A Topologically Inspired Path-Following Method With Intermittent State Feedback

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Abstract—Autonomous systems often operate in environments where state feedback may not be available, such as in anti-access and area denial environments. In these environments, it is often required that an agent track a path, despite interruptions in state feedback. As a result, the class of Relay-Explorer problems has emerged, where a switched system approach is used to account for intermittent state feedback. Past work on these problems established a framework for developing dwell-time conditions for stable tracking using these methods. However, existing work only applies to a limited class of reference paths and feedback region geometries. This letter advances a topologically inspired method for guaranteeing re-acquisition of feedback for nearly arbitrary geometries in arbitrary dimensions, all while relaxing the dwell-time conditions and retaining the uniformly ultimately bounded stability result from preceding work. Numerical experiments in the plane demonstrate an increase of hundreds of percentage points—even for fairly generic geometries—in the tracking error the agent could afford, using the proposed method, without sacrificing stability.

Index Terms—Computational geometry, hybrid logical/dynamical planning and verification, motion and path planning.

I. INTRODUCTION

STATE feedback is a critical component in designing path-planning methods used for guidance, navigation, and control of autonomous vehicles. Factors such as task definition, operating environment, sensor modality, and adversarial effects may result in intermittent state feedback, inhibiting a system’s ability to achieve its task. For mobile platforms, one prominent approach to overcoming difficulties with obtaining state feedback is to design controllers and planners that enable computation of the state from available observations (see [1] in the context of visual servoing, and [2] in a SLAM setting). Other approaches arise in network control, where limitations on communication and communication delays affect the stability of the system (see [3] in the context of time-varying connectivity and [4] for an event triggered approach).

An emerging body of work embraces these difficulties by considering them as a class of Relay-Explorer problems. Using a switched system approach, control laws are constructed which use state feedback for navigation when available (stabilizable mode), and open-loop state estimates otherwise (unstable mode). For example, in a single agent setting [5], to ensure stability, the agent has to repeatedly reacquire feedback by entering a fixed region where feedback is available. A time-varying feedback region may naturally arise in multi-agent settings. For example in [6], some agents may have the capability of relaying feedback information to feedback-denied members of their team by moving into communication radius.

Similar to the framework in [5], this letter considers a single agent tasked with tracking a desired path that may lie outside a known feedback region. State feedback is available when the agent is inside this region, and unavailable otherwise. Since the desired path may lie outside of the feedback region, the agent dead-reckons when feedback is not available. To achieve the task, a path-planning algorithm is designed to generate an auxiliary trajectory for the agent to track. The instabilities inherent to dead-reckoning impede the agent’s ability to return to the feedback region. This forces the auxiliary trajectory to alternate between following the desired path and returning to the feedback region. This framework relies on a Lyapunov-based switched system analysis [7] to derive dwell-time conditions dictating the duration a system can remain in each operating mode while ensuring stability.

The predominant factor affecting dwell-time conditions in this setting is the need to guarantee re-acquisition of state feedback. To furnish this guarantee, results in [5] and [8] require the agent to dead-reckon to a point where the region of state uncertainty is contained within the feedback region—the Inscribed Ball Criterion (IBC). Crucially, these results only consider circular feedback regions where the dwell-time conditions only depend on the distance of the agent to the feedback region. However, applying the same approach to more general geometries—the expected norm in real-life applications—results in unnecessarily conservative bounds. For example, when an agent moves towards a long and narrow rectangular feedback region, Fig. 1 shows that a better strategy is to aim at a point beyond the region, affording a larger error margin at the target point. Intuitively, small perturbations in the shape of this region should not result in a change of strategy, motivating a topological approach. Moreover, it is clear that the preceding considerations could not be easily replicated for an arbitrary geometry of the feedback region, because of complex interactions between local properties (e.g., curvature at nearby boundary points) and global properties (e.g., concavities, spirals, etc.).

This letter addresses the need for generating re-entry guarantees for arbitrary geometries of the feedback region. To this end,
II. PROBLEM STATEMENT

A. Notation

For any integer \( D \geq 1 \), the Euclidean distance between \( p, q \in \mathbb{R}^D \) is denoted by \( \text{dist}(p, q) \triangleq \|p - q\|_2 \). Let \( S^{D-1} \) and \( B^D \) denote the Euclidean unit sphere and closed unit ball in \( \mathbb{R}^D \), respectively. Given \( p \in \mathbb{R}^D \) and \( r > 0 \), the open ball \( B_r(p) \) of radius \( r \) about \( p \) is the set of \( q \in \mathbb{R}^D \) with \( \text{dist}(p, q) < r \). The shortest distance from a point to a set is \( \text{dist}(p, A) \triangleq \inf\{\text{dist}(p, q) : q \in A\} \). The diameter of \( A \subset \mathbb{R}^D \) is \( \text{diam}(A) \triangleq \sup\{\text{dist}(p, q) : p, q \in A\} \). The closure of \( A \subset \mathbb{R}^D \) is denoted by \( \overline{A}(A) \). A point is an interior point of a set if \( B_r(p) \subseteq A \) for some \( r > 0 \). The set of interior points of \( A \subset \mathbb{R}^D \) is denoted by \( A^\circ \). A different notion of interior is associated with embeddings of \( S^{D-1} \) in \( \mathbb{R}^D \), see (11).

Fig. 1. Given a feedback region \( F \), an initial starting point \( p \), and a target point \( q_1 \in F \), the ball \( B(p, q_1) \) represents the maximum allowed region of uncertainty based on IBC. For this geometry, a much larger region of uncertainty may be allowed for an overwhelming majority of starting points \( p \), provided one does not insist on \( q_1 \) to lie in \( F \), replacing \( q_1 \) with \( q_2 \), for example.

