Brief paper

Composite adaptive control for Euler–Lagrange systems with additive disturbances☆

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ABSTRACT

In a typical adaptive update law, the rate of adaptation is generally a function of the state feedback error. Ideally, the adaptive update law would also include some feedback of the parameter estimation error. The desire to include some measurable form of the parameter estimation error in the adaptation law resulted in the development of composite adaptive update laws that are functions of a prediction error and the state feedback. In all previous composite adaptive controllers, the formulation of the prediction error is predicated on the critical assumption that the system uncertainty is linear in the uncertain parameters (LP uncertainty). The presence of additive disturbances that are not LP would destroy the prediction error formulation and stability analysis arguments in previous results. In this paper, a new prediction error formulation is constructed through the use of a recently developed Robust Integral of the Sign of the Error (RISE) technique. The contribution of this design and associated stability analysis is that the prediction error can be developed even with disturbances that do not satisfy the LP assumption (e.g., additive bounded disturbances). A composite adaptive controller is developed for a general MIMO Euler–Lagrange system with mixed structured (i.e., LP) and unstructured uncertainties. A Lyapunov-based stability analysis is used to derive sufficient gain conditions under which the proposed controller yields semi-global asymptotic tracking. Experimental results are presented to illustrate the approach.

1. Introduction

Adaptive, robust adaptive, and function approximation methods typically use tracking error feedback to update the adaptive estimates. In general, the use of the tracking error is motivated by the need for the adaptive update law to cancel cross-terms in the closed-loop tracking error system within a Lyapunov-based analysis. As the tracking error converges, the rate of the update law also converges, but drawing conclusions about the convergent value (if any) of the parameter update law is problematic. Ideally, the adaptive update law would include some estimate of the parameter estimation error as a means to prove the parameter estimates

converge to the actual values; however, the parameter estimate error is unknown. The desire to include some measurable form of the parameter estimation error in the adaptation law resulted in the development of adaptive update laws that are driven, in part, by a prediction error (Middleton & Goodwin, 1988; Morse, 1980; Pomet & Praly, 1988; Sastry & Isidori, 1989; Slotine & Li, 1991) and also Q-modification techniques (Volyanskyy, Calise, & Yang, 2006; Volyanskyy, Haddad, & Calise, 2008).

The prediction error is defined as the difference between the predicted parameter estimate value and the actual system uncertainty. Including feedback of the estimation error in the adaptive update law enables improved parameter estimation. For example, some classic results (Krstic, Kanellakopoulos, & Kokotovic, 1995; Krstic & Kokotovic, 1995; Slotine & Li, 1991) have proven the parameter estimation error is square integrable and that the parameter estimates may converge to the actual uncertain parameters. Since the prediction error depends on the unmeasurable system uncertainty, the swapping lemma (Middleton & Goodwin, 1988; Morse, 1980; Pomet & Praly, 1988; Sastry & Isidori, 1989; Slotine & Li, 1991) is central to the prediction error formulation. The swapping technique (also described as input or torque filtering in some literature) transforms a dynamic parametric model into a static form where standard parameter estimation techniques can be applied. In Krstic and Kokotovic (1995) and Krstic et al. (1995), a nonlinear extension of the swapping lemma was de-

rived, which was used to develop the modular z-swapping and x-swapping identifiers via an input-to-state stable (ISS) controller for systems in parametric strict feedback form. The advantages provided by prediction error based adaptive update laws led to several results that use either the prediction error or a composite of the prediction error and the tracking error (cf. Abiko & Hirzinger, 2007; Christoforou, 2007; de Queiroz, Dawson, & Agarwal, 1999; Mrad & Majdalani, 2003; Wang & Chen, 2001; Zergeroglu, Dixon, Haste, & Dawson, 1999; Christoforou, 2001 and the references within).

Although prediction error based adaptive update laws have existed for approximately two decades, no stability result has been developed for systems with additive bounded disturbances with the exception of the result in Bartolini, Ferrara, and Stotsky (1999). However, Bartolini et al. (1999) considers a linear time invariant system with disturbances and the prediction error is defined only in the sliding mode while the resulting stability is uniformly ultimately bounded (UUB). In general, the inclusion of disturbances reduces the steady-state performance of continuous controllers to a UUB result. In addition to a UUB result, the inclusion of disturbances may cause unbounded growth of the parameter estimates (Lewis, Abdallah, & Dawson, 1993) for tracking error-based adaptive update laws without the use of projection algorithms or other update law modifications such as σ-modification (Reed & Ioannou, 1989). Problems associated with the inclusion of disturbances are magnified for control methods based on prediction error-based update laws because the formulation of the prediction error requires the swapping (or control filtering) method. Applying the swapping approach to dynamics with additive disturbances is problematic because the unknown disturbance terms also get filtered and included in the filtered control input. This problem motivates the question of how can a prediction error-based adaptive update law be developed for systems with additive disturbances.

