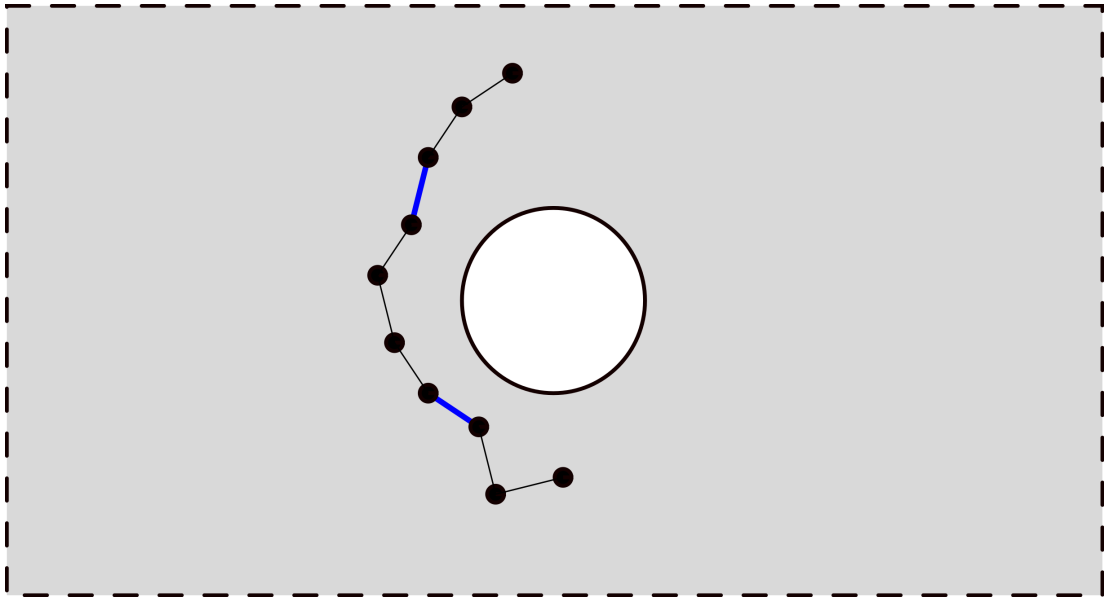

[they live happily ever after]

- Here, "very large combinatorial structure"=the space of all spanning trees over a varying set of agents.

Motivation

Emerging requirements:

- A rich formal "substrate" for symbolic representations of task domains

  $\rightsquigarrow$ *Do not treat tasks on a case-by-case basis*

  $\rightsquigarrow$ *Logic is not easily made mindful of geometry/topology*

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- Mitigate explosive complexities through combinatorial/topological analysis of the underlying space of discrete structures

  - Example: understandings about binary trees [AGK17] enabling efficient reactive collision-free navigation [AGK16] using a covering obtained by hierarchical clustering of configurations.

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  - Example: understandings about binary trees [AGK17] enabling efficient reactive collision-free navigation [AGK16] using a covering obtained by hierarchical clustering of configurations.
- More generally, underlying discrete structures must be mindful of local-to-global interactions between task and constraints.
  - Example: Erdmann [Erd10] constructs computable homotopy-theoretic invariants for guaranteed arrival in finite probabilistic/non-deterministic transition systems.

    ⇝ *We strongly suspect that these are functorial, and generalize broadly*

Proposed Program

We seek a framework combining:

- differential inclusions (continuous dynamics)
- jump/reset relations (discontinuous/switched dynamics)
- sequential and parallel composition (concatenation/coupling)
- maps between hybrid systems (HS)
- trajectories as such maps

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These accentuate the need in a CATEGORICAL FRAMEWORK, to enable programming of behaviors using TYPE-THEORETIC tools.

Proposed Program

"No Abstract Nonsense" Pledge. The proposed framework must enable the operationalization of the following:

- refinement/coarsening arguments to identify behaviors/tasks

  ⇝ *Template–Anchor pairs [FK99, CGKS19]*

  ⇝ *Other hierarchical compositions [RK18, Rev19b, Rev19a]*

- stability arguments for formal guarantees of robustness

  ⇝ *The hybrid differential inclusions framework [GST09] is an example*

- Computable invariants of task achievability

  ⇝ *Homological invariants à-la Erdmann?*

- temporal tameness analysis (noZeno / goodZeno / badZeno & worse. . .)