Fig. 2. The auxiliary trajectory \( x_\pi \), defined by a path plan \( \pi = (o, a, p, q) \), is superimposed over the desired path \( \pi_\alpha \), for a generic feedback region \( F \) (see Definition 1). Note that the point \( q \) need not lie in \( F \).

B. Agent Dynamics

This letter considers an agent with dynamics modeled by

\[
\dot{x} = f(x, t) + \zeta(t) + d(t),
\]

where \( x : \mathbb{R}^{20} \to \mathbb{R}^D, D \geq 2 \), denotes the state; \( f : \mathbb{R}^D \times \mathbb{R}^{20} \to \mathbb{R}^D \) denotes locally Lipschitz drift dynamics; \( d : \mathbb{R}^{20} \to \mathbb{R}^D \) denotes an exogenous disturbance; and \( \zeta : \mathbb{R}^{20} \to \mathbb{R}^D \) denotes a control input. The following assumption is used in the subsequent development.

Assumption 1: The exogenous disturbance \( d \) satisfies

\[
\|d(t)\| \leq \bar{d} \text{ for all } t \in [0, \bar{d}],
\]

where \( \bar{d} \in \mathbb{R}^{20} \) is known.

C. Control Objective

Let \( F \subset \mathbb{R}^D \) denote a known region where state feedback is available, i.e., state feedback is available to the agent if and only if \( x \in F \). The feedback region \( F \) is modeled as the closure of the interior of a polyhedral sphere \( C \). Equivalently, \( F \) is the image of an embedding of \( \mathbb{R}^D \) in \( \mathbb{R}^D \) (see Section IV). The feedback-denied region, \( F^c \triangleq \mathbb{R}^D \setminus F \), is the set of states where feedback is not available.

The agent is tasked with following a desired polygonal path \( X_d \), which is provided as a sequence of way points \( \{P_0, \ldots, P_M\} \) in \( \mathbb{R}^D \), some of which may lie outside of \( F \). Repeated dead-reckoning along the sequence of way points is inherently unstable outside of \( F \). This motivates an approach where the agent follows a sequence of auxiliary trajectories, relaying between the desired path and the feedback region (see Fig. 2), to ensure the error system remains bounded.

Definition 1 (Auxiliary Trajectory): There are two types of auxiliary trajectories. The auxiliary trajectory \( x_\pi : \mathbb{R}^{20} \to \mathbb{R}^D \) with path plan \( \pi = (n, o, p, q) \), is defined as the concatenation of three trajectories determined by four way points: \( n \) is the point of departure from the feedback region; \( o \) is the first point along a segment of \( X_d \) the agent selects to follow; \( p \) is the point of departure from \( X_d \); and \( q \) is the target point for the return trajectory to the feedback region. From \( n \) to \( o \), and from \( p \) to \( q \), \( x_\pi \) restricts to straight line trajectories\(^1\) with constant speed \( v_0 \). From \( o \) to \( p \), \( x_\pi \) coincides with the desired path \( X_d \), with a piecewise linear parametrization of constant speed \( v_0 \). For the second type of auxiliary trajectory—with path plan \( \pi = (p, q) \)—set

\(^1\)Curved trajectories may improve performance, but outside our scope.
$x_n$ to coincide with a linearly parametrized line segment from $p$ to $q$ with constant speed $v_0$.

Remark 1: Plans $\pi = (p,q)$ are used for acquisition of feedback from points $p \notin \mathcal{F}$, while plans $\pi = (n,o,p,q)$ are used for tracking $X_d$ and reacquiring feedback. A plan terminates the moment feedback is reacquired.

Assumption 2: It is assumed that feedback acquisition is instantaneous upon re-entry into the feedback region.

Remark 2: Between the executions of two plans, the agent travels through $\mathcal{F}$, taking advantage of the available feedback. Two cases may occur. If the next point $P_m$ along $X_d$ lies in $\mathcal{F}$, there is no need for auxiliary planning. Otherwise, the agent travels to the point $n \in \mathcal{F}$ closest to $P_m$.

To quantify the tracking objectives, define

\[ e \triangleq x - x_n, \quad \dot{e} \triangleq \dot{x} - x_n, \quad \ddot{e} \triangleq \ddot{x} - x_n, \quad (2) \]

where $\dot{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^D$ is the agent’s open-loop state estimate, and $e, \dot{e}, \ddot{e} : \mathbb{R}_{\geq 0} \to \mathbb{R}^D$ are the actual tracking error, estimated tracking error, and state estimation error, respectively. The challenge is to regulate the norm of the actual tracking error to remain, eventually, below a prescribed bound.

D. Switched Controller

Let $S \triangleq \{a,u\}$ be the set of indices denoting the operating modes, where $a$ and $u$ correspond to the modes where feedback is available and unavailable, respectively. Mode $a$ is active when $x \in \mathcal{F}^a$. Mode $u$ is active otherwise. Let $\sigma(x) \in S$ denote the switching signal indicating the active subsystem. The control input takes the form $\zeta \triangleq \zeta_a(x,t)$ when $\sigma(x) = a$, and $\zeta \triangleq \zeta_u(\tilde{x},t)$ when $\sigma(x) = u$, where $\zeta_a, \zeta_u : \mathbb{R}^D \times \mathbb{R}_{\geq 0} \to \mathbb{R}^D$ are the control inputs when feedback is available and unavailable, respectively. The closed loop error system for controllers of this kind were studied in [5], Section IV.

