Synthesis With Multiple Barrier Functions: Adaptive Control & Open Challenges

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Control Objective

$$\dot{x} = Y(x,t)\,\theta + g(x)\,u$$

- $\theta \in \mathbb{R}^p$ is unknown
- Design a controller so that

$$\mathcal{S} \triangleq \{x \in \mathbb{R}^n : B(x) \leq 0\}$$

is forward invariant, where

$$B(x) \triangleq [B_1(x), B_2(x), \ldots, B_d(x)]^T$$

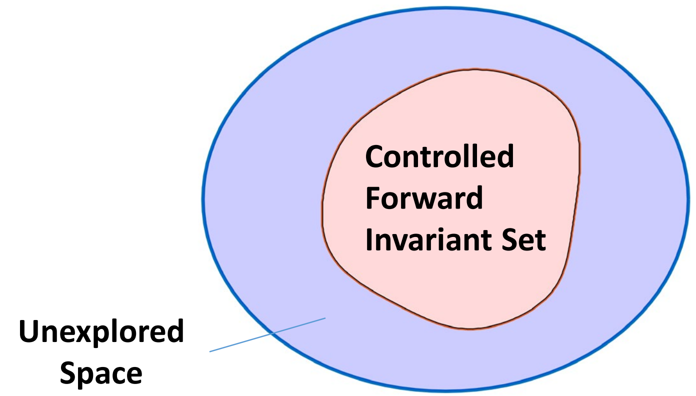
- Safe set described by multiple continuously differentiable functions

A. Isaly, O. Patil, R. Sanfelice, W. E. Dixon, "Adaptive Safety With Multiple Barrier Functions Using Integral Concurrent Learning", *Proc. Am. Control Conf.*, 2021

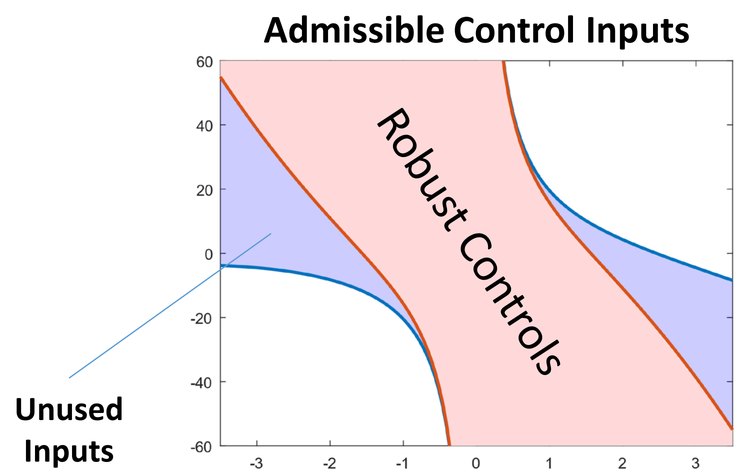
Motivation

- $\mathcal{S}$ is forward invariant using the controller:

$$\kappa\left(x,t\right) \triangleq \underset{u\in\mathbb{R}^m}{\arg\min} \;\left\|u-\kappa_{nom}\left(x,t\right)\right\|^2$$
$$s.t.\ \nabla B_i^T\left(x\right)\left(Y\left(x,t\right)\theta+g\left(x\right)u\right)$$
$$\leq -\gamma_i\left(x\right),\ \forall i\in\{1,2,\ldots,d\}$$



- Can't implement κ!



- Using robust control techniques to compensate for uncertainty leads to conservative control action

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Estimator Design

Integral Concurrent Learning

$$\tilde{\theta} \triangleq \theta - \hat{\theta}$$

We show that,

$$\left\|\tilde{\theta}(t)\right\| \leq \tilde{\theta}_{UB}(t)$$

for all $t \in \operatorname{dom} \phi$, where

$$\tilde{\theta}_{UB}(t) \triangleq \left\|\tilde{\theta}(0)\right\| \exp\left(-\int_0^t k_{CL}\lambda_{min}(\tau)\, \mathrm{d}\tau\right)$$

$$\lambda_{min}\left\{\sum_{i=1}^{N(t)} \mathcal{Y}_i^T \mathcal{Y}_i\right\}$$

Implementable Constraint

$$\nabla B_i^T(x) Y(x,t)\theta = \nabla B_i^T(x) Y(x,t)\hat{\theta} + \nabla B_i^T(x) Y(x,t)\tilde{\theta}$$

