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The Mittag Leffler reproducing kernel Hilbert spaces of entire and analytic functions

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ABSTRACT

This paper investigates the function theoretic properties of two reproducing kernel functions based on the Mittag-Leffler function that are related through a composition. Both spaces provide one parameter generalizations of the traditional Bargmann–Fock space. In particular, the Mittag-Leffler space of entire functions yields many similar properties to the Bargmann–Fock space, and several results are demonstrated involving zero sets and growth rates. The second generalization, the Mittag-Leffler space of the slitted plane, is a reproducing kernel Hilbert space (RKHS) of functions for which Caputo fractional differentiation and multiplication by z^q (for $q > 0$) are densely defined adjoints of one another.

1. Introduction

The genesis of fractional derivatives goes back to the 1600s with a letter between l'Hopital and Leibniz [9,20]. Since then, the fractional calculus has seen slow progress that lagged behind that of its integer order counterpart. In the 1960s fractional calculus had a tremendous impact on the study of mechanical properties of materials beginning with the works of Caputo and Mainardi in [5,6]. Subsequently, fractional calculus has been applied to wide ranging application domains including developments in control theory, psychology, mathematical physics, geophysics, and finance [9]. Many results analogous to integer order differential equations have been established for fractional order differential equations (FODEs), such as existence and uniqueness theorems [1,9,10,24], boundary value problems, Lyapunov stability analyses [15–17], and numerical methods [8,9,11,19].

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FODEs appear in fractional order models of quantum mechanical phenomenon, including fractional order harmonic oscillators and Schrödinger equations [14]. For classical problems in quantum mechanics, the Bargmann–Fock space has played a vital role. The Bargmann–Fock space is a particular realization of the Stone–Von Neumann theorem that transports the raising and lowering operators associated with the quantum harmonic oscillator to multiplication by z and differentiation in the Bargmann–Fock reproducing kernel Hilbert space (RKHS) via the Bargmann transform [2,12,25]. With recent developments in fractional order quantum mechanics [14] there is motivation to pursue a generalization of the Bargmann–Fock space that can play the same role for FODEs and fractional order quantum mechanics.

The Bargmann–Fock space has maintained significant interest independent of its applications to quantum mechanics. A large amount of research has investigated the study of properties of functions within the Bargmann–Fock space. In [21], Seip investigated interpolation sequences of the Bargmann–Fock space, and in [25, Theorem 5.9] Zhu produced examples of zero sets where the space of functions that have those zeros was finite dimensional. The finite dimensionality of this subspace is a property that sets the Bargman–Fock space apart from other RKHSs of analytic functions, such as the Hardy space and the Bergman space. The study of zero sets yields information concerning the growth rate of functions, helps determine the uniqueness sets of a function space, and has applications to signal processing.

In [19], the Mittag-Leffler RKHS of a real variable was introduced as a means to estimate the Caputo fractional derivative of a function of a real variable. The kernel associated with this RKHS arises from the Mittag-Leffler function, which is itself a generalization of the exponential function [13]. The objective of the present work is to extend this RKHS to functions of a complex variable and demonstrate that the resulting Mittag-Leffler RKHS of entire functions is a natural one parameter generalization of the Bargmann–Fock space.

This work explores the function theoretic aspects of the Mittag-Leffler RKHS, such as growth rates, zero sets, and determines an integral representation for the norm of the Mittag-Leffler space. The primary focus will be on the Mittag-Leffler RKHS of entire functions, which is defined in Section 3. Examination of the Mittag-Leffler space of entire functions allows for the utilization of a large amount of existing work on entire functions and zero sets (cf. [4,25]) in Section 4. The Mittag-Leffler RKHS of the slitted plane (which is composed of functions analytic everywhere except for the negative real line) is discussed in Section 5 and is obtained from the Mittag-Leffler space of entire functions via a composition with z^q (for $q > 0$). In particular, the Mittag-Leffler RKHS of the slitted plane is associated with a natural generalization of the Bargmann–Fock space where the Caputo fractional derivative becomes a densely defined operator and is adjoint to multiplication by z^q. Moreover, Theorem 5.3 indicates that the Mittag-Leffler space of the slitted plane is the unique space with this adjoint correspondence.

2. Preliminaries

This section presents several preliminary concepts related to the Caputo fractional derivative. The two more prominent time fractional derivatives extant in the literature are the Riemann–Liouville fractional derivative and the Caputo fractional derivative. Both fractional derivatives are realized through an interplay between the Riemann–Liouville fractional integral, denoted by J^q for $q > 0$, and integer order derivatives. The Riemann–Liouville fractional derivative first applies the fractional integral to a function followed by an integer order derivative, while the Caputo fractional derivative first applies an integer order derivative followed by the fractional integral. Since fractional integration and integer order differentiation do not commute, the two versions of the fractional derivatives lead to different fractional calculi. It can be seen through the Laplace transform of the two fractional differentiation operators, that the Riemann–Liouville fractional derivative results in initial value problems that require knowledge of initial conditions of fractional derivatives, whereas the Caputo fractional derivative yields initial value problems that require only integer order initial conditions [9]. This distinction has motivated the use of the Caputo fractional derivative for applications in engineering and science. Moreover, the Mittag-Leffler special function,[1] $E_q(z) := \sum_{n=0}^{\infty} \Gamma(qn+1)^{-1} z^n$, satisfies an eigenvalue problem for the Caputo fractional derivative, similar to that of the exponential function for integer order differentiation, that is leveraged in this work to generalize the Bargmann–Fock space.

Definition 1. The Riemann–Liouville fractional integral for $q \in \mathbb{R}_+$ is defined as

$$(J^q f)(t) := \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau) d\tau.$$

Definition 2. Let $n \in \mathbb{N}$. For an n-times differentiable function $f : \mathbb{R}_+ \to \mathbb{R}$, the Caputo fractional derivative of order q, where $n-1 < q \leq n$, is given by

$$\begin{aligned} D_*^q f(t) &:= J^{n-q} \frac{d^n}{dt^n} f(t) \\ &= \begin{cases} \dfrac{1}{\Gamma(n-q)} \displaystyle\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q+1-n}} d\tau & n-1 < q < n,\ n \in \mathbb{N} \\ f^{(n)}(t) & q = n \in \mathbb{N}. \end{cases} \end{aligned}$$

The real-valued Mittag-Leffler RKHS was first introduced to facilitate a numerical method for estimating the Caputo fractional derivative of a function in [19]. The real-valued Mittag-Leffler space possesses a universal property that positions it as a suitable venue for the approximation of continuous functions of a real variable and their fractional derivatives. Linear combinations of the Mittag-Leffler kernels were used in [19] to estimate fractional derivatives through interpolation.

