Global Asymptotic Stability of Nonlinear Systems while Exploiting Properties of Uncertified Feedback Controllers via Opportunistic Switching

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April 7, 2022

Introduction – Switched Controllers

Sometimes, a single continuous feedback controller cannot satisfy all design requirements.

- Switching between multiple controllers is necessary to achieve robust global asymptotic stability around topological obstructions.[1]
- Switching is used to unite local and global controllers.[2]
- Similarly, switching between a family of Lyapunov-certified controllers is used to achieve asymptotic stability.[3]
- Switching is used to provide a backup controller that guarantees safety when the primary controller is not provably safe.[4]

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[1] Mayhew, Sanfelice, and Teel (2011) and Sanfelice, Messina, et al. (2006).

[2] Prieur (2001) and Teel and Kapoor (1997).

[3] El-Farra, Mhaskar, and Christofides (2005).

[4] Seto et al. (1998).

Why Use an Uncertified Controller κ_1?

- Controller that almost always works (but sometimes does not)
  $\Longrightarrow$ MPC that occasionally fails to compute an update.
- Local optimal controller with unknown region of attraction
  $\Longrightarrow$ LQR for linearization about the origin.
- "Black box" controller
  $\Longrightarrow$ Neural Network Controllers.

Problem Setting

Consider a continuous-time plant

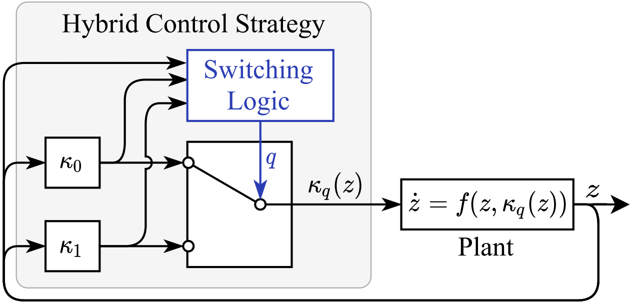
$$\dot{z} = f_P(z, u), \quad z \in \mathbb{R}^n, u \in \mathbb{R}^m.$$

Our goal is to make a compact set $\mathcal{A} \subset \mathbb{R}^n$ uniformly globally asymptotically stable (UGAS).

Given two controllers:

κ_0: a continuous Lyapunov-certified controller that renders $\mathcal{A}$ to be UGAS for $\dot{z} = f_P(z, \kappa_0(z))$

κ_1: an arbitrary (uncertified) continuous controller



We design the switching logic for $q \in \{0, 1\}$ such that

- $\mathcal{A}$ is UGAS
- κ_1 is preferred over κ_0

Problem Setting – Lyapunov-certified Controller κ_0

A set $\mathcal{A}$ is called *uniformly globally asymptotically stable* if for each $r > 0$, $\varepsilon > 0$,

- there is a uniform bound on the range of all trajectories that start within a distance r from $\mathcal{A}$, and
- there is a uniform bound on the time it takes all trajectories that start within a distance r from $\mathcal{A}$ to converge within a distance ε from $\mathcal{A}$.

Because κ_0 is Lyapunov-certified for

$$\dot{z} = f_P(z, \kappa_0(z))$$

and the set $\mathcal{A}$, there exists a Lyapunov function

$$V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$$

that guarantees $\mathcal{A}$ is UGAS.

Example – MPC

Suppose we are given a plant

$$\dot{z} = f_P(z, u)$$

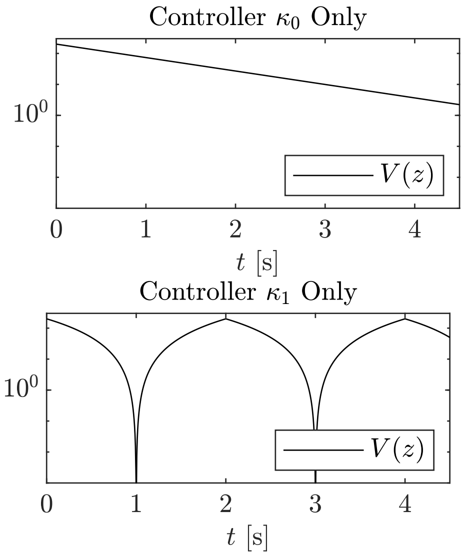
and two controllers:

κ_0: a Lyapunov-certified controller

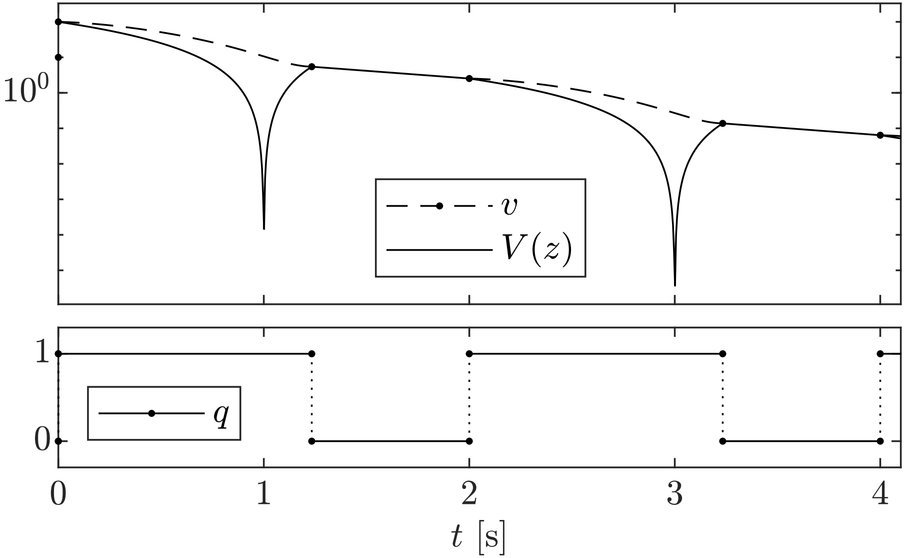
κ_1: a model predictive controller with a sampling period of 1 s

But, suppose the time required to compute the next MPC input value is 2 s.

- A new MPC feedback value is not available at every sample time.



Oppurtunistic Switching Between κ_0 and κ_1



Hybrid Control Strategy

We define a closed-loop system with state

$$x := (z, v, q) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \{0, 1\}$$

where

$z \in \mathbb{R}^n$: state of the plant

$v \in \mathbb{R}_{\geq 0}$: upper bound for $V(z)$ when $q = 1$

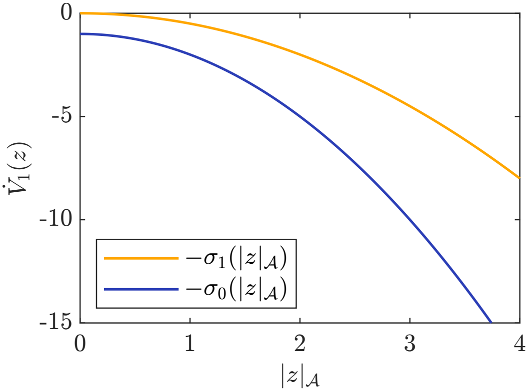
$q \in \{0, 1\}$: determines whether κ_0 or κ_1 is used

We aim to make the following compact set to be UGAS:

$$\mathcal{A}_{\mathcal{X}} := \{x \in \mathcal{X} \mid z \in \mathcal{A}, v = 0\} = \mathcal{A} \times \{0\} \times \{0, 1\}. \tag{1}$$

Hybrid Control Strategy – Switching Logic

$$\text{Let } \dot{V}_q(z) := \langle \nabla V(z), f_P(z, \kappa_q(z)) \rangle \,, \quad q \in \{0, 1\}.$$



Threshold functions:

Let $\sigma_0, \sigma_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be continuous functions such that

- σ_1 is positive definite and
- $\sigma_0(s) > \sigma_1(s)$ for all $s \geq 0$.