III. ERROR BOUNDS

To facilitate the analysis, the $i^{th}$ instant when $\sigma$ switches from $u$ to $a$ is denoted by $t_i^u \in \mathbb{R}_{\geq 0}$ for all $i \in \mathbb{N}$, i.e., the instant the agent enters the interior of the feedback region. The $i^{th}$ instant when $\sigma$ switches from $a$ to $u$ is denoted by $t_i^a \in \mathbb{R}_{\geq 0}$, i.e., the instant the agent exits $\mathcal{F}^a$. Based on the switching instants, dwell-times of the $i^{th}$ activation of the subsystems $a$ and $u$ are defined as $\Delta t_i^a \triangleq t_i^a - t_i^u$ and $\Delta t_i^u \triangleq t_{i+1}^u - t_i^a$, respectively.

To analyze the switched system, candidate Lyapunov-like functions are defined as

\[ V_e \triangleq \frac{1}{2} \| e \|^2, \quad V_{\dot{e}} \triangleq \frac{1}{2} \| \dot{e} \|^2, \quad V_{\ddot{e}} \triangleq \frac{1}{2} \| \ddot{e} \|^2, \quad (3) \]

where $V_e, V_{\dot{e}}, V_{\ddot{e}} : \mathbb{R}^D \to \mathbb{R}_{\geq 0}$. To ensure a bound on the error system, the following assumption is made.

Assumption 3: Based on the design of the control input in Section II-D, it is assumed that the time derivatives of (3) yields

\[ \dot{V}_e \leq -2\lambda_a V_e, \quad \ddot{V}_e \leq -2\lambda_u V_e, \quad (4) \]

\[ \dot{V}_e \leq \left\{ \begin{array}{ll} -2\lambda_a V_e, & \sigma = a, \\ 2\lambda_u V_e + \delta, & \sigma = u, \end{array} \right. \quad (5) \]

where $\lambda_a, \lambda_u, \delta \in \mathbb{R}_{\geq 0}$ are known constants.\(^2\)

While the agent is in the feedback-denied region (i.e., $\sigma = u$), solving the ordinary differential inequalities in (4)-(6) and substituting in (2) and (3) yields

\[ \| e(t) \| \leq \| e(t_f^u) \| e^{-\lambda_a \Delta t}, \quad (7) \]

\[ \| \dot{e}(t) \|^2 \leq \| \dot{e}(t_f^u) \|^2 e^{2\lambda_a \Delta t} + \frac{\delta}{\lambda_u} e^{2\lambda_u \Delta t - 1}, \quad (8) \]

for all $t \in [t_i^u, t_{i+1}^u]$, where $\Delta t = t - t_i^u$. Since $\dot{e} = \ddot{e} + \ddot{e}$, and using the reset maps\(^3\) from [5], the bounds in (7) and (8) yield

\[ \| e(t) \| \leq \rho(t - t_i^u), \quad t \in [t_i^u, t_{i+1}^u), \quad (9) \]

where

\[ (\rho(\Delta t))^2 \triangleq \frac{\delta}{\lambda_u} e^{2\lambda_u \Delta t - 1}, \quad (10) \]

and $\rho(\Delta t)$ is referred to as the radius of uncertainty at time $t$.

This bound holds for any trajectory under any controller satisfying Assumption 3, enabling the development of a dwell-time condition (Theorem 3).

IV. CRITERIA FOR GUARANTEED RE-ENTRY

A. Preliminaries: Embedded Spheres in Euclidean Space

An embedded sphere in $\mathbb{R}^D$ is defined to be the image $C \triangleq \gamma(S^{D-1})$ of an injective continuous map $\gamma : S^{D-1} \to \mathbb{R}^D$. For $D = 2$, the following classical result may be applied.

Theorem 1 (Jordan-Schönflies, see [9], Thm. EI): An injective continuous map $\gamma : S^1 \to \mathbb{R}^2$ extends to a homeomorphism $\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$, a continuous map with continuous inverse satisfying $\Gamma(p) = \gamma(p)$ for all $p \in S^1$. The mapping $\Gamma$ is called a Schönflies extension of $\gamma$.

For higher dimensions $D > 2$, Schönflies extensions exist under the additional condition that the embedding $\gamma$ is collared [10], [11]. The embedding $\gamma$ is collared if there is an injective continuous map $\gamma : S^{D-1} \times [-1,1] \to \mathbb{R}^D$ such that $\gamma(p,0) = \gamma(p)$ for all $p \in S^D-1$. It is well-known that polyhedral—and, more generally, piecewise-regular—maps $\gamma$ (with finitely many faces) are collared (see §1I in [12]).

The interior and exterior regions of a collared embedded sphere $C$ are defined as

\[ \text{int}(C) \triangleq \Gamma((B^\delta)^D), \quad \text{ext}(C) \triangleq \Gamma(\mathbb{R}^D \setminus B^D), \quad (11) \]

where $\Gamma$ is any Schönflies extension of $\gamma$. It is important to note $\text{int}(C)$ and $\text{ext}(C)$ do not depend on the choice of map $\gamma$—only on its image, $C$. They are also independent of the choice of the Schönflies extension $\Gamma$. Moreover, the boundary sets $\partial \text{int}(C)$ and $\partial \text{ext}(C)$ coincide with $C$, and $C$ separates every point $p \in \text{int}(C)$ from every point $q \in \text{ext}(C)$ in the sense that

\(^2\)An example of a controller satisfying the assumptions made thus far is given in Section VI of [5] where values for $\lambda_a, \lambda_u, \lambda_u$, and $\delta$ are shown to be functions of the agent dynamics and the disturbance bound $d$.