To address this motivating question, a general Euler–Lagrange-like MIMO system is considered with structured and unstructured uncertainties, and a gradient-based composite adaptive update law is developed that is driven by both the tracking error and the prediction error. The control development is based on the recent continuous Robust Integral of the Sign of the Error (RISE) (Patre, MacKinnis, Makkar, & Dixon, 2008) technique that was originally developed in Qu and Xu (2002) and Xian, Dawson, de Queiroz, and Chen (2004). The RISE architecture is adopted since this method can accommodate for C^2 disturbances and yield asymptotic stability. For example, the RISE technique was used in Cai, de Queiroz, and Dawson (2006) to develop a tracking controller for nonlinear systems in the presence of additive disturbances and parametric uncertainties. Based on the well accepted heuristic notion that the addition of system knowledge in the control structure yields better performance and reduces control effort, model-based adaptive and neural network feedforward elements were added to the RISE controller in Patre et al. (2008) and Patre, MacKinnis, Kaiser, and Dixon (2008), respectively. In comparison to these approaches that used the RISE method in the feedback component of the controller, the RISE structure is used in both the feedback and feedforward elements of the control structure to enable, for the first time, the construction of a prediction error in the presence of additive disturbances. Specifically, since the swapping method will result in disturbances in the prediction error (the main obstacle that has previously limited this development), an innovative use of the RISE structure is also employed in the prediction error update (i.e., the filtered control input estimate). Sufficient gain conditions are developed under which this unique double RISE controller guarantees semi-global asymptotic tracking. Experimental results

are presented to illustrate the performance of the proposed approach.

The paper is organized as follows. Section 2 describes the dynamic system and the assumptions required for the control development. Section 3 states the control objective and the defines the error states. Section 4 presents the control development and introduces the new RISE-based swapping procedure that is used to define the prediction error. A Lyapunov-based stability analysis is shown in Section 5 while Section 6 presents experimental results that demonstrate improved performance by the proposed method. Conclusions and future work are described in Section 7.

2. Dynamic system

Consider a class of MIMO nonlinear Euler–Lagrange systems of the following form:

$$x^{(m)} = f(x, \dot{x}, \ldots, x^{(m-1)}) + G(x, \dot{x}, \ldots, x^{(m-2)})u + h(t) \qquad (1)$$

where $(\cdot)^{(i)}(t)$ denotes the ith derivative with respect to time, $x^{(i)}(t) \in \mathbb{R}^n, i = 0, \ldots, m - 1$ are the system states, $u(t) \in \mathbb{R}^n$ is the control input, $f(x, \dot{x}, \ldots, x^{(m-1)}) \in \mathbb{R}^n$ and $G(x, \dot{x}, \ldots, x^{(m-2)}) \in \mathbb{R}^{n \times n}$ are unknown nonlinear C^2 functions, and $h(t) \in \mathbb{R}^n$ denotes a general nonlinear disturbance (e.g., unmodeled effects). Throughout the paper, $|\cdot|$ denotes the absolute value of the scalar argument, $\|\cdot\|$ denotes the standard Euclidean norm for a vector or the induced infinity norm for a matrix.

The subsequent development is based on the assumption that all the system states are measurable outputs. Moreover, the following assumptions will be exploited in the subsequent development.

Assumption 1. $G(\cdot)$ is symmetric positive definite, and satisfies the following inequality $\forall y(t) \in \mathbb{R}^n$:

$$\underline{g} \|y\|^2 \le y^T G^{-1} y \le \bar{g}(x, \dot{x}, \ldots, x^{(m-2)}) \|y\|^2 \qquad (2)$$

where $\underline{g} \in \mathbb{R}$ is a known positive constant, and $\bar{g}(x, \dot{x}, \ldots, x^{(m-2)}) \in \mathbb{R}$ is a known positive function.

Assumption 2. The functions $G^{-1}(\cdot)$ and $f(\cdot)$ are second order differentiable such that $G^{-1}, \dot{G}^{-1}, \ddot{G}^{-1}, f, \dot{f}, \ddot{f} \in \mathcal{L}_\infty$ if $x^{(i)}(t) \in \mathcal{L}_\infty, i = 0, 1, \ldots, m + 1$.

Assumption 3. The nonlinear disturbance term and its first two time derivatives (i.e., $h, \dot{h}, \ddot{h}$) are bounded by known constants.

Assumption 4. The unknown nonlinearities $G^{-1}(\cdot)$ and $f(\cdot)$ are linear in terms of unknown constant system parameters (i.e., LP).

Assumption 5. The desired trajectory $x_d(t) \in \mathbb{R}^n$ is assumed to be designed such that $x_d^{(i)}(t) \in \mathcal{L}_\infty, i = 0, 1, \ldots, m + 2$. The desired trajectory $x_d(t)$ need not be persistently exciting and can be set to a constant value for the regulation problem.

3. Control objective

The objective is to design a continuous composite adaptive controller which ensures that the system state $x(t)$ tracks a desired time-varying trajectory $x_d(t)$ despite uncertainties and bounded disturbances in the dynamic model. To quantify this objective, a tracking error, denoted by $e_1(t) \in \mathbb{R}^n$, is defined as

$$e_1 \triangleq x_d - x. \qquad (3)$$

To facilitate a compact presentation of the subsequent control development and stability analysis, auxiliary error signals denoted

by $e_i(t) \in \mathbb{R}^n$, $i = 3, \ldots, m$ are defined as

$$e_2 \triangleq \dot{e}_1 + \alpha_1 e_1, \qquad e_i \triangleq \dot{e}_{i-1} + \alpha_{i-1} e_{i-1} + e_{i-2} \tag{4}$$

where $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, m-1$ denote constant positive control gains. The error signals $e_i(t)$, $i = 2, 3, \ldots, m$ can be expressed in terms of $e_1(t)$ and its time derivatives as

$$e_i = \sum_{j=0}^{i-1} b_{i,j} e_1^{(j)}, \quad b_{i,i-1} = 1 \tag{5}$$

where the constant coefficients $b_{i,j} \in \mathbb{R}$ can be evaluated by substituting (5) in (4), and comparing coefficients. A filtered tracking error (Lewis et al., 1993), denoted by $r(t) \in \mathbb{R}^n$, is also defined as

$$r \triangleq \dot{e}_m + \alpha_m e_m \tag{6}$$

where $\alpha_m \in \mathbb{R}$ is a positive, constant control gain. The filtered tracking error $r(t)$ is not measurable since the expression in (6) depends on $x^{(m)}$.