Upper bound unknown term:

$$\nabla B_i^T(x) Y(x,t)\tilde{\theta}(t) \leq \left\|\nabla B_i^T(x) Y(x,t)\right\| \tilde{\theta}_{UB}(t)$$

Combining with $\nabla B_i^T(x) Y(x,t)\hat{\theta}$,

$$\nabla B_i^T(x) Y(x,t)\theta \leq \theta_{con,i}(x,t)$$

Control Design

$$\nabla B_i^T(x)\,Y(x,t)\,\theta \leq \theta_{con,i}(x,t)$$

Theorem 1. *Let $B:\mathbb{R}^n \to \mathbb{R}^d$ be a continuously differentiable BF candidate, and suppose that Assumption 1 holds. Let the function γ and the set $\mathcal{D}$ satisfy (C1) and (C2). Along any solution to the dynamic system, let κ^* be a control law generated by the following QP:*

$$\kappa^*(x,t) \triangleq \underset{u\in\mathbb{R}^m}{\arg\min}\ \|u-\kappa_{nom}(x,t)\|^2$$
$$s.t.\ \nabla B_i^T(x)\,g(x)u \leq -\gamma_i(x) - \theta_{con,i}(x,t)\,,\ \forall i \in [d]\,,$$

where $\kappa_{nom}:\mathbb{R}^n\times\mathbb{R}_{\geq 0}$ is a nominal controller. If $(x,t)\mapsto\kappa^*(x,t)$ is continuous, then the set $\mathcal{S}$ is forward pre-invariant for the closed-loop dynamics.

Feasibility Condition (C2)

For each $x\in\mathcal{D}$ and $t\in\mathbb{R}_{\geq 0}$, there exists $u\in\mathbb{R}^m$ such that, for every $i\in[d]$,

$$\nabla B_i^T(x)\,g(x)\,u < -\gamma_i(x) - \left\|\nabla B_i^T(x)\,Y(x,t)\right\|\bar{\theta}$$

Assumed that κ^* was continuous

Challenges

- Strong understanding of continuity/feasibility given one or two constraints
  - Xu, *Control Sharing Barrier Functions With Application to Constrained Control*
  - Ong, Cortés, *Universal Formula for Smooth Safe Stabilization*
  - Chai, Sanfelice, *Forward Invariance of Sets for Hybrid Dynamical Systems (Part II)*
- Less so when many constraints are present
  - Morris, Powell, Ames, *Continuity and Smoothness Properties of Nonlinear Optimization-Based Feedback Controllers*
    - Provides conditions for continuity at a given point in the state space, but results are inconclusive at points where active constraints change.
  - Powell, Ames, *Towards Real-Time Parameter Optimization for Feasible Nonlinear Control with Applications to Robot Locomotion*
    - Presents a technique to verify feasibility at a given point, but it's unclear how to search every point in the state space.

- More generally,

$$\dot{x} \in F(x,u), \; (x,u) \in C_u$$

combined with a barrier function B induces multiple constraints of the form

$$\Gamma_i(x,u) \triangleq \sup_{f \in F(x,u)} \langle \nabla B_i(x), f \rangle$$

$$K(x) \triangleq \{u \in \Psi(x) : \Gamma_i(x,u) \leq -\gamma_i(x), \; \forall i \in [d]\}$$

- Relevant for: obstacle avoidance, temporal logic, cruise control, input constraints

Current Efforts

- Investigating controllers of the form:

$$\kappa^{*}(x) \triangleq \underset{u \in K(x)}{\arg\min}\; Q(x,u)$$

1. Identify regularity of K and Q needed for continuous κ^{*}
   a) Berge's maximum theorem and generalizations

If $Q : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function and $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a continuous set-valued mapping with compact values, then the solution multifunction $\kappa^{*} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is upper semicontinuous and compact-valued.