Definition 3. The real-valued Mittag-Leffler RKHS of order $q > 0$ is the RKHS associated with the kernel functions $K_q(t, \lambda) := E_q(\lambda^q t^q)$, i.e.

$$ML^2(\mathbb{R}_+; q) := \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^{qn} \;\middle|\; \sum_{n=0}^{\infty} |a_n|^2 \Gamma(qn+1) < \infty \right\}.$$

The Mittag-Leffler function is entire and is a generalization of the exponential function. Similar to the relationship of the exponential to integer order differentiation, the Mittag-Leffler function serves as an eigen-function of the Caputo fractional derivative as $D_*^q E_q(\lambda t^q) = \lambda E_q(\lambda t^q)$. When $q = 1$, the resulting equation is the familiar relation $\frac{d}{dx} \exp(\lambda t) = \lambda \exp(\lambda t)$ [9,13].

The Bargmann–Fock space is the RKHS of entire functions which are L^2 under the Gaussian measure. More explicitly,

$$F^2(\mathbb{C}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \;\middle|\; \sum_{n=0}^{\infty} |a_n|^2 n! < \infty \right\}.$$

The kernel function for the Bargmann–Fock space $K(z, w) = \exp(\overline{w} z)$, which coincides with the Mittag-Leffler kernel function, $K_1(z, w)$, when z and w are restricted to $\mathbb{R}_+$. In the next section, the Mittag-Leffler space of entire functions will generalize the Bargmann–Fock space by offering a complexification of the (real-valued) Mittag-Leffler space.

[1] The function, $\Gamma : \mathbb{C} \setminus \{-1, -2, \ldots\} \to \mathbb{C}$, is the Gamma function given by $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$ when z is a complex number with a positive real part and satisfies the functional equation $\Gamma(z+1) = z\Gamma(z)$.

3. The Mittag-Leffler RKHS of entire functions

Definition 4. For $q > 0$, the Mittag-Leffler space of entire functions of order $q > 0$, denoted $ML^2(\mathbb{C}; q)$, is the RKHS of entire functions associated with the kernel functions given by[2]

$$K_q(z, w) = E_q(\overline{w}z) = \sum_{n=0}^{\infty} \frac{\overline{w}^n z^n}{\Gamma(qn+1)},$$

i.e.

$$ML^2(\mathbb{C}; q) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \;\middle|\; \sum_{n=0}^{\infty} |a_n|^2 \Gamma(qn+1) < \infty \right\}. \tag{1}$$

The major difference between the real-valued Mittag-Leffler RKHS and the Mittag-Leffler space of entire functions is that the former space is composed of functions that are represented by series in t^q whereas the latter space is composed of functions that are represented as series in z. Consideration of the Mittag-Leffler space of entire functions allows for direct comparison with properties of the Bargmann–Fock space, itself a space of entire functions. However, by considering only functions that are series in z as opposed to series in z^q, an important property is lost in translation from the Bargmann–Fock space to the Mittag-Leffler space of entire functions; the operator D_*^q does not act as a lowering operator for the Mittag-Leffler space of entire functions. Therefore, in some cases a modified Mittag-Leffler RKHS will be considered, namely the Mittag-Leffler space of the slitted plane.

From (1), the following result holds, which will be used to establish several integral and norm relations for the rest of the functions in $ML^2(\mathbb{C}; q)$.

Lemma 3.1. *The set of functions*

$$\{g_n(z)\}_{n=0}^{\infty} = \left\{ \frac{z^n}{\sqrt{\Gamma(qn+1)}} \right\}_{n=0}^{\infty} \tag{2}$$

is an orthonormal basis for $ML^2(\mathbb{C}; q)$.

For $q = 1$, $ML^2(\mathbb{C}; q) = F^2(\mathbb{C})$ and the norm can be expressed as

$$\|f\|_{F^2} = \sqrt{\frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dA(z)}, \tag{3}$$

where the integral is taken with respect to Lebesgue area measure [25, Section 2.1]. The integral representation of the norm for the Bargmann–Fock space is pivotal in the study of solutions to the heat equation, Toeplitz operators, growth rates, zero sets, atomic decompositions, and many other properties of the Bargmann–Fock space. Therefore, it is desirable to establish an analogous integral formula for the norm of $ML^2(\mathbb{C}; q)$.

Theorem 3.2. *Given $q > 0$, the norm for the space $ML^2(\mathbb{C}; q)$ can be expressed as*

$$\|f\|_{ML^2(\mathbb{C};q)}^2 = \frac{1}{q\pi} \int_{\mathbb{C}} |f(z)|^2 |z|^{\frac{2}{q}-2} e^{-|z|^{\frac{2}{q}}} dz < \infty. \tag{4}$$

[2] Henceforth, the function given by K_q will denote the kernel associated with the Mittag-Leffler space of entire functions rather than the function given in Definition 3.

Proof. For the sake of argument, assume that there exists a radially symmetric measure, μ, for which $\|f\|^2 = \int_{\mathbb{C}} |f(x)|^2 d\mu(z)$. From the Mittag-Leffler kernel function and the series definition of the norm given in (1) it can be seen that $\|z^n\|^2_{ML^2(\mathbb{C};q)} = \Gamma(qn+1)$. Thus,

$$\Gamma(qn+1) = \int_{\mathbb{C}} |z|^{2n} d\mu(z) = 2\pi \int_0^\infty r^{2n+1} d\hat{\mu}(r) \tag{5}$$

where $\hat{\mu}$ is a measure on the real line.

However, the Γ function can be expressed as an integral over the positive real line as $\Gamma(qn+1) = \int_0^\infty t^{qn} e^{-t} dt$. Letting $r^2 = t^q$, yields $r^{2/q} = t$ and $\frac{2}{q} r^{\frac{2}{q}-1}\, dr = dt$, so that

$$\int_0^\infty r^{2n+1} d\hat{\mu}(r) = \int_0^\infty r^{2n+1} \cdot \left(\frac{1}{q\pi} r^{\frac{2}{q}-2} e^{-r^{\frac{2}{q}}} \right) dr. \tag{6}$$

Thus, the norm determined by the measure given by

$$d\mu = \frac{1}{q\pi} |z|^{\frac{2}{q}-2} e^{-|z|^{\frac{2}{q}}}\, dz, \tag{7}$$

agrees with $\|f\|^2_{ML^2(\mathbb{C};q)}$ when $f(z) = z^n$. Since z^n is an orthogonal basis for $ML^2(\mathbb{C};q)$, the theorem is complete. $\square$

Note that when $q = 1$ the measure in (4) becomes $d\mu = \frac{1}{\pi} e^{-|z|^2}$, and thus the measure for the Bargmann–Fock space is recovered. From Theorem 3.2, $ML^2(\mathbb{C};q)$ consists of entire functions f for which

$$\|f\|^2_{ML^2(\mathbb{C};q)} = \int_{\mathbb{C}} |f(z)|^2 \frac{1}{q\pi} |z|^{\frac{2}{q}-2} e^{-|z|^{\frac{2}{q}}}\, dz < \infty.$$

Since the Gaussian measure $d\lambda(z) = \frac{1}{\pi} e^{-|z|^2}$ bounds the measure given by (7) the following proposition is immediate.