  ⇝ *Generalized hybrid time domains / hybrid arcs*

  ⇝ *Weaker topology on the space of hybrid arcs, à-la Conley*

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Initial Goal: Fuse the Hybrid Differential Inclusions (HDI) framework [GST09] with the categorical formulations of Culbertson *et. al.* [CGKS19] and Lerman–Schmidt [LS20].

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$\rightsquigarrow$ *A categorical viewpoint to further empower HDI in applications*

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Next Steps: Develop an instance dedicated specifically to distributed control of mobile agent networks, using categorical descriptions for:

- Kruskal graph structure (edge exchages) on the space of spanning trees
- Other computationally accessible classes of graphs, e.g., chordal graphs [CGS17]
- Parallel composition operations representing interactions among agent coalitions.

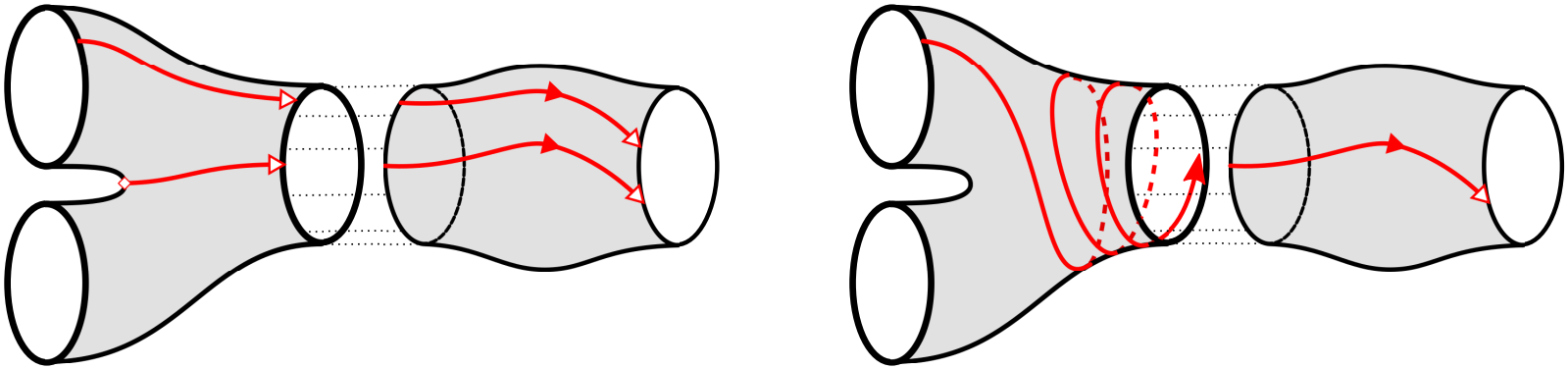
Existing Categorical Frameworks

General mantra: "Hybrid System=Graph of Dynamical Systems"

- Ames [Ame06]:
  - general "hybridization" construction for *any* category;
  - applies to smooth dynamical systems (no composition).
- Haghverdi–Tabuada–Pappas [HTP05]:
  - an open system version (both discrete and continuous control).
  - weakened notion of equivalence: bisimulation.
- Lerman, Lerman–Schmidt [Ler18, LS20]:
  - open systems as hybrid submersions;
  - interconnections via hybrid submersions between products.
- Culbertson–Gustafson–Koditschek–Stiller [CGKS19]:
  - hybrid semiconjugacies to construct template-anchor pairs;
  - Sequential composition using weakened notion of trajectory.

Informal Tidbits: Compositions

Sequential composition may be thought of as a concatenation operator on the trajectories of a pair of systems:

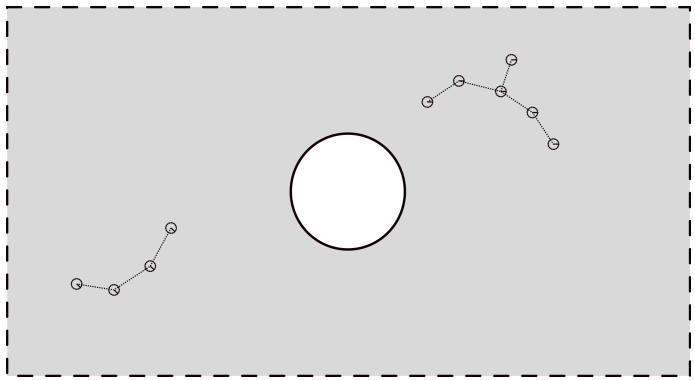


CGKS [CGKS19]: discuss difficulties with sequential composition of piecewise smooth (hybrid) trajectories, establishing the need for coarse notions of (1) hybrid trajectory and/or (2) hybrid time domains.