4. Control development

To develop the open-loop tracking error system, the filtered tracking error in (6) is premultiplied by $G^{-1}(\cdot)$, and (5) is used to yield

$$G^{-1} r = G^{-1} \sum_{j=0}^{m-1} b_{m,j} e_1^{(j+1)} + G^{-1} \alpha_m e_m. \tag{7}$$

By separating the last term from the summation, using the fact that $b_{m,m-1} = 1$, and making substitutions from (1) and (3), the expression in (7) is rewritten as

$$G^{-1} r = Y_d \theta + S_1 - G_d^{-1} h - u. \tag{8}$$

In (8), the auxiliary function $S_1(x, \dot{x}, \ldots, x^{(m-1)}, t) \in \mathbb{R}^n$ is defined as

$$S_1 \triangleq G^{-1} \left(\sum_{j=0}^{m-2} b_{m,j} e_1^{(j+1)} + \alpha_m e_m \right) + G^{-1} x_d^{(m)}$$
$$- G_d^{-1} x_d^{(m)} - G^{-1} f + G_d^{-1} f_d - G^{-1} h + G_d^{-1} h. \tag{9}$$

Also in (8), $Y_d \theta \in \mathbb{R}^n$ is defined as

$$Y_d \theta \triangleq G_d^{-1} x_d^{(m)} - G_d^{-1} f_d \tag{10}$$

where $Y_d(x_d, \dot{x}_d, \ldots, x_d^{(m)}) \in \mathbb{R}^{n \times p}$ is a desired regression matrix, and $\theta \in \mathbb{R}^p$ contains the constant unknown system parameters. In (10), the functions $G_d^{-1}(x_d, \dot{x}_d, \ldots, x_d^{(m-2)}) \in \mathbb{R}^{n \times n}$ and $f_d(x_d, \dot{x}_d, \ldots, x_d^{(m-1)}) \in \mathbb{R}^n$ are defined as

$$G_d^{-1} \triangleq G^{-1}(x_d, \dot{x}_d, \ldots, x_d^{(m-2)})$$
$$f_d \triangleq f(x_d, \dot{x}_d, \ldots, x_d^{(m-1)}). \tag{11}$$

4.1. RISE-based swapping

A measurable form of the prediction error $\varepsilon(t) \in \mathbb{R}^n$ is defined as the difference between the filtered control input $u_f(t) \in \mathbb{R}^n$ and the estimated filtered control input $\hat{u}_f(t) \in \mathbb{R}^n$ as

$$\varepsilon \triangleq u_f - \hat{u}_f \tag{12}$$

where the filtered control input $u_f(t) \in \mathbb{R}^n$ is generated by Slotine and Li (1991)

$$\dot{u}_f + \omega u_f = \omega u, \quad u_f(0) = 0 \tag{13}$$

where $\omega \in \mathbb{R}$ is a known positive constant, and $\hat{u}_f(t) \in \mathbb{R}^n$ is subsequently designed. The differential equation in (13) can be directly solved to yield

$$u_f = v * u, \qquad v \triangleq \omega e^{-\omega t} \tag{14}$$

where $v(t) \in \mathbb{R}$, and $*$ is used to denote the standard convolution operation. Using (1), the expression in (14) can be rewritten as

$$u_f = v * \left(G^{-1} x^{(m)} - G^{-1} f - G^{-1} h \right). \tag{15}$$

Since the system dynamics in (1) include non-LP bounded disturbances $h(t)$, they also get filtered and included in the filtered control input in (15). To compensate for the effects of these disturbances, the typical prediction error formulation is modified to include a RISE-like structure in the design of the estimated filtered control input. With this motivation, the structure of the open-loop prediction error system is engineered to facilitate the RISE-based design of the estimated filtered control input.

Adding and subtracting the term $G_d^{-1} x_d^{(m)} + G_d^{-1} f_d + G_d^{-1} h$ to the expression in (15) and using (10) yields

$$u_f = Y_{df} \theta + v * S - v * S_d + h_f \tag{16}$$

where $S(x, \dot{x}, \ldots, x^{(m)})$, $S_d(x_d, \dot{x}_d, \ldots, x_d^{(m)}) \in \mathbb{R}^n$ are defined as

$$S \triangleq G^{-1} x^{(m)} - G^{-1} f - G^{-1} h \tag{17}$$
$$S_d \triangleq G_d^{-1} x_d^{(m)} - G_d^{-1} f_d - G_d^{-1} h, \tag{18}$$

the filtered regressor matrix $Y_{df}(x_d, \dot{x}_d, \ldots, x_d^{(m)}) \in \mathbb{R}^{n \times p}$ is defined as