Proposition 3.3. *For $0 < q \le p$ if $f \in ML^2(\mathbb{C};p)$ then $f \in ML^2(\mathbb{C};q)$. Moreover,*

$$\lim_{q \to p^-} \|f\|_{ML^2(\mathbb{C};q)} = \|f\|_{ML^2(\mathbb{C};p)}.$$

In particular, $F^2(\mathbb{C}) \subset ML^2(\mathbb{C};q)$ for $0 < q \le 1$.

Proof. Note that $\left\{ \frac{z^n}{\sqrt{\Gamma(pn+1)}} \right\}$ is a orthonormal basis for the Fock space. If $f(z) = \sum a_n z^n \in ML^2(\mathbb{C};p)$, then

$$f(z) = \sum_{n=0}^\infty a_n \frac{z^n}{\sqrt{\Gamma(pn+1)}} = \sum_{n=0}^\infty \left(a_n \sqrt{\frac{\Gamma(qn+1)}{\Gamma(pn+1)}} \right) \frac{z^n}{\sqrt{\Gamma(qn+1)}}.$$

Moreover, it follows that

$$\|f\|_{ML^2(\mathbb{C};p)} = \sum |a_n|^2 \ge \sum |a_n|^2 \frac{\Gamma(qn+1)}{\Gamma(pn+1)} = \|f\|_{ML^2(\mathbb{C};q)}, \tag{8}$$

since $\Gamma(z)$ is a monotonically increasing function on the real line and

$$\frac{\Gamma(qn+1)}{\Gamma(pn+1)} \leq 1$$

for all $n \in \mathbb{N}$. The proposition follows by way of the dominated convergence theorem and (8). □

4. Growth properties and zero sets

While the Mittag-Leffler function lacks some of the properties of the exponential function, such as the semi-group property, it retains similar growth characteristics. This section establishes theorems concerning the order and type of functions in $ML^2(\mathbb{C}; q)$ that are analogous to theorems on the Bargmann–Fock space. Foremost, this section begins with a standard point-wise estimate for functions in $ML^2(\mathbb{C}; q)$.

Lemma 4.1. *For all $z \in \mathbb{C}$ and $f \in ML^2(\mathbb{C}; q)$,*

$$|f(z)|^2 \leq E_q(|z|^2)\|f\|^2_{ML^2}.$$

Proof. By an application of Cauchy–Schwarz:

$$\begin{aligned} |f(z)|^2 &\leq \left(\sum_{n=0}^{\infty} |a_n||z^n| \right)^2 \\ &= \left(\sum_{n=0}^{\infty} |a_n|\sqrt{\Gamma(qn+1)} \cdot \frac{|z^n|}{\sqrt{\Gamma(qn+1)}} \right)^2 \\ &\leq \left(\sum_{n=0}^{\infty} |a_n|^2 \Gamma(qn+1) \right) \cdot \left(\sum_{n=0}^{\infty} \frac{|z^n|^2}{\Gamma(qn+1)} \right) \\ &= \|f\|^2_{ML^2(\mathbb{C};q)} E_q(|z|^2). \quad \square \end{aligned}$$

Lemma 4.1 establishes a bound on f in terms of the Mittag-Leffler function of order q. Growth rates for entire functions are often expressed in terms of order and type. Recall the following definition which can be found in [7].

Definition 5.

(a) Let f be an entire function. The function f is said to be of finite order if there is a positive constant $a > 0$ and $r_0 > 0$ for which $|f(z)| < e^{|z|^a}$ for all $|z| > r_0$. If f is of finite order, then the order of f is given by $\lambda = \inf\{a : |f(z)| < e^{|z|^a}$ for $|z|$ sufficiently large$\}$.
(b) If f is an entire function of order a, then f is of finite type if there some $0 < \gamma < \infty$ for which $|f(z)| \leq e^{\gamma|z|^a}$ for all z sufficiently large. For a function of order a and finite type, the quantity $\gamma' = \inf\{\gamma > 0 : |f(z)| < e^{\gamma|z|^a}$ for $|z|$ sufficiently large$\}$ is the type of f.

The functions in the Bargmann–Fock space are of order at most 2, which is the result of the Gaussian measure in (3). Similarly, it will be shown that a bound on the order of a function in $ML^2(\mathbb{C}; q)$ may be established that is connected to the measure in (6). There is a strong connection between the zeros of an entire function and its growth rate [4,7]. The relationship between zero sets and the growth rate will be considered in Theorem 4.8.

The following estimate may be found in [13, Proposition 3.1 and Corollary 3.7], and it will be used to refine Lemma 4.1 by establishing the order and type of entire functions in $ML^2(\mathbb{C};q)$.

Lemma 4.2. *For $q>0$ the Mittag-Leffler function, E_q, is of order $1/q$ and type 1. Specifically, $|E_q(z)| \le Ce^{|z|^{\frac{1}{q}}}$ for some $C>0$.*

Proposition 4.3. *Let $q>0$. If $f \in ML^2(\mathbb{C},q)$, then f is of order at most $\frac{2}{q}$, and if f is of order $\frac{2}{q}$ then it is of type at most $\frac{1}{2}$.*

Proof. By Lemma 4.2, $|E_q(|z|^2)| \le Ce^{|z|^{\frac{2}{q}}}$ for some $C>0$. Therefore, $|f(z)| \le C\|f\|_{ML^2(\mathbb{C};q)}\, e^{\frac{1}{2}|z|^{\frac{2}{q}}}$ by Lemma 4.1, which establishes both the order and type of f. $\square$

The following restriction on order and type for functions in $ML^2(\mathbb{C};q)$ has a well established analogue for the Bargmann–Fock space [3,23,25]. Proposition 4.4 is facilitated by the integral equation developed in Theorem 3.2, where the measure allows for a clear determination of order and type for functions in $ML^2(\mathbb{C};q)$.

Proposition 4.4. *Let $q>0$. If f is entire and of order less than $\frac{2}{q}$, or of order equal to $\frac{2}{q}$ and of type less than $\frac{1}{2}$, then f is in $ML^2(\mathbb{C},q)$.*

Proof. Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function with order less than $\frac{2}{q}$. Therefore, $|f(z)| \le Ce^{|z|^{\frac{2}{q}-\epsilon}}$ for some $\epsilon>0$. By Theorem 3.2 the norm of f may be written as

$$\|f\|^2_{ML^2(\mathbb{C};q)} = \frac{1}{q\pi}\int_{\mathbb{C}} |f(z)|^2|z|^{\frac{2}{q}-2}e^{-|z|^{\frac{2}{q}}}\,dz \le \frac{1}{q\pi}\int_{\mathbb{C}} |C|^2|z|^{\frac{2}{q}-2}e^{2|z|^{\frac{2}{q}-\epsilon}-|z|^{\frac{2}{q}}}\,dz.$$

For sufficiently large $|z|$, $2|z|^{\frac{2}{q}-\epsilon} - |z|^{\frac{2}{q}} < 0$, and the integrand decays exponentially. Consequently, $\|f\|_{ML^2(\mathbb{C};q)} < \infty$, and $f \in ML^2(\mathbb{C};q)$. The case of f being of order $\frac{2}{q}$ with type less than $\frac{1}{2}$ follows by a similar argument. $\square$

Theorem 4.5. *Let $q>0$ and $\frac{2}{q} > \epsilon > 0$. If $\{z_n\}$ is a sequence of complex numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\frac{2}{q}-\epsilon}} < \infty,$$

then $\{z_n\}$ is a zero set for $ML^2_*(\mathbb{C},q)$.