Informal Tidbits: Compositions

Parallel compositions.

- The simplest example is a decoupled Cartesian product of systems.
- In mobile agent networks, interconnection may be intermittent.



[While far apart, the two agent coalitions do not interact]

Ames [Ame06] and Lerman–Schmidt [LS20]: enable interconnections, but need to be reconciled with HDI and sequential compositions.

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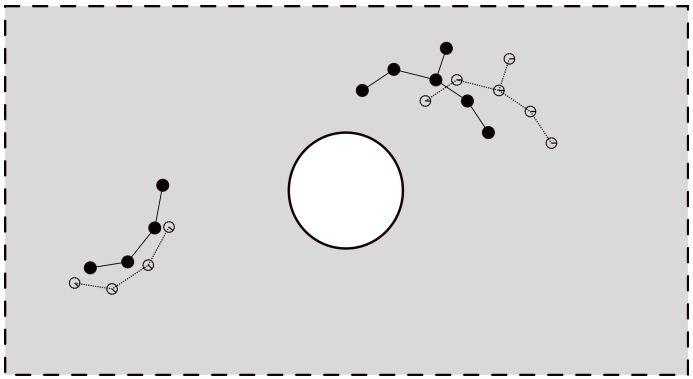

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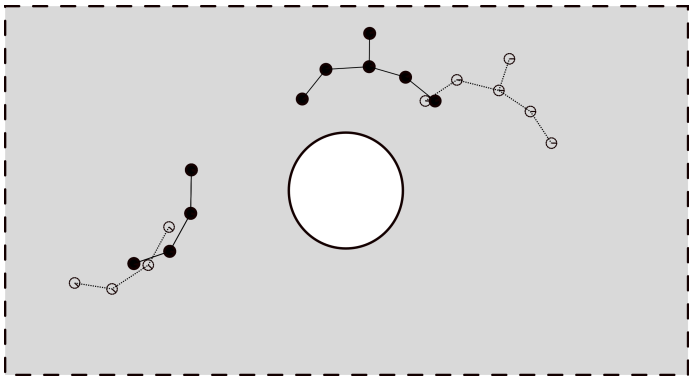

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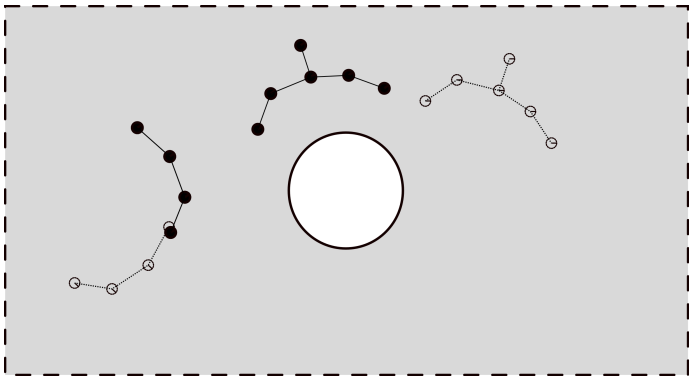

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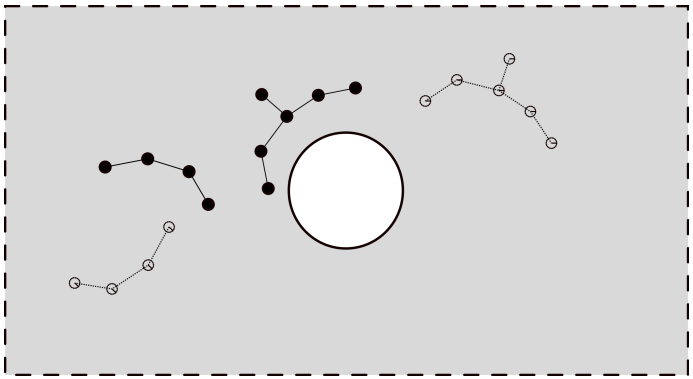

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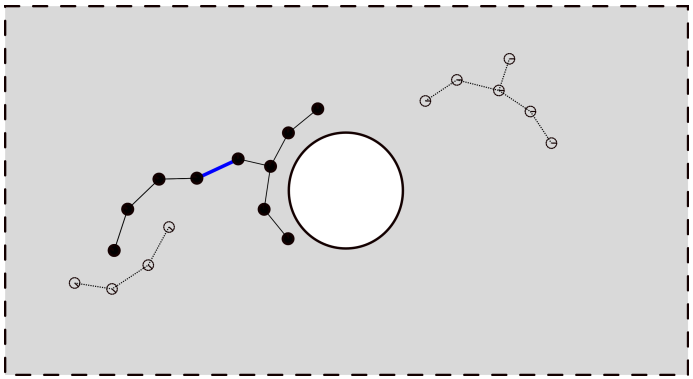

[Once communication is established, their motion may be coupled. . .]