$$Y_{df} \triangleq v * Y_d, \tag{19}$$

and the disturbance $h_f(t) \in \mathbb{R}^n$ is defined as

$$h_f \triangleq -v * G_d^{-1} h.$$

The term $v * S(x, \dot{x}, \ldots, x^{(m)}) \in \mathbb{R}^n$ in (16) depends on $x^{(m)}$. Using the following property of convolution (Lewis et al., 1993):

$$g_1 * \dot{g}_2 = \dot{g}_1 * g_2 + g_1(0) g_2 - g_1 g_2(0) \tag{20}$$

an expression independent of $x^{(m)}$ can be obtained. Consider

$$v * S = v * \left(G^{-1} x^{(m)} - G^{-1} f - G^{-1} h \right)$$

which can be rewritten as

$$v * S = v * \left(\frac{d}{dt} (G^{-1} x^{(m-1)}) - \dot{G}^{-1} x^{(m-1)} - G^{-1} f - G^{-1} h \right). \tag{21}$$

Applying the property in (20) to the first term of (21) yields

$$v * S = S_f + W \tag{22}$$

where the state-dependent terms are included in the auxiliary function $S_f(x, \dot{x}, \ldots, x^{(m-1)}) \in \mathbb{R}^n$, defined as

$$S_f \triangleq \dot{v} * \left(G^{-1} x^{(m-1)} \right) + v(0) G^{-1} x^{(m-1)}$$
$$- v * \dot{G}^{-1} x^{(m-1)} - v * G^{-1} f - v * G^{-1} h \tag{23}$$

and the terms that depend on the initial states are included in $W(t) \in \mathbb{R}^n$, defined as

$$W \triangleq -v G^{-1} \left(x(0), \dot{x}(0), \ldots, x^{(m-2)}(0) \right) x^{(m-1)}(0). \tag{24}$$

Similarly, following the procedure in (21)–(24), the expression $v * S_d$ in (16) is evaluated as

$$v * S_d = S_{df} + W_d \tag{25}$$

where $S_{df}(x_d, \dot{x}_d, \ldots, x_d^{(m-1)}) \in \mathbb{R}^n$ is defined as

$$S_{df} \triangleq \dot{v} * (G_d^{-1} x_d^{(m-1)}) + v(0) G_d^{-1} x_d^{(m-1)}$$
$$- v * \dot{G}_d^{-1} x_d^{(m-1)} - v * G_d^{-1} f_d - v * G_d^{-1} h \tag{26}$$

and $W_d(t) \in \mathbb{R}^n$ is defined as

$$W_d \triangleq -v G_d^{-1}(x_d(0), \dot{x}_d(0), \ldots, x_d^{(m-2)}(0)) x_d^{(m-1)}(0). \tag{27}$$

Substituting (22)–(27) into (16), and then substituting the resulting expression into (12) yields

$$\varepsilon = Y_{df} \theta + S_f - S_{df} + W - W_d + h_f - \hat{u}_f. \tag{28}$$

4.2. Composite adaptation

The composite adaptation for the adaptive estimates $\hat{\theta}(t) \in \mathbb{R}^p$ in (39) is given by

$$\dot{\hat{\theta}} = \Gamma \dot{Y}_d^T r + \Gamma \dot{Y}_{df}^T \varepsilon \tag{29}$$

where $\Gamma \in \mathbb{R}^{p \times p}$ is a positive definite, symmetric, constant gain matrix and the filtered regressor matrix $Y_{df}(x_d, \dot{x}_d, \ldots, x_d^{(m)}) \in \mathbb{R}^{n \times p}$ is defined in (19). The update law in (29) depends on the unmeasurable signal $r(t)$, but the parameter estimates are independent of $r(t)$ as can be shown by directly solving (29) as in Zhang, Dawson, de Queiroz, and Dixon (2000).

4.3. Closed-loop prediction error system

Based on (30) and the subsequent analysis, the filtered control input estimate is designed as

$$\hat{u}_f = Y_{df}\hat{\theta} + \mu_2 \tag{30}$$

where $\mu_2(t) \in \mathbb{R}^n$ is a RISE-like term defined as

$$\mu_2(t) \triangleq \int_0^t [k_2 \varepsilon(\sigma) + \beta_2 sgn(\varepsilon(\sigma))]d\sigma \tag{31}$$

where $k_2, \beta_2 \in \mathbb{R}$ denote constant positive control gains. In a typical prediction error formulation, the estimated filtered control input is designed to include just the first term $Y_{df}\hat{\theta}$ in (30). But as previously discussed, the presence of non-LP disturbances in the system model results in filtered disturbances in the unmeasurable form of the prediction error in (28). Hence, the estimated filtered control input is augmented with an additional RISE-like term $\mu_2(t)$ to cancel the effects of disturbances in the prediction error and the subsequent design and stability analysis. Substituting (30) into (28) yields the following closed-loop prediction error system:

$$\varepsilon = Y_{df}\tilde{\theta} + S_f - S_{df} + W - W_d + h_f - \mu_2 \tag{32}$$

where $\tilde{\theta}(t) \in \mathbb{R}^p$ denotes the parameter estimate mismatch defined as

$$\tilde{\theta} \triangleq \theta - \hat{\theta}. \tag{33}$$

To facilitate the subsequent composite adaptive control development and stability analysis, the time derivative of (32) is expressed as

$$\dot{\varepsilon} = \dot{Y}_{df}\tilde{\theta} - Y_{df}\Gamma \dot{Y}_{df}^T \varepsilon + \tilde{N}_2 + N_{2B} - k_2 \varepsilon - \beta_2 sgn(\varepsilon) \tag{34}$$

where (29) is utilized. In (34), the unmeasurable/unknown auxiliary terms $\tilde{N}_2(e_1, e_2, \ldots, e_m, r, t)$, $N_{2B}(t) \in \mathbb{R}^n$ are defined as

$$\tilde{N}_2 \triangleq \dot{S}_f - \dot{S}_{df} - Y_{df}\Gamma \dot{Y}_d^T r, \qquad N_{2B} \triangleq \dot{W} - \dot{W}_d + \dot{h}_f. \tag{35}$$