Proof. Let $p = \lfloor \frac{2}{q} - \epsilon \rfloor - 1$ and let

$$F_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}\right).$$

The function,

$$f(z) = \prod_{n=1}^{\infty} F_p\left(\frac{z}{z_n}\right)$$

is of order less than or equal to $\frac{2}{q}-\epsilon$ (see [7, p. 287]). An appeal to Proposition 4.4 completes the proof. $\square$

Theorem 4.6. *Let $\{z_n\}$ be the zero sequence, repeated according to multiplicity and arranged so that $0 < |z_1| \leq |z_2| \leq \ldots$, of a function $f \in ML^2(\mathbb{C}, q)$ such that $f(0) \neq 0$. If $q > 0$ then there exists a positive constant c such that $|z_n| \geq cn^{\frac{q}{2}}$.*

The following proof is a modification of the one found in [25, Section 5.1].

Proof. Fix $r > 0$, such that f has no zero on $|z| = r$ and let $n(r)$ denote the number of zeros in $|z| < r$. By Jensen's formula [7]

$$\sum_{k=1}^{n(r)} \log \frac{r}{|z_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta.$$

By Proposition 4.3, $|f(re^{i\theta})| \leq C\|f\|_{ML^2(\mathbb{C};q)}\, e^{\frac{1}{2}r^{\frac{2}{q}}}$. Hence,

$$\sum_{k=1}^{n(r)} \log \frac{r}{|z_k|} \leq \tilde{C} + \frac{r^{\frac{2}{q}}}{2}, \tag{9}$$

where $\tilde{C} = \log(C\|f\|_{ML^2})$. The inequality in (9) may be rewritten as

$$\prod_{n=1}^{n(r)} \frac{r}{|z_k|} \leq \exp\left(\frac{r^{\frac{2}{q}}}{2} + \tilde{C}\right).$$

If $n < n(r)$, then

$$\sum_{k=1}^{n} \log \frac{r}{|z_k|} \leq \sum_{k=1}^{n(r)} \log \frac{r}{|z_k|}.$$

Hence,

$$\prod_{k=1}^{n} \frac{r}{|z_k|} \leq \prod_{k=1}^{n(r)} \frac{r}{|z_k|}.$$

In addition, if $n > n(r)$, then since $0 < |z_1| \leq |z_2| \leq \ldots$ and $r/|z_k| < 1$ when $k > n(r)$ it follows that

$$\prod_{k=1}^{n} \frac{r}{|z_k|} \leq \prod_{k=1}^{n(r)} \frac{r}{|z_k|}.$$

Therefore, for any n (independent of r) the inequality

$$\prod_{n=1}^{n} \frac{r}{|z_k|} \leq \exp\left(\frac{r^{\frac{2}{q}}}{2} + \tilde{C}\right)$$

holds for all $r > 0$ such that f has no zeros of $|z| = r$. Since $\{|z_n|\}_{n=1}^{\infty}$ is non-decreasing

$$\frac{r^n}{|z_n|^n} \leq \exp\left(\frac{r^{\frac{2}{q}}}{2} + \tilde{C}\right),$$

and therefore,

$$\frac{1}{|z_n|} \leq \frac{1}{r} \exp\left(\frac{r^{\frac{2}{q}}}{2n} + \frac{\tilde{C}}{n}\right). \tag{10}$$

Choose a sequence $\{r_k\}$ such that $r_k \to n^{\frac{q}{2}}$ and f has no zero on $|z| = r_k$. By (10) it follows that

$$\frac{1}{|z_n|} \leq \frac{1}{n^{\frac{q}{2}}} \exp\left(\frac{1}{2} + \frac{\tilde{C}}{n}\right). \quad \square$$

The analysis of zero sets for the Bargmann–Fock space is incomplete and often requires delicate arguments. As described in [25, Section 5.3] only some square lattices will be zero sets for the Bargmann–Fock space. There is a lower bound on the spacing of a square lattice that allows it to be a zero set for the Bargmann–Fock space. As will be seen in the following discussion, order is a sufficient tool to differentiate between the zero sets of Mittag-Leffler spaces of different parameters q.

Definition 6. Let

$$\Lambda_\alpha := \left\{\omega_{mn} = \sqrt{\frac{\pi}{\alpha}}(m + in) \ : \ (m,n) \in \mathbb{Z}^2\right\}$$

denote the square lattice in the complex plane with fundamental region

$$\Omega_\alpha := \left\{z = x + iy \ : \ |x| < \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}, |y| < \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\right\}.$$

The Weierstrass σ-function associated to Λ_α is[3] $\sigma_\alpha(z) = z\prod'_{m,n}\left(1 - \frac{z}{\omega_{m,n}}\right)\exp\left(\frac{z}{\omega_{m,n}} + \frac{1}{2}\frac{z^2}{\omega^2_{m,n}}\right)$ (cf. [25, Equation 1.14]).

Lemma 4.7. *For $q > 1$ and $\alpha > 0$, the function $\sigma_\alpha(z)$ is not a member of $ML^2(\mathbb{C}; q)$.*

Proof. For $1 < q$ since the σ_α-function is of order 2 (cf. [25, Corollary 1.21]) then it cannot lie in the space $ML^2(\mathbb{C}; q)$ since the maximum order of functions in those spaces is $\frac{2}{q}$ (which is less than 2) by Proposition 4.3. $\square$

Lemma 4.7 establishes that the Weierstrass sigma function is not in $ML^2(\mathbb{C}; q)$ for a specific range of q's. This suggests that square lattices may not be zero sets for $ML^2(\mathbb{C}; q)$ when $1 < q$. Theorem 4.8 demonstrates that square lattices are zero sets for $ML^2(\mathbb{C}; q)$ when $0 < q < 1$ but not when $1 < q$. This is significant, since square lattices are fundamental for the description of sampling and interpolation sequences in the Bargmann–Fock space [21,25], and many of the pathological properties of zero sets of the Bargmann–Fock space, such as finite dimensional zero subspaces, utilize square lattices as a foundation [25, Section 5.3].

Theorem 4.8.

(a) *For $0 < q < 1$ every square lattice is a zero set for $ML^2(\mathbb{C}; q)$.*
(b) *For $q > 1$ no square lattice is a zero set for $ML^2(\mathbb{C}; q)$.*
(c) *For $q = 1$ some square lattices are zero sets for $ML^2(\mathbb{C}; 1) = F^2(\mathbb{C})$.*

[3] The notation $\prod'_{n,m}$ indicates that the product is to be taken over all $n, m \in \mathbb{Z}$ except for $(n,m) = (0,0)$.