\(^3\)Upon each re-entry into $\mathcal{F}$, it is possible to reset the auxiliary path $x_n$ to a new path and have $\hat{x}(t_{i+1}^u) = x(t_{i+1}^u)$ at $t_{i+1}^u$. In other words, the auxiliary path is updated based on the re-entry location of the agent instead of having to travel to the desired re-entry location before returning to the desired path to follow. Resetting the errors to zero results in the elimination of the minimum dwell-time condition and the vanishing of the initial conditions from the maximum dwell-time condition.
any continuous curve from \( p \) to \( q \) must intersect \( C \). For the rest of this letter we make the following assumption.

**Assumption 4.** The feedback region \( \mathcal{F} \) is the closure of \( \text{int}(C) \), where \( C \) is a collared embedded sphere in \( \mathbb{R}^D \).

### B. Construction of a “Target Region”

In the notation of Section III, given \( t_i^p \), consider a plan \( \pi = (n, o, p, q) \). Alternatively, consider \( \pi = (p, q) \), while setting \( t_i^q = 0 \). Let \( x_{\pi} \) be the associated auxiliary trajectory. Let \( p = x_\pi(t_i^p) \) and \( q = x_\pi(t_i^q) \), then the initial and final uncertainty radii for the plan \( \pi \) are defined as

\[
\rho_{\text{init}}(\pi) \triangleq \rho(t_i^p - t_i^q), \quad \rho_{\text{fin}}(\pi) \triangleq \rho(t_i^q - t_i^p),
\]

where \( \rho \) is defined in (10). The regions of uncertainty are

\[
U_{\text{init}}(\pi) \triangleq B_{\rho_{\text{init}}(\pi)}(p), \quad U_{\text{fin}}(\pi) \triangleq B_{\rho_{\text{fin}}(\pi)}(q).
\]

In [5] and [8], re-entry is guaranteed by selecting \( q \in \mathcal{F} \) so that \( U_{\text{fin}}(\pi) \subseteq \mathcal{F} \). This method is referred to as the **Inscribed Ball Criterion** (IBC). Treating this inclusion as a constraint results in a bound on \( \rho_{\text{fin}}(\pi) \), and hence also on \( \rho_{\text{init}}(\pi) \), since the function \( \rho \) is known. The example in Fig. 1 illustrates the need for a target region much larger than \( \mathcal{F} \), in which to fit \( U_{\text{fin}}(\pi) \), to obtain less conservative bounds. In this section such target regions are introduced.

Recall that \( \mathcal{F}^* = \text{int}(C) \). Given a point \( p \in \text{ext}(C) \) and a closed, connected region \( R \subseteq \mathbb{R}^D \) with \( p \in R \), define \( T_{C,R}(p) \) to be the collection of all points \( q \in R \) for which any smooth curve \( \gamma : [0, 1] \to R \) from \( p \) to \( q \) must pass through a point of \( C \). Let \( E_{C,R}(p) \) denote the set of all \( y \in C \) such that, for some \( q \in T_{C,R}(p) \) there is a smooth curve \( \gamma : [0, 1] \to R \) from \( p \) to \( q \) that crosses \( C \) exactly once at \( y \).

**Remark 3:** In the above, one is allowed to restrict attention to smooth curves \( \gamma \) which intersect \( C \) transversely, i.e., their tangent line at any point of intersection with \( C \) and the tangent to \( C \) at that point span \( \mathbb{R}^D \) (see §10 of [12]).

**Lemma 1:** If \( q \in T_{C,R}(p) \), then any curve in \( R \) from \( p \) to \( q \) must pass through a point of \( E_{C,R}(p) \).

**Proof:** Suppose a curve \( \gamma : [0, 1] \to R \) starts at point \( p = \gamma(0) \) and terminates at the point \( q = \gamma(1) \). Let \( t^* \triangleq \inf \{t \in [0, 1] : \gamma(t) \in C\} \) be the first time \( C \) crosses \( \gamma \). One may assume \( \gamma \) only crosses \( C \) transversely. Set \( q^* \triangleq \gamma(t^*) \), and let \( \mathcal{U} \) be a neighborhood of \( q^* \) not containing any other intersection point of \( \gamma \) and \( C \) such that \( C \cap \mathcal{U} \) is a single interval. Find \( \Delta t > 0 \) such that \( \gamma([t^* + \Delta t, t^*]) \subseteq \mathcal{U} \) and now set \( q'' \triangleq \gamma(t^* + \Delta t/2) \). Since the curve \( \gamma' \triangleq \gamma|_{[t^*, t^* + \Delta t/2]} \) crosses \( C \) exactly once, one finds that \( q'' \in \text{int}(C) \) and \( q'' \in E_{C,R}(p) \), as required (see Fig. 3 (left)).

### C. Guarantee of Re-Entry Into the Feedback Region

Assume once more the agent is executing a plan \( \pi = (n, o, p, q) \) over the time period \( t \in [t_i^p, t_i^q) \). At time \( t_i^p \), the agent departs in the direction of \( \mathcal{F} \) by dead-reckoning to a point \( q \in D \) (note that this is possible for \( q \) to lie in \( \mathcal{F} \)). For all \( t \in [t_i^p, t_i^q) \), \( t_i^p - t_i^q = \frac{4v_0}{v_0^2 - p} \), one has \( x_\pi(t) = p + (t - t_i^p)v \), where \( v = v_0 - \frac{g - p}{\sqrt{v_0^2 - p}} \). The true position \( x(t) \) is guaranteed to lie in the ball \( B_{\rho}(\Delta t)(x_\pi(t)) \), where \( \Delta t \) and \( \rho \) are defined in (7) and (10), respectively.