Hybrid Control Strategy – Switching Logic

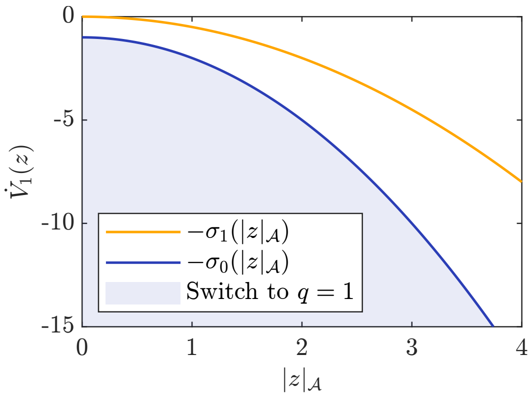
$$\dot{V}_1(z) := \langle \nabla V(z), f_P(z, \kappa_1(z)) \rangle$$

For $q = 0$, we say $\dot{V}_1$ is "small enough to switch to $q = 1$" at $z \in \mathbb{R}^n$ if

$$\dot{V}_1(z) \leq -\sigma_0(|z|_{\mathcal{A}})$$

and $\dot{V}_1$ is "large enough to hold $q = 0$" if

$$\dot{V}_1(z) \geq -\sigma_0(|z|_{\mathcal{A}}).$$



Hybrid Control Strategy – Switching Logic

For $q = 1$, we say that $\dot{V}_1$ is "small enough to hold $q = 1$" at $z \in \mathbb{R}^n$ if

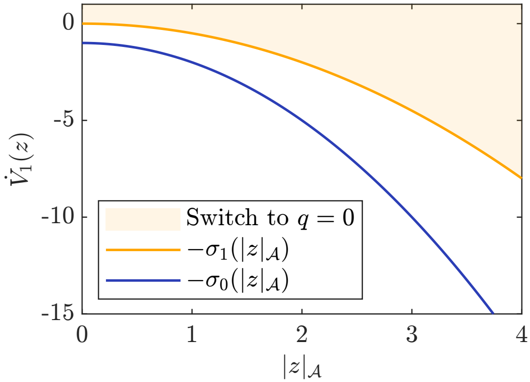
$$\dot{V}_1(z) \leq -\sigma_1(|z|_{\mathcal{A}})$$

and "large enough to switch to $q = 0$" if

$$\dot{V}_1(z) \geq -\sigma_1(|z|_{\mathcal{A}}).$$

The condition that $\dot{V}_1(z) \geq -\sigma_1(|z|_{\mathcal{A}})$ is **necessary** but **not sufficient** to switch to $q = 0$.

- We also require that $V(z) \geq v$ for a switch to occur.



Switching Logic – Example

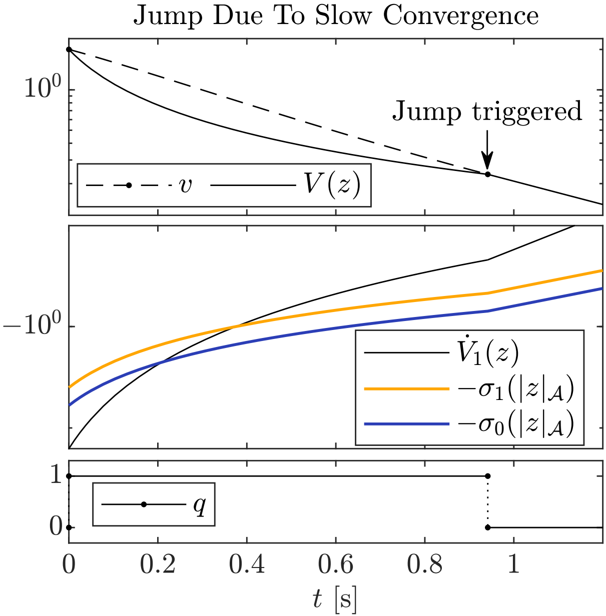
Consider the plant

$$\dot{z} = u$$

with $z, u \in \mathbb{R}$ and controllers

$$\kappa_0(z) := -z,$$
$$\kappa_1(z) := -z^3$$

Pick $(z_0, v_0, q_0) := (2, 0, 0)$.