Proof. (a) For $0 < q < 1$, square lattices, $\Lambda_\alpha = \{z_n\}$, when arranged in order of magnitude satisfy $c\sqrt{n} < |z_n| < C\sqrt{n}$ for some pair constants, $c, C > 0$. Theorem 4.5 demonstrates that Λ_α is a zero set, since $\sum_{z \in \Lambda_\alpha} \frac{1}{|z|^r}$ converges for any $r > 2$. Alternatively, note that σ_α is an entire function of order 2, thus by Proposition 4.4, $\sigma_\alpha \in ML^2(\mathbb{C}; q)$ and Λ_α is a zero set for $ML^2(\mathbb{C}; q)$.

(b) Points in a square lattice, when ordered according to modulus, satisfy $|z_n| < C\sqrt{n}$ for some $C > 0$. However, if $\{z_n\}$ were a zero set for $ML^2(\mathbb{C}; q)$ with $1 < q$, then by Theorem 4.6 $cn^{\frac{q}{2}} < |z_n|$ for some $c > 0$. Since $\frac{q}{2} > \frac{1}{2}$, $|z_n|$ does not grow fast enough to be a zero set for $ML^2(\mathbb{C}; q)$ with $1 < q$.

(c) For $q = 1$, $ML^2(\mathbb{C}; 1) = F^2(\mathbb{C})$. It is well established that there is a minimum spacing required for a square lattice to be a zero set for $F^2(\mathbb{C})$ [25, Lemma 5.6 and Lemma 5.7]. Thus, when the spacing is too small, the lattice is no longer a zero set for the Bargmann–Fock space. □

The square lattice is a limiting case of zero sets for the Bargmann–Fock space. In particular, a square lattice fails the condition for Theorem 4.5. Indeed, $\sum_{n=1}^{\infty} \frac{1}{|z_n|^2} = \infty$ when $\Lambda_\alpha = \{z_n\}$. However, for a wide range of $\alpha > 0$, Λ_α is a zero set for $F^2(\mathbb{C})$. Note that for the same sequence of zeros $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{2+\epsilon}}$ converges for all $\epsilon > 0$. An open question remains as to the determination of a similar limiting configuration of zeros for the Mittag-Leffler space of entire functions. Such a collection of points, $\Omega_{q,\alpha}$, should satisfy

1. $\Omega_{1,\alpha} = \Lambda_\alpha$.
2. $\Omega_{q,\alpha}$ is a zero set for $ML^2(\mathbb{C}; q)$ for sufficiently large $\alpha > 0$.
3. $\Omega_{q,\alpha}$ is a zero set for $ML^2(\mathbb{C}; p)$ for all $0 < p < q$ and all $\alpha > 0$.
4. Letting $\Omega_{q,\alpha} = \{z_n\}$, $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\frac{2}{q}}} = \infty$ and for every $\epsilon > 0$, $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\frac{2}{q}+\epsilon}}$ converges.
5. There is an $\alpha > 0$ such that $\Omega_{q,\alpha}$ is not a zero set for $ML^2(\mathbb{C}; q)$ and there is a finite collection of points $\{a_1, ..., a_n\} \in \Omega_{q,\alpha}$ such that $\Omega_{q,\alpha} \setminus \{a_1, ..., a_n\}$ is a zero set for $ML^2(\mathbb{C}; q)$.

The main challenge to determine limiting configurations lies is establishing Property 5 for such a set. The difficulty in establishing a sufficient $\Omega_{q,\alpha}$ comes from the lack of a corresponding Weierstrass sigma function. The properties of the Weierstrass sigma function arise from leveraging the periodicity of the square lattice in a nontrivial way, and thus establishes the property that $|\sigma_\alpha(z)|e^{-\frac{\alpha}{2}|z|^2}$ is doubly periodic [25, Corollary 1.21]. The double periodicity allows for the examination of zero sets for which Property 5 holds. In particular, $\frac{\sigma_\alpha(z)}{z(z-z_1)}$ is a function in the Bargmann–Fock space, but has two fewer zeros than σ_α. For $q \neq 1$, it is clear that the set $\Omega_{q,\alpha}$ will not have periodic spacing of its points, since that would lead to a square lattice, which cannot be a zero set for $ML^2(\mathbb{C}; q)$ when $1 < q$. One possible first step would be to establish a function analogous to the Weierstrass sigma function which would make the integrand in (4) nonzero and of polynomial growth with a resulting divergent integral.

Proposition 4.9. *For $q > 0$ and $\alpha > 0$, the set*

$$\Omega_{q,\alpha} := \{z \in \mathbb{C} : z^{1/q} \in \Lambda_\alpha\}$$

satisfies Properties 1–4 in the paragraphs above.

Proof. From the definition of $\Omega_{q,\alpha}$, $\Omega_{1,\alpha} = \Lambda_\alpha$. Let $\{z_n\} = \Omega_{q,\alpha}$ be a sequence ordered by modulus. Since $|z_n|^{2/q} \approx n$, Property 4 follows as well.

Property 4 gives the exponent of convergence for $\Omega_{q,\alpha}$ (cf. [4]), and by Theorem 2.6.5 in [4], the function $f_{q,\alpha}(z) := \prod_{n=1}^{\infty} F_p\left(\frac{z}{z_n}\right)$ has order $\frac{2}{q}$ where p is an integer such that $p < \frac{2}{q} < p+1$. If $2/q \notin \mathbb{N}$ then Lemma 2.9.5 in [4] guarantees that $f_{q,\alpha}$ is of finite type, since

$$n(r) := \{z \in \mathbb{C} : f_{q,\alpha}(z) = 0 \text{ and } |z| < r\} = O\left(r^{\frac{2}{q}}\right).$$

Thus, $|f_{q,\alpha}(z)| \leq Ce^{\beta|z|^{2/q}}$ for $\beta > 0$ from which Property 3 follows. Note that for $\lambda > 0$, $f_{q,\lambda\alpha}(z) = f_{q,\alpha}(z\lambda^{q/2})$. Therefore, $|f_{q,\lambda\alpha}(z)| \leq Ce^{(\beta/\lambda)z^{2/q}}$ and there exists $\lambda > 0$ such that $\beta/\lambda < 1/2$. Property 2 then follows from Proposition 4.4.