![Fig. 3. Illustration of the proofs of Lemma 1 (left) and Theorem 2 (right).](image-url)

**Theorem 2:** Suppose \( U(\pi) \triangleq \bigcup_{t \in [t_i^p, t_i^q]} B_{\rho(t - t_i^p)}(x_\pi(t)) \) is contained in the interior of a region \( R \). If \( U_{\text{fin}}(\pi) \subseteq T_{C,R}(p) \) then there exists \( t \in (t_i^p, t_i^q) \) with \( x(t) \in \mathcal{F} \).

**Proof:** If \( U_{\text{fin}}(\pi) \subseteq T_{C,R}(p) \), then \( q = x(t_i^q) \). Then \( \rho_{\text{fin}}(\pi) \triangleq \rho(t_i^q - t_i^p) \) and \( x(t) \) entered int \( C \) some time \( t \) in \( (t_i^p, t_i^q) \) (see Fig. 3 (right)).

**Remark 4:** If \( R \) is selected as a convex region, then ensuring \( U_{\text{init}}(\pi) \subseteq R \) and \( U_{\text{fin}}(\pi) \subseteq T_{C,R}(p) \) suffices for meeting the requirements of Theorem 2.

### D. Path-Planning With Infinite Cylinders

Theorem 2 states region \( R \) should be selected so the error growth in (9) of the agent is accounted for in \( R \) up until re-entry can be guaranteed. Since an upper bound on the error growth rate is given in (9), \( R \) may be designed to contain all possible trajectories of \( x(t) \), given a plan \( \pi = (n, o, p, q) \). In principle, given the reference point of departure \( p = x_\pi(t_i^p) \) and a reference target velocity vector \( v \) with \( ||v|| = v_0 \), \( R \) could always be selected to be the (unbounded) cone of uncertainty \( \bigcup_{t \in [t_i^p, \infty]} B_{\rho(t - t_i^p)}(p + (t - t_i^p)v) \) in that direction. However, this choice is challenging from a computational perspective, because of the non-linearity of \( \rho \).

Given any point \( p \in \mathcal{F}^* \), the planner needs to select \( q \) and a region \( R \) best suited for guaranteeing the agent’s return to \( \mathcal{F} \). In general, to apply Theorem 2 a rich collection \( R(p) \) of regions \( R \) satisfying \( p \in \mathcal{R}(p) \) with “sufficiently large” \( T_{C,R}(p) \) for some \( R \in \mathcal{R}(p) \) must be designed.

Fig. 4 presents two alternatives to the approach using cones of uncertainty: the cylindrical geometry in Fig. 4(A) and a conical geometry in Fig. 4(B). The subsequent development only considers the cylindrical geometry, in the interest of reducing computational cost. Specifically, consider

\[
\mathcal{R}(p) \triangleq \left\{ R_{p,v,w} : v \in \mathbb{R}^D, ||v|| = v_0, w > 0 \right\},
\]

where \( R_{p,v,w} \) is the solid cylinder of radius \( w \) whose axis passes through \( p \) and has direction \( v \) (see Fig. 4(A)),

\[
R_{p,v,w} \triangleq \left\{ z \in \mathbb{R}^D : \left\| (z - p) - \frac{(z - p) \cdot v}{v_0^2} v \right\| \leq w \right\}.
\]

A pair \((v, w)\) will be referred to as **return parameters**. For \( R = R_{p,v,w} \), the following observations are made.
For excessively large values of \( w \) (e.g., \( w \) big enough for \( C \subset R \)) one has \( T_{C,R}(p) = \mathcal{F} \). Therefore, any ball contained in \( T_{C,R}(p) \) provides little improvement upon IBC of guaranteeing return.

More generally, note that \( R \setminus C \) has no more than two unbounded components. When there is only one such component, the existence of a ball of radius \( w \) contained in \( T_{C,R}(p) \) cannot be guaranteed.

In all other cases, the collection of balls of radius \( w \) contained in \( T_{C,R}(p) \) is non-empty: a point of the form \( \bar{q}_{i} \) for each \( i \geq 1 \) such that \( B_{\bar{q}_{i}}(p) \subset T_{C,R}(p) \) is defined in (12). The developed planner seeks to maximize the arc length from \( o \) to \( p \) along \( X_{d} \) subject to the constraint of re-entry being guaranteed by \( R(p) \). As a result, maximizing \( t_{q}^{p} - t_{q}^{u} \) subject to the existence of a point \( q = p + \alpha w \) with \( B_{\bar{q}_{i}}(q) \subset T_{C,R}(p) \) and \( R = R_{p,v,w} \) provides a lower bound on the time the agent could spend tracking \( X_{d} \), which the planner seeks to maximize.

\[
\Delta t^{*} \triangleq \max_{p,v,w,\alpha} \left( \rho^{-1}(w) - \alpha \right),
\]

subject to the same constraints as (16), and noting that \( p = x_{r}(\Delta t^{*}) \) is an implicit constraint on \( \pi^{*} \).

### E. Dwell-Time Analysis

**Definition 2 (Feasible Region):** For every point \( p \in \mathcal{F} \), let the feasible region \( \mathcal{G} \) be defined by

\[
\mathcal{G} \triangleq \mathcal{F} \cup \{ p \in \mathcal{F} : \rho^{-1}(\text{dist}(p, \mathcal{F})) < \tau(p) \},
\]

where \( \tau(p) \triangleq \max_{v,w,\alpha} \{ \rho^{-1}(w) - \alpha \} \), subject to \( B_{w}(p + \alpha v) \subset T_{C,R}(p) \) and \( R = R_{p,v,w} \). The feasible region for initialization is defined by

\[
\mathcal{G}_{0} \triangleq \mathcal{F} \cup \{ p \in \mathcal{F} : \tau(p) > 0 \}.
\]