In a similar manner as in Xian et al. (2004), the Mean Value Theorem can be used to develop the following upper bound for the expression in (35):

$$\left\| \tilde{N}_2(t) \right\| \le \rho_2(\|z\|)\|z\| \tag{36}$$

where the bounding function $\rho_2(\cdot) \in \mathbb{R}$ is a positive, globally invertible, nondecreasing function, and $z(t) \in \mathbb{R}^{n(m+1)}$ is defined as

$$z(t) \triangleq \begin{bmatrix} e_1^T & e_2^T & \ldots & e_m^T & r^T \end{bmatrix}^T. \tag{37}$$

Using Assumption 3, and the fact that $v(t)$ is a linear, strictly proper, exponentially stable transfer function, the following

inequality can be developed based on the expression in (35) with a similar approach as in Lemma 2 of Middleton and Goodwin (1988):

$$\|N_{2B}(t)\| \le \xi \tag{38}$$

where $\xi \in \mathbb{R}$ is a known positive constant.

4.4. Closed-loop tracking error system

Based on the open-loop error system in (8), the control input is composed of an adaptive feedforward term plus the RISE feedback term as

$$u \triangleq Y_d\hat{\theta} + \mu_1 \tag{39}$$

where $\mu_1(t) \in \mathbb{R}^n$ denotes the RISE feedback term defined as

$$\mu_1(t) \triangleq (k_1 + 1)e_m(t) - (k_1 + 1)e_m(0)$$
$$+ \int_0^t \{(k_1 + 1)\alpha_m e_m(\sigma) + \beta_1 sgn(e_m(\sigma))\}d\sigma \tag{40}$$

where $k_1, \beta_1 \in \mathbb{R}$ are positive constant control gains, and $\alpha_m \in \mathbb{R}$ was introduced in (6). In (39), $\hat{\theta}(t) \in \mathbb{R}^p$ denotes a parameter estimate vector for unknown system parameters $\theta \in \mathbb{R}^p$, generated by a subsequently designed gradient-based composite adaptive update law (Slotine & Li, 1987, 1989; Tang & Arteaga, 1994).

The closed-loop tracking error system can be developed by substituting (39) into (8) as

$$G^{-1}r = Y_d\tilde{\theta} + S_1 - G_d^{-1}h - \mu_1. \tag{41}$$

To facilitate the subsequent composite adaptive control development and stability analysis, the time derivative of (41) is expressed as

$$G^{-1}\dot{r} = -\frac{1}{2}\dot{G}^{-1}r + \dot{Y}_d\tilde{\theta} - Y_d\Gamma \dot{Y}_{df}^T \varepsilon + \tilde{N}_1 + N_{1B}$$
$$- (k_1 + 1)r - \beta_1 sgn(e_m) - e_m \tag{42}$$

where (29) was utilized. In (42), the unmeasurable/unknown auxiliary terms $\tilde{N}_1(e_1, e_2, \ldots, e_m, r, t)$ and $N_{1B}(t) \in \mathbb{R}^n$ are defined as

$$\tilde{N}_1 \triangleq -\frac{1}{2}\dot{G}^{-1}r + \dot{S}_1 + e_m - Y_d\Gamma \dot{Y}_d^T r \tag{43}$$

where (29) was used, and

$$N_{1B} \triangleq -\dot{G}_d^{-1}h - G_d^{-1}\dot{h}. \tag{44}$$

The structure of (42) and the introduction of the auxiliary terms in (43) and (44) are motivated by the desire to segregate terms that can be upper bounded by state-dependent terms and terms that can be upper bounded by constants. In a similar fashion as in (36), the following upper bound can be developed for the expression in (43):

$$\left\| \tilde{N}_1(t) \right\| \le \rho_1(\|z\|)\|z\| \tag{45}$$

where the bounding function $\rho_1(\cdot) \in \mathbb{R}$ is a positive, globally invertible, nondecreasing function, and $z(t) \in \mathbb{R}^{n(m+1)}$ was defined in (37). Using Assumptions 2 and 3, the following inequalities can be developed based on the expression in (44) and its time derivative:

$$\|N_{1B}(t)\| \le \zeta_1, \qquad \left\| \dot{N}_{1B}(t) \right\| \le \zeta_2 \tag{46}$$

where $\zeta_i \in \mathbb{R}$, $i = 1, 2$ are known positive constants.