When $\frac{2}{q} \in \mathbb{N}$, it must be established that the sums $S(r) = \sum_{|z_n|<r} \frac{1}{z_n^{2/q}}$ are bounded to ensure that $f_{p,\alpha}$ is of finite type (cf. [4, Theorem 2.10.1]). When $q = 1$, $\Omega_{q,\alpha} = \Lambda_\alpha$ and the sum, $\sum_{|z_n|<r} \frac{1}{z_n^2}$, can be seen to be bounded through the symmetry of the square lattice. For $q \neq 1$, $z_n^{1/q} \in \Lambda_\alpha$ by definition. Thus, the sums are bounded for the same reason they are bounded for Λ_α. □

The function, $f_{q,\alpha}$, defined in Proposition 4.9 seems to be a viable candidate for a generalized Weierstrass function. A lower bound can be determined when the argument lies outside of a neighborhood of its zeros. Indeed, given any $\epsilon > 0$ there exists $R > 0$ and $\sigma > 0$ such that

$$e^{-|z|^{\frac{2}{q}+\epsilon}} < |f_{q,\alpha}(z)| \tag{11}$$

for all $z \in \mathbb{C}$ satisfying $|z| > R$ and $z \notin B_{|z_n|^{-\sigma}}(z_n)$ (cf. Lemma 2.6.19 [4]). If the lower bound of (11) can be replaced by an equation of the form $Ce^{-\beta|z|^{\frac{2}{q}}}$ then $f_{q,\alpha}$ may be used to prove Property 5 of $\Omega_{q,\alpha}$ just as the Weierstrass sigma function was used to prove Property 5 for Λ_α. However, the authors are not aware of a result of this kind.

5. Mittag-Leffler space of the slitted plane

This section considers the Mittag-Leffler space of the slitted plane, $ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$. The Mittag-Leffler space of the slitted plane is demonstrated to be a more natural analogue of the Bargmann–Fock space than the Mittag-Leffler space of entire functions by demonstrating an adjoint relationship between fractional differentiation and multiplication by z^q in this section. Each function in $ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$ space can be obtained from functions in the former space via the composition $f(z^q)$, where z^q is determined via the principle branch cut of the logarithm [7]. Specifically, Definition 7 defines the Mittag-Leffler space of the slitted plane.

Definition 7. For $q > 0$, the Mittag-Leffler space on the slitted plane is defined as

$$ML^2(\mathbb{C} \setminus \mathbb{R}_-; q) := \{f : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C} \ : f(z^{1/q}) \in ML^2(\mathbb{C}; q)\}$$

i.e.

$$ML^2(\mathbb{C} \setminus \mathbb{R}_-; q) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^{qn} \ : \ \sum_{n=0}^{\infty} |a_n|^2 \Gamma(qn+1) < \infty \right\}.$$

The Mittag-Leffler space of the slitted plane is a RKHS equipped with the inner product

$$\langle f(z),\ g(z) \rangle_{ML^2(\mathbb{C}\setminus\mathbb{R};q)} = \left\langle f(z^{1/q}),\ g(z^{1/q}) \right\rangle_{ML^2(\mathbb{C};q)}.$$

In a sense, even though it is not composed of entire functions, the Mittag-Leffler space of the slitted plane is a more natural generalization of the Bargmann–Fock space. The operation D_*^q is densely defined over $ML^2(\mathbb{C} \setminus \mathbb{R}_-)$ just as $\frac{d}{dz}$ is densely defined over $F^2(\mathbb{C})$ as demonstrated in Proposition 5.2. Moreover, D_*^q can act as a lowering operator for $ML^2(\mathbb{C} \setminus \mathbb{R}_-)$ in a fashion similar to $\frac{d}{dz}$ for the Bargmann–Fock space. Consequently, the structure of the Mittag-Leffler space of the slitted plane bears a stronger resemblance to that of the Bargmann–Fock space.

In previous sections, the Caputo fractional derivative has only been defined for positive real values; however, it is not restricted to real arguments as can be seen in [20]. Since the Mittag-Leffler space of the slitted plane is composed of functions analytic in $\mathbb{C} \setminus \mathbb{R}_-$, there is only one choice of Caputo fractional derivative that agrees with the traditional Caputo fractional derivative on the real line while also yielding a function analytic in $\mathbb{C} \setminus \mathbb{R}_-$. The following definition and theorem from [9] motivates a definition of the Caputo fractional derivative that extends to the rest of the slitted plane.

Definition 8. Let $q > 0$ and let ν be an entire function with power series expansion $\nu(x) = \sum_{n=0}^{\infty} a_n x^n$. Then, the operator $\mathcal{G}^q$ that maps the function ν to the function $\mathcal{G}^q v$ with

$$\mathcal{G}^q \nu(x) := \sum_{n=1}^{\infty} a_n \frac{\Gamma(qn+1)}{\Gamma(q(n-1)+1)} x^{n-1}$$

is called the Gel'fond–Leont'ev operator of order q [9].

Theorem 5.1. *Let $q > 0$ and let ν be an entire function with power series expansion $\nu(x) = \sum_{n=0}^{\infty} a_n x^n$. Moreover, let $f(x) := \nu(x^q)$ for $x \geq 0$. Then $D_*^q f(x) = \mathcal{G}^q v(x^q)$ [9].*

Definition 8 and Theorem 5.1 establish that if a function is expressible as a power series in x^q, the Caputo fractional derivative of the function is simply the term by term Caputo differentiation of the series. This yields a natural and obvious definition for a complex Caputo fractional derivative for functions in the Mittag-Leffler space of the slitted plane.

Definition 9. Let $q > 0$ and $f \in ML^2(\mathbb{C} \setminus \mathbb{R}; q)$ be given by $f(z) = \sum_{n=0}^{\infty} a_n z^{qn}$, then the Caputo fractional derivative of f at $z \in \mathbb{C} \setminus \mathbb{R}_-$, denoted $D_*^q f(z)$, is given by

$$D_*^q f(z) = \sum_{n=1}^{\infty} a_n \frac{\Gamma(qn+1)}{\Gamma(q(n-1)+1)} z^{q(n-1)} \tag{12}$$

The function $D_*^q f$ is called the Caputo fractional derivative of f of order q.

According to Theorem 5.1, Definition 9 agrees with the traditional Caputo fractional derivative for functions in $ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$ for positive reals. Note that $D_*^q f$ is analytic over $\mathbb{C} \setminus \mathbb{R}_-$. Therefore $D_*^q f$ is uniquely determined by its values on the positive reals via the identity theorem [7]. The correspondence between the series definition of the Caputo fractional derivative given in Definition 9 and an integral representation similar to that given in Definition 2 over the complex plane is further explored in Appendix A.

In [2], it was demonstrated that differentiation $\frac{d}{dz}$ and multiplication by z were adjoint operators on the Bargmann–Fock space. In [23], it was further shown that the Gaussian weight is the only continuous radial weight such that differentiation and multiplication by z are adjoints on polynomials (a common domain for the unbounded operators). The next proposition establishes an analogous result for the Caputo fractional derivative and polynomials in z^q.