Adapted from: Feinberg, Kasyanov, Voorneveld, *Berge's maximum theorem for noncompact image sets*

- Investigating controllers of the form:

$$\kappa^*(x) \triangleq \underset{u \in K(x)}{\arg\min}\ Q(x,u)$$

$$K(x) \triangleq \{u \in \mathbb{R}^m : \mathcal{C}_i(x,u) \le 0,\ \forall i \in [d]\}$$

2. Identify regularity of $\mathcal{C}_i$'s leading to regularity of K
3. Identify properties of dynamics leading to regularity of $\mathcal{C}_i$'s
4. Need tools for verifying that $K(x) \neq \emptyset$ for all *x* in a given set
   a) Sum of squares?

Duality Approach to Safety

Work by: Masoumeh Ghanbarpour

Suppose

- $\dot{x} = f(x) \quad x \in \mathbb{R}^n$ where f is a locally Lipschitz continuous function.
- $\mathcal{X}$ is a nonempty, closed convex set,
- Let ψ be defined as $\psi(x) = \phi(x) + \delta_{\mathcal{X}}(x)$, where
  - ϕ is a proper, lsc, strongly convex, and twice differentiable,
  - $\delta_{\mathcal{X}}$ is the convex indicator function of $\mathcal{X}$
- Dual system is defined as

$$\dot{z} = f(\nabla\psi^*(z))$$

- $x(t) = \nabla\psi^*(z(t))$ for all $t \in \operatorname{dom} z$ is well-defined
- A new system in the primal space is obtained:

$$\begin{aligned} \dot{s} &= \langle \nabla^2\psi^*(z), f(\nabla\psi^*(z)) \rangle \\ &= \langle \nabla^2\psi^*(z), f(s) \rangle \end{aligned}$$

- Since for all $z \in \partial\psi(x)$, $\nabla^2\psi^*(z) = \nabla^2\psi^*(\nabla\phi(x))$,
- Since the range of $\nabla\psi^*$ is equal to $\mathcal{X}$
- Then,
  - The s system is given by $\dot{s} = \langle \nabla^2\psi^*(\nabla\phi(s)), f(s) \rangle$
  - $\mathcal{X}$ is forward pre-invariant.

Application - Example, Safety And Stability

Given $\dot{x} = u$, the set $\mathcal{X} = \{x \in \mathbb{R}^2 \,|\, x_2^2 + x_1 - 1 \leq 0\}$, and the system

$$\dot{x} = f(x, u_1) = \begin{cases} \dot{x}_1 = -x_1 - x_2 \\ \dot{x}_2 = x_1^3 + x_2 u_1 \end{cases} \qquad (1)$$

We want to design a control law κ to make the set $\mathcal{X}$ forward pre-invariant for the system such that the solutions comply with system (1) in the interior of the set $\mathcal{X}$ and the origin is asymptotically stable.

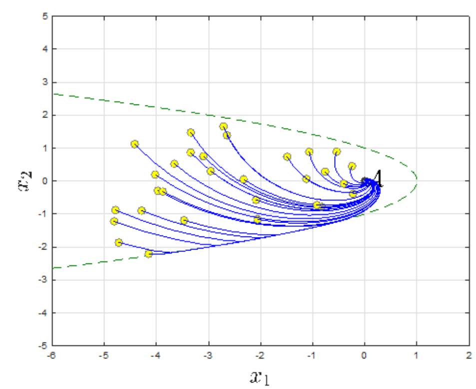
- First, we design u_1 such that the origin is asymptotically stable for (1). Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ be the control lyapunov function, then based on that $u_1 = \kappa_1(x)$ is given by

$$\kappa_1(x) = \begin{cases} 0 & if\ x_2 = 0, \\ -1 + x_1/x_2 - x_1^3/x_2 & if\ otherwise \end{cases}$$

- Next, the κ can be defined by

$$\kappa(x) = \begin{cases} f(x, \kappa_1(x)) & if \quad x \in int(\mathcal{X}) \\ \alpha v(x) & if \quad otherwise \end{cases}$$

where $v(x)$ is the eigenvector of the Hessian matrix $\nabla^2\psi^*(\nabla\phi(x))$ and $\alpha \in \mathbb{R}_{>0}$.



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Conclusion

- Developing systematic tools for guaranteeing safety in controlled systems
- Feasibility of safe controllers is a significant challenge
- Next steps:
  - Continue collaborating!
  - Understand least restrictive conditions for safety
    - Do controllers need to be continuous?