**Theorem 3:** Suppose an agent is given whose motion is governed by (1) and the controller from Section II-D. Moreover, suppose Assumptions 1–3 are satisfied. Let \( \mathcal{F} \) be a region satisfying Assumption 4. Also suppose that, given \( x(0) \in \mathcal{G}_{0} \) and \( e(0) = \bar{e}(0) = 0 \), the agent executes a sequence of plans as follows: (a) if \( x(0) \in \mathcal{F} \) then let \( \pi_{i} = (u_{i}, a_{i}, p_{i}, q_{i}), i \geq 1 \) each with an associated region \( R_{i} = R(p_{i}) \) as defined in (14); (b) otherwise, let \( \pi_{0} = (p_{0}, q_{0}) \) with \( p_{0} = x(0) \) and an associated region \( R_{0} = R(p_{0}) \), followed by a sequence of plans as in (a). If, between plan executions the agent is confined to \( \mathcal{F} \), and all the plans satisfy the re-entry condition of Theorem 2—that is \( U_{\text{fin}}(\pi_{i}) \subset T_{C,R}(p_{i}) \) for all \( i \)—then the actual tracking error is ultimately bounded, uniformly over \( \mathcal{G}_{0} \) and the set of plans, provided the switching signal satisfies the maximum dwell-time condition \( \rho(\Delta t^{*}) \leq \rho_{\text{fin}}(\pi_{i}) \), written explicitly as

\[
\frac{1}{2\lambda_{u}} \ln \left( \frac{\lambda_{u}}{\rho_{\text{fin}}(\pi_{i})} \right) + 1 \right),
\]

where \( \rho_{\text{fin}}(\pi_{i}) \) is defined in (12).

**Proof:** For \( x(0) \in \mathcal{G}_{0} \setminus \mathcal{F} \) it takes at most \( \rho^{-1}(\text{diam}(\mathcal{F})) \) time to acquire feedback. Therefore, without loss of generality assume \( x(0) \in \mathcal{F} \). With Assumptions 1–3 satisfied, suppose that plan \( \pi_{i} \) also satisfies \( U_{\text{fin}}(\pi_{i}) \subset T_{C,R}(p_{i}) \). By Theorem 2, \( x \) is guaranteed to re-enter into the feedback region \( \mathcal{F} \), while \( \rho_{\text{fin}}(\pi_{i}) \) bounds the error growth from above. One can then apply the proof of Theorem 1 in [5] to deduce that the tracking error is ultimately bounded uniformly over \( \mathcal{G}_{0} \) and the set of plans, since it satisfies the dwell-time condition (23).

**Theorem 4:** Suppose \( X_{d} \) is a polygonal curve contained in the feasible region \( \mathcal{G} \). Then \( X_{d} \) can be tracked with a guarantee of re-entry, for any initial condition \( x(0) \in \mathcal{G}_{0} \), provided the error system is initialized with \( e(0) = \bar{e}(0) = 0 \).

**Proof:** Suppose \( X_{d} \) is a polygonal curve contained in the feasible region \( \mathcal{G} \). For each \( m \), let \( \nu(m) \) denote the smallest \( m' > m \) such that \( P_{\nu(m)} \notin \mathcal{F} \). The proof proceeds by induction. For the base step, at time \( t = 0 \), if \( x(0) \in \mathcal{F} \) then set \( t_{q}^{u} = 0 \);
otherwise, the agent executes \( p_\theta = (p_0, q_0) \) with \( p_0 = x(0) \) and \( q_0 = x(0) + \alpha_0 v_0 \), where \((v_0, w_0, \alpha_0)\) realizes the maximum in the definition of \( \tau(x) \). By Theorem 2, entry into \( F \) is guaranteed, resulting in \( t_{F}^a \in [0, \alpha_0] \). For the induction hypothesis, assume plans \( \pi_i = (n_i, O_i, p_i, q_i, i = 1, \ldots, k) \) have been constructed so that (a) \( O_i = \mathcal{P}_O \); (b) \( p_i = \mathcal{P}_{m_i} \) with \( 1 < m_1 < \cdots < m_k \); (c) \( \alpha_{i+1} = \mathcal{P}_{w(m_i)} \) for \( i = 1, \ldots, k - 1 \); and (d) the conditions of Theorem 3 hold.

Let \( x : [0, t_{k+1}^a] \to \mathbb{R}^D \) be an execution of these plans and note \( x(t_{k+1}^a) \in \mathcal{F} \). For the induction step, define a point \( n_{k+1} \in \mathcal{F} \) at minimum distance to \( \alpha_{k+1} \triangleq \mathcal{P}_{w(m_k)} \). Select \( \pi_{k+1} \) to be the optimal plan \( \pi^* = (n^*, \alpha, p^*, q^*, \alpha^* v^*) \) from Section IV-D, for a choice of \( n^* = n_{k+1} \) and \( \alpha = \alpha_{k+1} \). Set \( p_{k+1} = p^*, \alpha_{k+1} = \alpha^*, v_{k+1} = v^*, \) and \( q_{k+1} = q^* \).

Now extend \( x \) as follows: first, the agent proceeds from \( x(t_{k+1}^a) \) to \( n_{k+1} \) through \( \mathcal{F} \) — while tracking \( X_d \), if \( \nu(m_k) > m_k + 1 \) — defining the behavior of \( x \) over the time interval \([t_{k+1}^a, t_{k+1}^b] \); next, the agent executes the plan \( \pi_{k+1} \), which guarantees re-entry into \( \mathcal{F} \) at some time \( t_{k+1}^b \). It is required to show that the dwell-time condition \( \rho(\Delta t_{k+1}) \leq \rho_{f,m,j} \) of Theorem 3 is satisfied.