Definition 10. The operators Z_q and Y_q are defined on $ML^2(\mathbb{C} \setminus \mathbb{R}; q)$ as

$$Z_q : \mathrm{Dom}(Z_q) \subseteq ML^2(\mathbb{C} \setminus \mathbb{R}; q) \to ML^2(\mathbb{C} \setminus \mathbb{R}; q) \text{ where } (Z_q f)(z) = z^q f(z)$$

and

$$Y_q : \mathrm{Dom}(Y_q) \subseteq ML^2(\mathbb{C} \setminus \mathbb{R}; q) \to ML^2(\mathbb{C} \setminus \mathbb{R}; q) \text{ where } (Y_q f)(z) = D_*^q f(z).$$

The domains for Z_q and Y_q are defined explicitly as

$$\mathrm{Dom}(Z_q) := \{f \in ML^2(\mathbb{C} \setminus \mathbb{R}_-) : z^q f \in ML^2(\mathbb{C} \setminus \mathbb{R}_-)\}$$

and

$$\mathrm{Dom}(Y_q) := \{f \in ML^2(\mathbb{C} \setminus \mathbb{R}_-) : D^q_* f \in ML^2(\mathbb{C} \setminus \mathbb{R}_-)\}.$$

Proposition 5.2. *Let $q > 0$. The operators Z_q and Y_q are closed, moreover $Z_q^* = Y_q$, and $Y_q^* = Z_q$.*

Proof. The operator Z_q is a multiplication operator over a RKHS, which means it is closed [22]. Moreover, since polynomials in z^q are dense in $ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$, Z_q is densely defined. Therefore, Z_q^*, the unbounded adjoint of Z_q, is both densely defined and closed [18]. The domain of Z_q^* is given by

$$\mathrm{Dom}(Z_q^*) = \{g \in ML^2(\mathbb{C} \setminus \mathbb{R}_-; q) : L_g(f) := \langle Z_q f, g\rangle \text{ is continuous on } \mathrm{Dom}(Z_q)\}$$

(cf. [18, Chapter 5]). For each $g \in \mathrm{Dom}(Z_q^*)$ there is a unique $h \in ML^2(\mathbb{C}\setminus\mathbb{R}_-; q)$ for which $\langle Z_q f, g\rangle = \langle f, h\rangle$ for all $f \in \mathrm{Dom}(Z_q)$. The vector h is then declared as $Z_q^* g$ [18].

For $f \in \mathrm{Dom}(Z_q)$ and $g \in ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$ with $f(z) = \sum_{n=0}^\infty a_n z^{qn}$ and $g(z) = \sum_{n=0}^\infty b_n z^{qn}$ consider the inner product,

$$\langle Z_q f, g\rangle = \sum_{n=0}^{\infty} a_{n-1} \overline{b_n} \Gamma(qn + 1).$$

If a function h were to satisfy $\langle f, h\rangle = \sum_{n=0}^\infty a_{n-1}\overline{b_n}\Gamma(qn + 1)$ for all $f \in \mathrm{Dom}(Z_q)$, then $h(z) = \sum_{n=0}^\infty b_{n+1} \frac{\Gamma(q(n+1)+1)}{\Gamma(qn+1)} z^{qn}$ by the density of $\mathrm{Dom}(Z_q)$. Through the inspection of (12), $h(z) = D^q_* g(z) = Y_q g$. This places a constraint on $\mathrm{Dom}(Z_q^*)$ to consist of functions $g \in ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$ for which $D^q_* g \in ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$ (i.e. $\mathrm{Dom}(Z_q^*) \subset \mathrm{Dom}(Y_q)$). Moreover, if $D^q_* g \in ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$, then $L_g(f)$ is a continuous functional (i.e. $\mathrm{Dom}(Y_q) \subset \mathrm{Dom}(Z_q^*)$). Thus, $\mathrm{Dom}(Z_q^*) = \mathrm{Dom}(Y_q)$, and $Z_q^* = Y_q$. Since Z_q is closed, $Y_q^* = Z_q^{**} = Z_q$ [18]. □

As seen in [23] the Bargmann–Fock space is essentially the unique space with a radially symmetric measure such that multiplication by z and differentiation with respect to z are adjoint operations to each other. The analogous result holds for the slitted Mittag-Leffler space as established by Theorem 5.3.

Theorem 5.3. *For $q > 0$ the function $w(z) = \frac{1}{q\pi}|z|^{\frac{2}{q}-2} e^{-|z|^{\frac{2}{q}}}$ is the unique radial weight, continuous on $\mathbb{C} \setminus \{0\}$, such that for polynomials $p(z) = a_n z^{qn} + \cdots + a_0$ and $s(z) = b_m z^{qm} + \cdots + b_0$ under the inner product*

$$\langle p(z), s(z)\rangle = \int_{\mathbb{C}} p(z^{1/q})\overline{s(z^{1/q})} w(z) dz \tag{13}$$

the operations of multiplication by z^q and Caputo differentiation are adjoint, and the function $f \equiv 1$ has norm 1.

Proof. Consider the monomials in z^q, $p(z) = z^{q(n-1)}$ and $s(z) = z^{qm}$, and suppose that Z_q and D^q_* are adjoint operations on ML^2 under the inner product (13). Specifically,

$$\langle Z_q p(z), s(z)\rangle = \langle p(z), D^q_* s(z)\rangle. \tag{14}$$

Under condition (14)

$$\int_{\mathbb{C}} z^n \bar{z}^m w(z) dz = \frac{\Gamma(qm+1)}{\Gamma(q(m-1)+1)} \int_{\mathbb{C}} z^{n-1} \bar{z}^{m-1} w(z) dz \tag{15}$$

where dz is the normal area measure in $\mathbb{C}$. Let $z = re^{i\theta}$ and suppose that $w(z)$ is a radially symmetric weight (i.e. $w(z) = w(r)$). The integral relation in (15) becomes

$$2\pi \int_0^\infty r^{2n+1} w(r) dr = 2\pi \frac{\Gamma(qn+1)}{\Gamma(q(n-1)+1)} \int_0^\infty r^{2n-1} w(r) dr.$$

Letting

$$\gamma_n := 2\pi \int_0^\infty r^{2n+1} w(r) dr, \qquad \text{for } n \in \mathbb{N}$$

the following recursive relation is established

$$\gamma_n = \frac{\Gamma(qn+1)}{\Gamma(q(n-1)+1)} \gamma_{n-1} \qquad \text{for } n \in \mathbb{N}. \tag{16}$$

Since the function $f \equiv 1$ has norm 1, then

$$1 = \int_{\mathbb{C}} w(z) dz = 2\pi \int_0^\infty r w(r) dr = \gamma_0.$$

From the recursive relation in (16) on γ_n, it can be concluded that $\gamma_n = \Gamma(qn+1)$. Hence, from the adjoint condition in (14) along with the condition that the function $f \equiv 1$ has norm 1, $\|z^{qn}\|^2 = \Gamma(qn+1)$. Assuming continuity of the measure on $\mathbb{C} \setminus \{0\}$, noting for all functions $f(z) \in ML^2(\mathbb{C} \setminus \mathbb{R}^-; q)$ that $\int_{\mathbb{C}} f(z)w(z)dz = \int_{\mathbb{C}\setminus\{0\}} f(z)w(z)dz$, and by performing the same calculations as in (5) and (6) it is established that

$$w(z) = \frac{1}{q\pi} |z|^{\frac{2}{q}-2} e^{-|z|^{\frac{2}{q}}} \qquad \text{for } q > 0. \quad \square$$

Theorem 5.3 is stronger than Theorem 3.2 in that it establishes the measure to be the unique measure that yields the Mittag-Leffler space of the slitted plane. The establishment of this uniqueness theorem was made possible through the adjoint relationship between multiplication by z^q and the Caputo fractional derivative in this space, which in itself justifies the investigation of the Mittag-Leffler space of the slitted plane in addition to the Mittag-Leffler space of entire functions.