5. Stability analysis

Theorem 1. *The controller given in* (39) *and* (40) *in conjunction with the composite adaptive update law in* (29), *where the prediction error is generated from* (12), (13), (30) *and* (31), *ensures that all system signals are bounded under a closed-loop operation and that the position tracking error and the prediction error are regulated in the sense that*

$$\|e_1(t)\| \to 0 \quad and \quad \|\varepsilon(t)\| \to 0 \quad as \ t \to \infty$$

provided the control gains k_1 and k_2 introduced in (40) *and* (31) *are selected sufficiently large based on the initial conditions of the system (see the subsequent proof), and the following conditions are satisfied:*

$$\alpha_m, \alpha_{m-1} > \frac{1}{2}, \qquad \beta_1 > \zeta_1 + \frac{1}{\alpha_m}\zeta_2, \qquad \beta_2 > \xi \qquad (47)$$

where the gains α_{m-1} and α_m were introduced in (4), *β_1 was introduced in* (40), *β_2 was introduced in* (31), *ζ_1 and ζ_2 were introduced in* (46), *and ξ was introduced in* (38).

Proof. Let $\mathcal{D} \subset \mathbb{R}^{n(m+2)+p+2}$ be a domain containing $y(t) = 0$, where $y(t) \in \mathbb{R}^{n(m+2)+p+2}$ is defined as

$$y \triangleq [\ z^T \quad \varepsilon^T \quad \sqrt{P_1} \quad \sqrt{P_2} \quad \tilde{\theta}^T \ ]^T. \qquad (48)$$

In (48), the auxiliary function $P_1(t) \in \mathbb{R}$ is defined as

$$P_1(t) \triangleq \beta_1 \sum_{i=1}^n |e_{mi}(0)| - e_m(0)^T N_{1B}(0) - \int_0^t L_1(\tau)\mathrm{d}\tau \qquad (49)$$

where $e_{mi}(0) \in \mathbb{R}$ denotes the ith element of the vector $e_m(0)$, and the auxiliary function $L_1(t) \in \mathbb{R}$ is defined as

$$L_1 \triangleq r^T(N_{1B} - \beta_1 sgn(e_m)) \qquad (50)$$

where $\beta_1 \in \mathbb{R}$ is a positive constant chosen according to the sufficient condition in (47). Provided the sufficient condition introduced in (47) is satisfied, the following inequality is obtained (Xian et al., 2004):

$$\int_0^t L_1(\tau)\mathrm{d}\tau \le \beta_1 \sum_{i=1}^n |e_{mi}(0)| - e_m(0)^T N_{1B}(0). \qquad (51)$$

Hence, (51) can be used to conclude that $P_1(t) \ge 0$. Also in (48), the auxiliary function $P_2(t) \in \mathbb{R}$ is defined as

$$P_2(t) \triangleq - \int_0^t L_2(\tau)\mathrm{d}\tau \qquad (52)$$

where the auxiliary function $L_2(t) \in \mathbb{R}$ is defined as

$$L_2 \triangleq \varepsilon^T(N_{2B} - \beta_2 sgn(\varepsilon)) \qquad (53)$$

where $\beta_2 \in \mathbb{R}$ is a positive constant chosen according to the sufficient condition in (47). Provided the sufficient condition introduced in (47) is satisfied, then $P_2(t) \ge 0$.

Let $V_L(y, t) : \mathcal{D} \times [0, \infty) \to \mathbb{R}$ be a continuously differentiable, positive definite function defined as

$$V_L(y, t) \triangleq \frac{1}{2}\sum_{i=1}^m e_i^T e_i + \frac{1}{2}r^T G^{-1}r + \frac{1}{2}\varepsilon^T \varepsilon$$
$$+ P_1 + P_2 + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1}\tilde{\theta} \qquad (54)$$

which satisfies the inequalities

$$U_1(y) \le V_L(y, t) \le U_2(y) \qquad (55)$$

provided the sufficient conditions introduced in (47) are satisfied. In (55), the continuous positive definite functions $U_1(y)$, $U_2(y) \in \mathbb{R}$ are defined as $U_1(y) \triangleq \lambda_1 \|y\|^2$ and $U_2(y) \triangleq \lambda_2(x, \dot{x}, \dots, x^{(m-2)}) \|y\|^2$, where $\lambda_1, \lambda_2(x, \dot{x}, \dots, x^{(m-2)}) \in \mathbb{R}$ are defined as

$$\lambda_1 \triangleq \frac{1}{2} \min \left\{ 1, \underline{g}, \lambda_{\min} \left\{ \Gamma^{-1} \right\} \right\} \qquad (56)$$

$$\lambda_2 \triangleq \max \left\{ \frac{1}{2}\bar{g}(x, \dot{x}, \dots, x^{(m-2)}), \frac{1}{2}\lambda_{\max} \left\{ \Gamma^{-1} \right\}, 1 \right\}$$

where $g, \bar{g}(x, \dot{x}, \dots, x^{(m-2)})$ are introduced in (2), and $\lambda_{\min} \{\cdot\}$ and $\lambda_{\max} \{\cdot\}$ denote the minimum and maximum eigenvalue of the arguments, respectively.