Since \( \rho \) is strictly increasing, this is equivalent to requiring \( \Delta t_{k+1} \leq \rho^{-1}(\rho_{f,m,j}(\pi_{k+1})) = t_{k+1}^b - t_{k+1}^a \). Recalling that \( \Delta t_{k+1} = t_{k+1}^b - t_{k+1}^a \), this finishes the argument.

**Remark 5:** Note the path plan \( \pi_{k+1} \) in the proof covers the range of consecutive way points \( \alpha_{k+1} = \mathcal{P}_{w(m_k)} \), \( \mathcal{P}_{m_{k+1}} = p_{k+1} \) along \( X_d \), \( \nu(m_k) \geq m_k + 1 \), while the preceding plan \( \pi_k \) covered the range \( \alpha_k = \mathcal{P}_{w(m_{k-1})} \), \( \cdots \), \( \mathcal{P}_{m_k} = p_k \subset X_d \). Hence, progress will be made along \( X_d \) by any execution of \( \pi_{k+1} \). In particular, there are no Zeno executions.

V. PRECOMPUTATION AND PLAN GENERATION

Since the geometry of the feedback region and the evolution of the region of uncertainty are known, a brute-force algorithm can be used to obtain all of the needed information for future path-planning, provided the way points \( \mathcal{D}_m \), \( m = 0, \ldots, M - 1 \), form a subdivision of \( X_d \) of sufficiently fine mesh \( \epsilon > 0 \). For each \( m \), an approximation to the solution of the optimization problem (16) for \( \mathcal{P}_m \) is obtained by solving for MAUR \((\mathcal{P}_m, v, w)\) — see (17) — over a sufficiently dense range of possible return parameters \((v, w)\), and storing the solution in a data-set. The plans \( \pi_i \) constructed in the preceding section are then selected based on the information in the data-set to satisfy the mission objectives. To select a return trajectory, the agent solves the optimization problem in (19) by executing a search over the data-set.

To construct the data-set, for each \( \mathcal{P}_m \), \( m = 0, \ldots, M - 1 \), return parameters \((v_j, w_j)\), \( v_j \triangleq v_0 [\cos(\theta_j), \sin(\theta_j)] \) are used, \( \Theta \triangleq \{\theta_j \triangleq 2\pi j / J : j = 0, \ldots, J - 1\} \), and \( W \triangleq \{v_0, \ldots, w_{K-1}\} \subset \mathbb{R}_{\geq 0} \). Given the feedback region \( \mathcal{F} \), Algorithm 1 computes \( T_{C,R}(\mathcal{P}_m) \) for each \( \mathcal{R} = \mathcal{R}_{\mathcal{P}_m, v, w} \).

**Algorithm 1:** Data-Set for \( \mathcal{P} \).

Require: \( X_d \) as a list of points \( \mathcal{P}_0, \ldots, \mathcal{P}_{M-1} \in \mathbb{R}^D \)

Require: \( \mathcal{F} = \{C \cap \text{int}(C)\}, C \) given as a sequence of vertices

1: for \( m = 0 \) to \( M - 1 \) do
2: \( p \leftarrow \mathcal{P}_m \)
3: for \( j = 0 \) to \( J - 1 \) do
4: \( v \leftarrow v_0 \cdot [\cos 2\pi j / J, \sin 2\pi j / J] \)
5: for \( k = 0 \) to \( K - 1 \) do
6: \( w \leftarrow w_k \)
7: \( \mathcal{R} \leftarrow \mathcal{R}_{\mathcal{P}_m, v, w} \)
8: \( T_{C,R}(\mathcal{R}) \leftarrow \text{FindTargetRegion}(p, \mathcal{R}, v, w) \)
9: \( q_{m,j,k} \leftarrow \text{FindTargetPoint}(v, w, T_{C,R}(\mathcal{R})) \)
10: \( \rho_{m,j,k} \leftarrow \text{Equation (26)} \)
11: return \( \{q_{m,j,k}, \rho_{m,j,k}\} \in \mathbb{R}^D \)
12: function \text{FindTargetRegion}(p, \mathcal{R}, v, w) \)
13: \( A \leftarrow R \setminus \mathcal{F} \)
14: for all \( \alpha \in \text{Regions}A \) do
15: \( B \leftarrow \alpha \)
16: \( \text{Exit} \)
17: \( B \leftarrow \text{Return} R \setminus B \)
18: function \text{FindTargetPoint}(p, v, w, t) \)
19: \( \alpha^* \leftarrow \min \{\alpha > 0 : B_w(p + \alpha v) \subseteq T\} \)
20: return \( p + \alpha^* v \)
21: function RegionsA \)
22: return the list of connected components of \( \mathcal{A} \)

\( \alpha_{m,j,k} \triangleq \min \{\alpha > 0 : B_{w_k}(q_{m,j,k}) \subset T_{C,R}(\mathcal{P}_m)\} \), \( \rho_{m,j,k} \triangleq \rho(\rho^{-1}(w_k) - \alpha_{m,j,k}) \).

Note that \( \rho^{-1}(w_k) - \alpha_{m,j,k} \leq 0 \) means \( \mathcal{P}_m \notin \mathcal{G} \), in which case \( X_d \) cannot be tracked with a guarantee of re-entry using this method. Also note that \( v_0 \alpha_{m,j,k} = \text{dist}(\mathcal{P}_m, q_{m,j,k}) \).

Upon termination, the result is a three dimensional array denoted by \( \rho_{m,j,k} \in \mathbb{R}^{M-1 \times J \times K} \), where each element of the array is the MAUR at point \( \mathcal{P}_m \) in \( X_d \), associated with a return trajectory \( \theta_j \in \Theta \), and a width \( w_k \in W \).