6. Conclusion

This paper introduced and examined the function theoretic properties of two RKHSs associated with the Mittag-Leffler functions. Each space was obtained through the invocation of the Aroszajn-Moore theorem and two methods were used to determine integral representations of their norms. Collections of zero sets were studied for the Mittag-Leffler space of entire functions with a particular emphasis on square lattices which are important for the Bargmann–Fock space. A replacement for the square lattice as important zero sets for the Mittag-Leffler space of entire functions was developed when $q \neq 1$.

Following the study of the Mittag-Leffler space of entire functions, the Mittag-Leffler space of the slitted plane was investigated, where the Mittag-Leffler space of the slitted plane allows for a generalization of the classical adjoint relationship of multiplication by z and $\frac{d}{dz}$ present in the Bargmann–Fock space. In particular, it was demonstrated that in the Mittag-Leffler space of the slitted plane that multiplication by z^q and D_*^q are closed densely defined adjoints of one another.

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Appendix A. Caputo derivatives of a complex argument

This appendix explores the direct extension of the Riemann–Liouville fractional integral to complex arguments and compares the resulting Caputo fractional derivative to the series definition given in (12). Similar results may be found in [9,20]. The results of this appendix are well known in the community, however the authors were unable to locate a reference that included a treatment comparing the two definitions of the Caputo fractional derivative given by (12) and (A.1) for complex arguments. Therefore, this appendix is included for the sake of completeness. The development in this section leverages the construction found in [20] and makes adjustments suitable for the Caputo fractional derivative.

Suppose a function f is defined on the slitted complex plane $\mathbb{C} \setminus \mathbb{R}_-$. Note that $\mathbb{C} \setminus \mathbb{R}_-$ is starlike relative to the origin [7]. Therefore, line segments connecting points in $\mathbb{C} \setminus \mathbb{R}_-$ to the origin remain in the set $\mathbb{C} \setminus \mathbb{R}_-$.

Definition 11. Let $\alpha > 0$, define the Riemann–Liouville fractional integral of a function $f : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ of order α at $z \in \mathbb{C} \setminus \mathbb{R}_-$ as

$$(I_0^\alpha f)(z) := \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\alpha}}\, d\xi$$

where the integration is over the line-segment from 0 to z.

The use of the slitted complex plane arises from the selection of the principle branch of the multivalued function $(z-\xi)^{1-\alpha}$. Moreover, since integration is taken over a line segment connecting the origin and the point z, $\theta := \arg(z) = \arg(\xi)$, where $-\pi < \theta < \pi$. The following is given as an alternate definition to (12) for the Caputo fractional derivative of a complex argument.

Definition 12. Let $f : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ and $m - 1 < q < m$ where $m \in \mathbb{N}$ and $q > 0$. Define

$$(\mathcal{D}^q f)(z) = I_0^{m-q} f^{(m)}(z) \tag{A.1}$$

to be the Caputo fractional derivative of complex argument.

Theorem A.1 demonstrates that the Caputo fractional derivative given in (12) coincides with that given in (A.1). Therefore, the integral representation and the series representation for the Caputo fractional derivative agree for complex arguments.

Theorem A.1. *Let $m \in \mathbb{N}$ and let $q \in (m-1, m)$. If $f \in ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$, then $\mathcal{D}^q f = D_*^q f$.*

Proof. Given a function $f(z) = \sum_{n=0}^{\infty} a_n z^{qn}$, it will be demonstrated that $\mathcal{D}^q f(z) = D_*^q f(z)$ for all $z \in \mathbb{T} \setminus \{-1\}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. By the identity theorem (cf. [7]) the $\mathcal{D}^q f(z) = D_*^q f(z)$ for all $z \in \mathbb{C} \setminus \mathbb{R}_-$. Moreover, since the collection of functions $e_n(z) := z^{qn}$ for $n \in \mathbb{N}$ constitutes a basis for $ML^2(\mathbb{C} \setminus \mathbb{R}_-; q)$, it is sufficient to establish the theorem for each e_n and then apply the result term by term to f via a dominated convergence argument. Consider the power function $f_d(z) := z^d$ where $d \in \mathbb{R}^+$ and let $\alpha > 0$. In addition, assume that $d > m$. Note that

$$(I_0^\alpha f^{(m)})(z) = \frac{\prod_{i=0}^{m-1}(d-i)}{\Gamma(\alpha)} \int_0^z \frac{\xi^{d-m}}{(z-\xi)^{1-\alpha}}\, d\xi. \tag{A.2}$$

Since $z = e^{i\theta} \in \mathbb{T} \setminus \{-1\}$, the path connecting 0 to z may be parameterized as $\xi = (1-\rho)e^{i\theta}$, $\rho \in [0, 1]$. Therefore, after a change of variables (A.2) becomes

$$(I_0^\alpha f_d^{(m)})(z) = \frac{e^{i(\alpha+d-m)\theta} \prod_{i=0}^{m-1}(d-i)}{\Gamma(\alpha)} \int_0^1 \rho^{\alpha-1}(1-\rho)^{(d-(m-1))-1}\, d\rho.$$

Utilizing the following well-known identity of the Beta function (cf. [13, Equation A.2.4]):

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

and properties of the Gamma function

$$\begin{aligned}(I_0^\alpha f_d^{(m)})(z) &= \frac{\prod_{i=0}^{m-1}(d-i) \cdot \Gamma(d-(m-1))}{\Gamma(\alpha+d-(m-1))} z^{\alpha+d-m} \\ &= \frac{\Gamma(d+1)}{\Gamma(\alpha+d-(m-1))} z^{\alpha+d-m}.\end{aligned}$$

Let $\alpha = m - q$ and $d = qn$ for $n \in \mathbb{N}$ the equation,

$$\mathcal{D}^q(z^{qn}) = \frac{\Gamma(qn+1)}{\Gamma(q(n-1)+1)} z^{q(n-1)} \quad \text{for } z \in \mathbb{T} \setminus \{-1\},$$

is established by the argument above. For $q > 0$ (12) yields

$$D_*^q(z^{qn}) = \frac{\Gamma(qn+1)}{\Gamma(q(n-1)+1)} z^{q(n-1)} \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}_-.$$

Since the two functions agree on $\mathbb{T} \setminus \{-1\}$, which has cluster points, the functions $\mathcal{D}^q e_n$ and $D_*^q e_n$ must agree over $\mathbb{C} \setminus \mathbb{R}_-$ by the identity theorem. Hence, $\mathcal{D}^q f(z) = D_*^q f(z)$ for all $z \in \mathbb{C} \setminus \mathbb{R}_-$ by a dominated convergence argument. □

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