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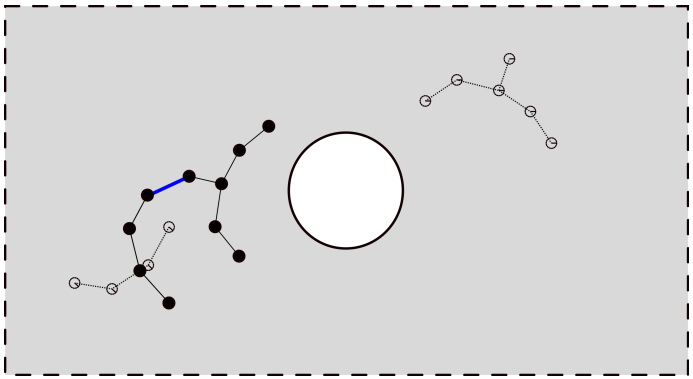

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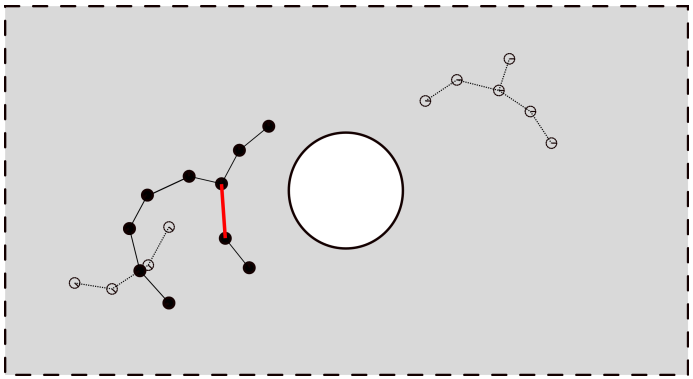

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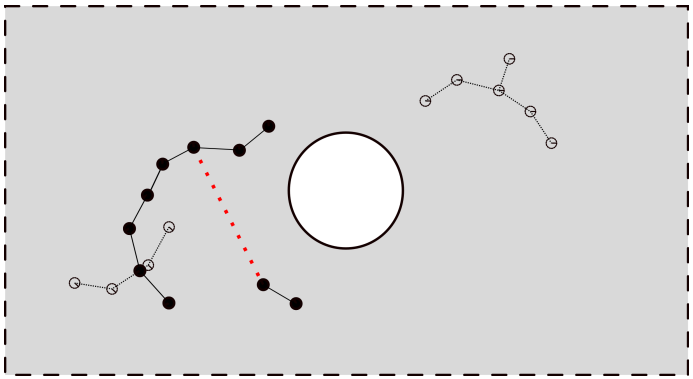

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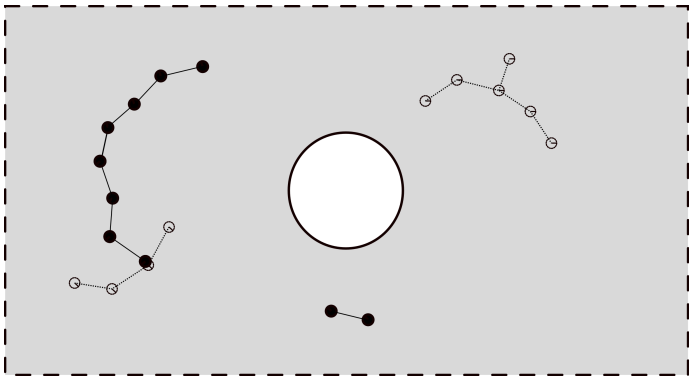