**Remark 6:** In some applications, it may be necessary to further restrict the MAUR to ensure a certain degree of accuracy while tracking \( X_d \). If this is the case, the user may use this predetermined desired upper bound as long as it is less than the MAUR at the point of departure.

A. Algorithm Simplification (Algorithm 2)

The size of the data-set, \( M \times J \times K \), can easily become prohibitive for optimization by brute force search. To reduce the search space, some of the iterations for the different \( R \in \mathcal{R}(\mathcal{P}_m) \) may be bypassed by selecting a single return trajectory and/or single width. In the numerical experiments in Section VI, a single pair of return parameters, \((v, w)\) is assigned to each \( \mathcal{P}_m \). Specifically, \( v \) is set to equal the vector which bisects the smallest sector emanating from \( \mathcal{P}_m \) and containing \( \mathcal{F} \); next, \( w \) is selected as the largest \( w_k \in W \) such that \( R = \mathcal{R}_{\mathcal{P}_m, v, w} \) satisfies \( T_{C,R}(p) \neq \mathcal{F} \). This results in the data-set having size
M. Algorithm 2 assigns the largest possible radius of uncertainty $\rho_m$ to the points $q_m$, but it may fail to account for the effects of the distance traveled from $P_m$ to $q_m$. In (26), maximizing $w_k$ may come at the expense of $\alpha_m$ having to become large, as well. Ultimately, this simplification trades geometric information for computational efficiency.

VI. NUMERICAL EXPERIMENTS

Experiments were conducted in MATLAB to investigate different geometries for the feedback region. In each experiment the MAURs $\rho_m$ generated by the Algorithm 2 (Section V-A) are compared with the corresponding MAURs—denoted by $\rho'_m$—generated by IBC.

Experiment, A Generic Example. The task space is a rectangular region in the plane with unitless dimensions $2.048 \times 1.536$. Generic polygonal feedback region $F$ and desired path $X_d$ were hand-drawn (see Fig. 5). $X_d$ was then subdivided to ensure a mesh size at most 1, resulting in 8,449 way points. The parameters are $\lambda_u = 3, \delta = 5$, and $v_0 = 2,000$, to ensure $\rho'_m > 0$ exists for all $m$.

Fig. 6 plots the percent increase $\mu_m \triangleq \frac{\rho'_m - \rho_m}{\rho_m} \times 100$ as a function of $m$. The plot indicates that Algorithm 2 provides a significant improvement over IBC for this specific geometry. However, IBC outperforms Algorithm 2 ($\mu_m < 0$) for points $P_{3592}$ to $P_{3991}$ (see Fig. 6; also see the points marked in Fig. 5). This reduction in the performance of Algorithm 2 results from the MAUR being generated from only one pair of return parameters. Had Algorithm 1 been deployed instead of Algorithm 2, the list of possible return trajectories would have included the one suggested by IBC, guaranteeing $\mu_m \geq 0$ at every point. Despite its sub-optimal performance, for this geometry, Algorithm 2 provides an average improvement of 209% over IBC, with the largest value of $\mu_m$ being 65.6%.

The typical run time for Algorithm 2 in the experiments presented is about an hour. Consequently, running Algorithm 1 on all the experiments would require weeks of computation time, motivating Algorithm 2.
The observation made in Section VI, motivated “rounding” the corners of all the feedback region shapes. Given these experiments (see Fig. 9), with a total of 46,855 data points (including the generic example), the mean-average increase in the maximum allowed radius of uncertainty was found to be 233%. The largest mean-average for a single geometry was 354%. This was observed in the “star” geometry (see Fig. 7(D)). The smallest mean-average for a single geometry was 168%. This was observed in the “switchback” geometry (see Fig. 7(E)). The largest increase at a single point was found to be 969%. This was observed in the “horse-shoe” geometry (see Fig. 7(C)). The smallest was found to be −42%. This was observed in the “dumbbell” geometry (see Fig. 7(A)).

Overall, it was shown that, even across a range of unique geometries, the topological criterion for guaranteed re-entry in Theorem 2 is superior to IBC, in terms of MAUR. The topological criterion was shown to render large improvements even though the sub-optimal Algorithm 2 was used instead of Algorithm 1. It is expected that Algorithm 1, with a much richer data-set, would generate larger values for the optimal MAUR. Moreover, Algorithm 1 is guaranteed to remove any instances where IBC outperforms the topological method. This guarantee is due to the fact the richer data-set would include the suggested return trajectories generated by IBC.

VII. CONCLUSION

Given an autonomous agent in Euclidean space, a topologically motivated method for guaranteeing re-entry of the agent into a feedback region was developed, with the aim of extending the reach of existing methods to include arbitrary geometries of the feedback region. This method was integrated into an existing framework for developing dwell-times for an autonomous system tasked with following a desired path in the presence of intermittent state feedback. A path-planning algorithm leveraging the new topological re-entry criterion is presented and evaluated against the same planner using IBC for synthetic and generic example geometries. The new topological method was shown to increase the time an agent is able to safely operate in a feedback-denied region, for a variety of geometries. A simplified and sub-optimal implementation of the new method yields improvements in the allowed error growth by hundreds of percentage points. This outcome, as well as computational inefficiencies in the current algorithm, motivate future investigation along several lines of inquiry. Among these, more efficient methods for computing MAURs and optimizing the choice of return trajectories, as well as a method for implementing the “cone of uncertainty”—rather than an infinite strip—as a target region (Section IV-D), are of great interest to the future development of an optimal implementation. Future research will also focus on applying the topological criterion for guaranteed re-entry to the task of optimizing traditional long-term tracking objectives such as total time to task completion, rather than merely maximizing each individual segment.

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