Dynamics of Closed-Loop System

At each jump:

- z is constant
- v is set to $V(z)$
- q is toggled to the opposite value in $\{0, 1\}$

During flows:

- z evolves according to $\dot{z} = f_P(z, \kappa_q(z))$
- q is constant
- v evolves according to the dynamics chosen here:

$$\dot{v} = f_v(z, v, q) := \begin{cases} -v, & \text{if } q = 0, \\ -\sigma_1(|z|_{\mathcal{A}}) + \mu(V(z) - v), & \text{if } q = 1, \end{cases}$$

where $\mu > 0$ is parameter.

Hybrid Control Strategy — Design of f_v

$$\dot{v} = f_v(z, v, q) := \begin{cases} -v, & \text{if } q = 0, \\ -\sigma_1(|z|_{\mathcal{A}}) + \mu(V(z) - v), & \text{if } q = 1 \end{cases}$$

- If $q = 0$ or $q = 1$ and no switches occur, then v converges to 0.
- Each switch from $q = 0$ to $q = 1$ is followed by an interval where $V(z) < v$.
- If $q = 1$, $V(z) < v$, then v is decreasing.
- If $q = 1$, $V(z) < v$, and $\dot{V}_1(z)$ is large enough to switch to $q = 0$ then v will eventually catch up to $V(z)$, causing a switch to $q = 0$.
- The parameter μ determines how closely v follows $V(z)$.

Example: Linear Quadratic Regulator

Consider the nonlinear plant

$$\dot{z} = A_1 z + h(\|z\|) A_2 z + u \tag{2}$$

with $z, u \in \mathbb{R}^2$,

$$A_1 := \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad A_2 := 4I, \quad \text{and} \quad h(s) = \min\{s, 1\}.$$

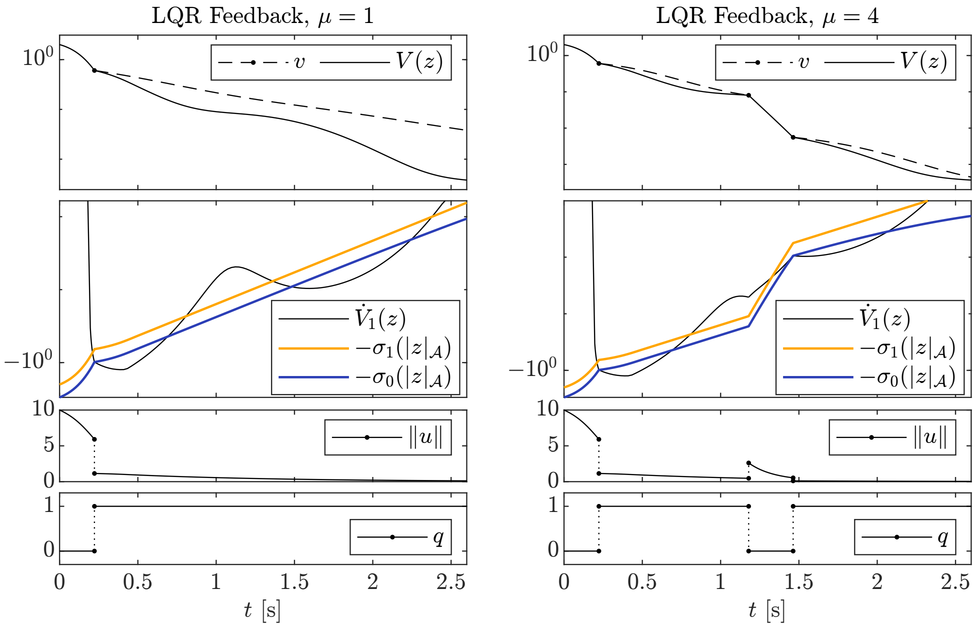
The origin is UGAS for

$$\kappa_0(z) := \begin{bmatrix} -5 & 0 \\ 0 & -6 \end{bmatrix} z.$$

For κ_1, we use the LQR feedback $u = \kappa_1(z) := -z$, which is the solution to the following LQR problem:

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & \int_0^\infty \|z(t)\|^2 + \|u(t)\|^2 \, dt \\ \text{subject to} \quad & \dot{z} = A_1 z + u. \end{aligned}$$

Example: Linear Quadratic Regulator



Theorem 1

Suppose that

- f_P, κ_0, *and* κ_1 *are continuous;*
- V *is continuously differentiable.*

Then, $\mathcal{A}_{\mathcal{X}} := \mathcal{A} \times \{0\} \times \{0,1\}$ *is UGAS for the closed-loop system* $\mathcal{H}$.

Proof sketch. The proof proceeds by showing that

$$\widetilde{V}(x) := \max\{V(z), v\},$$

is a (nonsmooth) Lyapunov function for $\mathcal{H}$.

Outside $\mathcal{A}_{\mathcal{X}}$, $\widetilde{V}(x)$ decreases along flows:

- if $q = 0$, then both $V(z)$ and v are decreasing;
- if $q = 1$, then whichever is larger of $V(z)$ or v, that value is decreasing. or

At jumps, $\widetilde{V}(x)$ does not increase.

Therefore, $\mathcal{A}_{\mathcal{X}}$ is UGAS. □

Remark. The asymptotic stability of $\mathcal{A}_{\mathcal{X}}$ is robust to vanishing noise.

Conclusion

- Introduced a hybrid control strategy for using a Lyapunov-certified controller as a backup for an uncertified controller while ensuring convergence.
- Illustrated with examples that method allows us to take advantage of useful properties of uncertified controllers while guaranteeing convergence.

Future work

- Analyze our hybrid control strategy when applied to systems with disturbances and noisy signals, and
- Combine our strategy with existing supervisory control strategies for constraint safety.

Questions?

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Definition 1 (Sanfelice, 2021, Definition 3.7)

A nonempty set $\mathcal{A} \subset \mathbb{R}^n$ is said to be

- *uniformly globally stable* if there exists a continuous, strictly increasing function α such that every solution x to $\mathcal{H}$ satisfies $|x(t,j)|_{\mathcal{A}} \leq \alpha\left(|x(0,0)|_{\mathcal{A}}\right)$ for each $(t,j) \in \operatorname{dom} x$; and
- *uniformly globally attractive* for $\mathcal{H}$ if every maximal solution is complete and for all $\varepsilon > 0$ and $r > 0$, there exists $T > 0$ such that every solution x to $\mathcal{H}$ with $|x(0,0)|_{\mathcal{A}} \leq r$ satisfies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom} x$ such that $t + j \geq T$.
- If $\mathcal{A}$ is both uniformly globally stable and uniformly globally attractive for $\mathcal{H}$, then it is said to be *uniformly globally asymptotically stable* (UGAS) for $\mathcal{H}$.

Preliminaries – Hybrid Systems

We consider hybrid systems modeled as

$$\mathcal{H} \begin{cases} \dot{x} = f(x) & x \in C \\ x^+ = g(x) & x \in D \end{cases}$$

with

- flow set $C \subset \mathbb{R}^n$
- flow map $f : C \to \mathbb{R}^n$
- jump set $D \subset \mathbb{R}^n$
- jump map $g : D \to \mathbb{R}^n$

Clarke Generalized Gradient

$$\widetilde{V}^{\circ}(x,w) = \begin{cases} \langle \nabla_z V(z), w_z \rangle & \text{if } V(z) > v, \\ \max \left\{ \langle \nabla_z V(z), w_z \rangle, w_v \right\} & \text{if } V(z) = v, \\ w_v & \text{if } V(z) < v. \end{cases} \tag{3}$$