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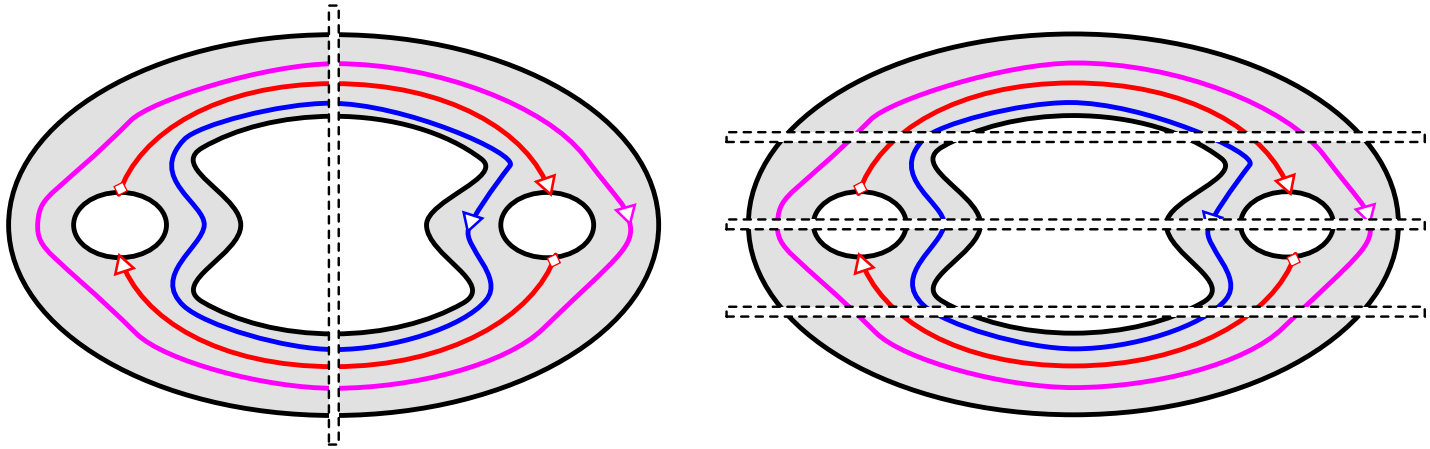
Informal Tidbits: Refinement/Coarsening

Refinement: Splitting and recombining continuous modes is useful:



- Time as a hybrid system, trajectories as maps of time into a state space.

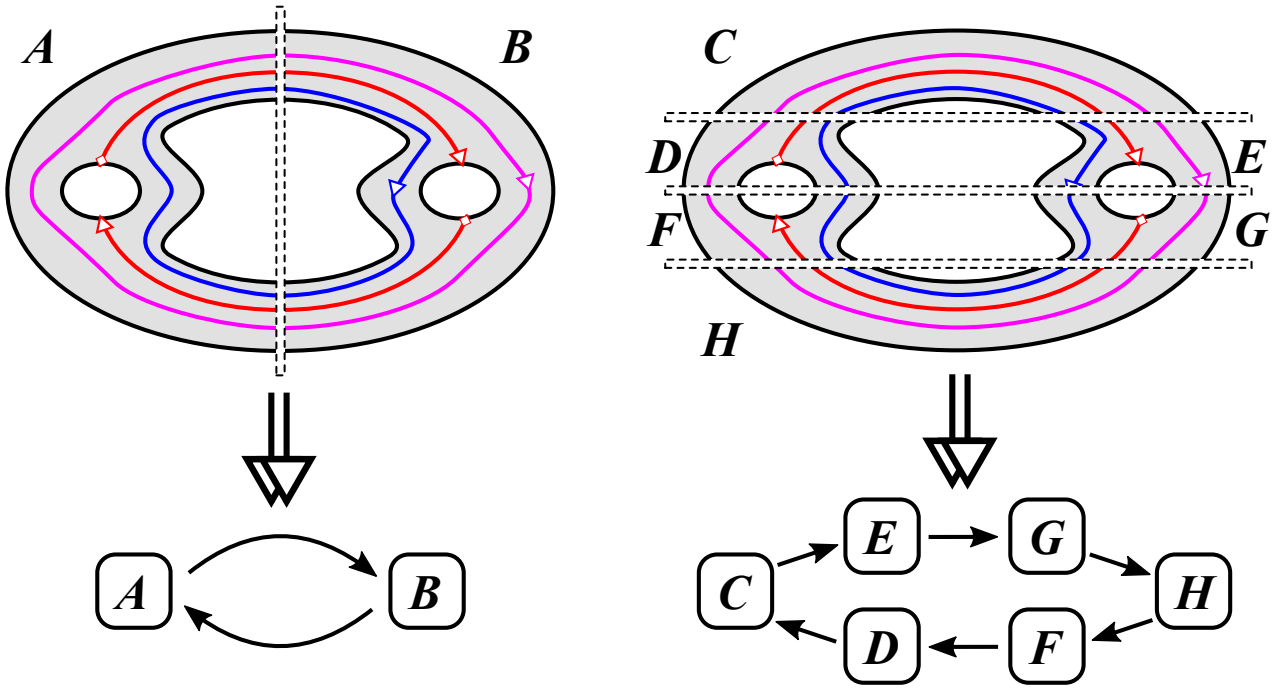
  ⇝ *A central principle in all approaches*