Remark 1. From (34), (42), (49), (50), (52) and (53), some of the differential equations describing the closed-loop system for which the stability analysis is being performed have discontinuous right-hand sides as

$$G^{-1}\dot{r} = -\frac{1}{2}\dot{G}^{-1}r + Y_d\tilde{\theta} - Y_d\Gamma\dot{Y}_{df}^T\varepsilon + \tilde{N}_1 + N_{1B}$$
$$- (k_1 + 1)r - \beta_1 sgn(e_m) - e_m \qquad (57)$$

$$\dot{\varepsilon} = \dot{Y}_{df}\tilde{\theta} - Y_{df}\Gamma\dot{Y}_{df}^T\varepsilon + \tilde{N}_2 + N_{2B} - k_2\varepsilon$$
$$- \beta_2 sgn(\varepsilon) \qquad (58)$$

$$\dot{P}_1 = -L_1 = -r^T(N_{1B} - \beta_1 sgn(e_m)) \qquad (59)$$

$$\dot{P}_2 = -L_2 = -\varepsilon^T(N_{2B} - \beta_2 sgn(\varepsilon)). \qquad (60)$$

Let $f(y, t) \in \mathbb{R}^{n(m+2)+p+2}$ denote the right-hand side of (57)–(60). Since the subsequent analysis requires that a solution exists for $\dot{y} = f(y, t)$, it is important to show the existence and uniqueness of the solution to (57), (59) and (60). As described in Polycarpou and Ioannou (1993) and Qu (1998), the existence of Filippov's generalized solution can be established for (57)–(60). First, note that $f(y, t)$ is continuous except in the set $\{(y, t) | e_m = 0\}$. Let $\mathcal{F}(y, t)$ be a compact, convex, upper semicontinuous set-valued map that embeds the differential equation $\dot{y} = f(y, t)$ into the differential inclusions $\dot{y} \in \mathcal{F}(y, t)$. From Theorem 2.7 of Qu (1998), an absolute continuous solution exists to $\dot{y} \in \mathcal{F}(y, t)$ that is a generalized solution to $\dot{y} = f(y, t)$. A common choice for $\mathcal{F}(y, t)$ that satisfies the above conditions is the closed convex hull of $f(y, t)$ (Polycarpou & Ioannou, 1993; Qu, 1998). A proof that this choice for $\mathcal{F}(y, t)$ is upper semicontinuous is given in Gutman (1979). Moreover, note that the differential equation describing the original closed-loop system (i.e., after substituting (39) into (1)) has a continuous right-hand side; thus, satisfying the condition for existence of classical solutions.

After using (4), (6), (29), (34), (42), (49), (50), (52) and (53), the time derivative of (54) can be expressed as

$$\dot{V}_L(y, t) = -\sum_{i=1}^m \alpha_i e_i^T e_i + e_{m-1}^T e_m - r^T r - k_1 r^T r$$
$$+ r^T \dot{Y}_d\tilde{\theta} + r^T \tilde{N}_1 + r^T N_{1B} - r^T Y_d\Gamma\dot{Y}_{df}^T\varepsilon$$
$$- \beta_1 r^T sgn(e_m) + \varepsilon^T \dot{Y}_{df}\tilde{\theta} + \varepsilon^T \tilde{N}_2 + \varepsilon^T N_{2B}$$
$$- k_2\varepsilon^T\varepsilon - \varepsilon^T Y_{df}\Gamma\dot{Y}_{df}^T\varepsilon - \beta_2\varepsilon^T sgn(\varepsilon)$$
$$- r^T(N_{1B} - \beta_1 sgn(e_m)) - \varepsilon^T N_{2B}$$
$$+ \varepsilon^T \beta_2 sgn(\varepsilon) - \tilde{\theta}^T \Gamma^{-1}(\Gamma\dot{Y}_d^T r + \Gamma\dot{Y}_{df}^T\varepsilon). \qquad (61)$$

After canceling similar terms, using the fact that $a^T b \le \frac{1}{2}(\|a\|^2 + \|b\|^2)$ for some $a, b \in \mathbb{R}^n$, and using the following upper bounds

$$\left\| Y_d\Gamma\dot{Y}_{df}^T \right\| \le c_1, \qquad \left\| Y_{df}\Gamma\dot{Y}_{df}^T \right\| \le c_2$$

where $c_1, c_2 \in \mathbb{R}$ are positive constants, $\dot{V}_L(y, t)$ is upper bounded using the squares of the components of $z(t)$ as

$$\dot{V}_L(y, t) \leq -\lambda_3 \|z\|^2 - k_1 \|r\|^2 + \|r\| \left\|\tilde{N}_1\right\| + c_1 \|\varepsilon\| \|r\|$$
$$+ \|\varepsilon\| \left\|\tilde{N}_2\right\| - (k_2 - c_2) \|\varepsilon\|^2, \tag{62}$$

where

$$\lambda_3 \triangleq \min \left\{ \alpha_1, \alpha_2, \ldots, \alpha_{m-2}, \alpha_{m-1} - \frac{1}{2}, \alpha_m - \frac{1}{2}, 1 \right\}.$$

Letting $k_2 = k_{2a} + k_{2b}$ where $k_{2a}, k_{2b} \in \mathbb{R}$ are positive constants, using the inequalities in (36) and (45), and completing the squares, the expression in (62) is upper bounded as

$$\dot{V}_L(y, t) \leq -\lambda_3 \|z\|^2 + \frac{\rho^2(\|z\|) \|z\|^2}{4k} - k_{2b} \|\varepsilon\|^2 \tag{63}$$

where $k \in \mathbb{R}$ is defined as

$$k \triangleq \frac{k_1 (k_{2a} - c_2)}{\max \{k_1, (k_{2a} - c_2)\}}, \quad k_{2a} > c_2 \tag{64}$$

and $\rho(\cdot) \in \mathbb{R}$ is a positive, globally invertible, nondecreasing function defined as

$$\rho^2(\|z\|) \triangleq \rho_1^2(\|z\|) + (\rho_2(\|z\|) + c_1)^2.$$

The expression in (63) can be further upper bounded by a continuous, positive semi-definite function

$$\dot{V}_L(y, t) \leq -U(y) = c \left\| [z^T \; \varepsilon^T]^T \right\|^2 \quad \forall y \subset \mathcal{D} \tag{65}$$

for some positive constant c, where

$$\mathcal{D} \triangleq \left\{ y(t) \in \mathbb{R}^{n(m+2)+p+2} \mid \|y\| \leq \rho^{-1} \left(2\sqrt{\lambda_3 k} \right) \right\}.$$