- Need *generalized* trajectories to support ill-behaved time sudivisions

  ⇝ *b/c mode-transitions are only allowed at jump times!*

Informal Tidbits: Refinement/Coarsening

Coarsening: When is "projection" of a HS to the underlying discontinuous structure *more* informative?

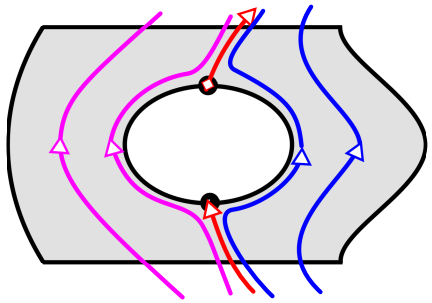


- Methods for bringing topology and hybrid structure into sync?

  ⇝ *This is precisely what happened to us in [AGK16]!*

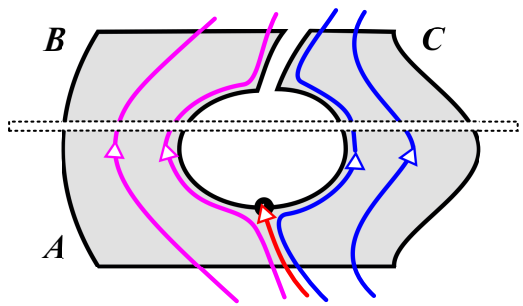
Informal Tidbits: Refinement/Coarsening

- Moving away from graphs as discrete models of hybrid structure? (a "Conley decomposition"?)



Fixed points are two-dimensional simplices?

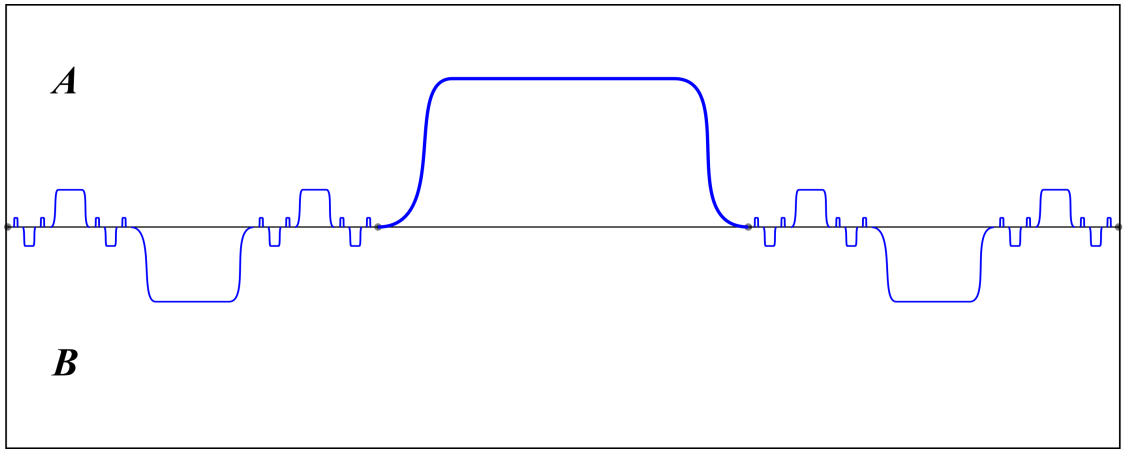
- Probabilistic aspects of hybrid structure? (Entropy??)



Probability of arrival in B given A or given C?

Informal Tidbits: It's About Time

- Generalized Hybrid Time Domains (HDT)?



A smooth "Cantor-themed" curve between two domains...

- Reformulate HDTs to facilitate trajectories of this form?

  ⇝ *MORE admissible solutions!*

- Then we need to replace graphs-of-modes with covers-by-modes!

  ⇝ *Another vote in favor of replacing graphs with complexes?*

  ⇝ *An opening into measure-theoretic machinery?*

Thank You!


UF UNIVERSITY of FLORIDA
Duke UNIVERSITY
TEXAS
The University of Texas at Austin
UC SANTA CRUZ

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