Larger values of k will expand the size of the domain $\mathcal{D}$. The inequalities in (55) and (63) can be used to show that $V_L(y, t) \in \mathcal{L}_\infty$ in $\mathcal{D}$; hence, $e_i(t) \in \mathcal{L}_\infty$ and $\varepsilon(t), r(t), \tilde{\theta}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. The closed-loop error systems can now be used to conclude that all the remaining signals are bounded in $\mathcal{D}$, and the definitions for $U(y)$ and $z(t)$ can be used to prove that $U(y)$ is uniformly continuous in $\mathcal{D}$. Let $\mathcal{S} \subset \mathcal{D}$ denote a set defined as

$$\mathcal{S} \triangleq \left\{ y(t) \in \mathcal{D} \mid U_2(y(t)) < \lambda_1 \left(\rho^{-1} \left(2\sqrt{\lambda_3 k} \right) \right)^2 \right\}. \tag{66}$$

The region of attraction in (66) can be made arbitrarily large to include any initial conditions by increasing the control gain k (i.e., a semi-global stability result). Theorem 8.4 of Khalil (2002) can now be invoked to state that

$$c \left\| [z^T \; \varepsilon^T]^T \right\|^2 \to 0 \quad \text{as } t \to \infty \; \forall y(0) \in \mathcal{S}. \tag{67}$$

Based on the definition of $z(t)$, (67) can be used to show that

$$\|e_1(t)\|, \|\varepsilon(t)\| \to 0 \quad \text{as } t \to \infty \; \forall y(0) \in \mathcal{S}. \quad \square \tag{68}$$

6. Experiment

A testbed was used to implement the developed controller. The testbed consists of a circular disc of unknown inertia mounted on a direct-drive switched reluctance motor. A rectangular nylon block was mounted on a pneumatic linear thruster to apply an external friction load to the rotating disk. A pneumatic regulator maintained a constant pressure of 20 psi on the circular disk. The dynamics for the testbed are given as follows:
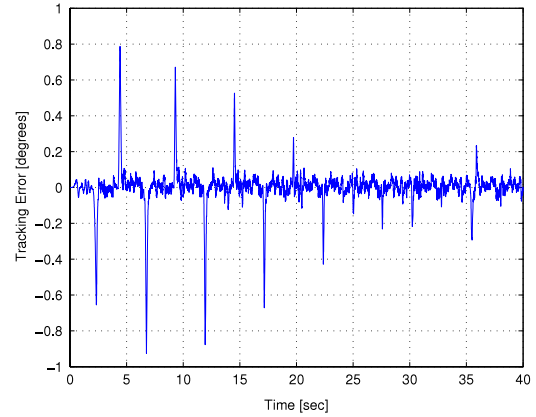
$$J\ddot{q} + f(\dot{q}) + \tau_d(t) = \tau(t) \tag{69}$$

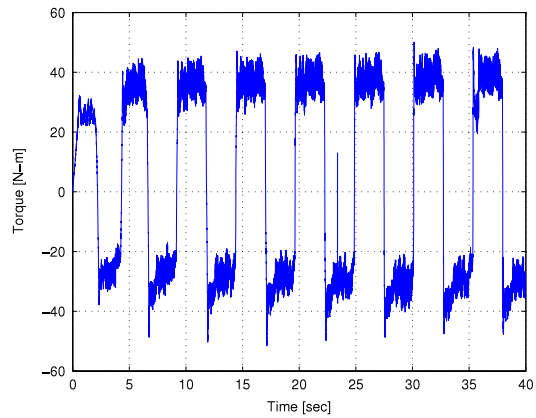**Fig. 1.** Tracking error for the proposed composite adaptive control law (RISE + CFF).

Fig. 2. Control torque for the proposed composite adaptive control law (RISE + CFF).

where $J \in \mathbb{R}$ denotes the combined inertia of the circular disk and rotor assembly, $f(\dot{q}) \in \mathbb{R}$ is the nonlinear friction, and $\tau_d(t) \in \mathbb{R}$ denotes a general nonlinear disturbance (e.g., unmodeled effects). The nonlinear friction term $f(\dot{q})$ is assumed to be modeled as a continuously differentiable function as described in Makkar, Hu, Sawyer, and Dixon (2007), Patre et al. (2008). The desired link trajectory is selected as follows (in degrees):

$$q_d(t) = 60.0 \sin(1.2t)(1 - \exp(-0.01t^3)). \tag{70}$$

Three different experiments were conducted to demonstrate the efficacy of the proposed controller. For each controller, the gains were not retuned (i.e., the common control gains remain the same for all controllers). First, no adaptation was used and the controller with only the RISE feedback was implemented. For the second experiment, the prediction error component of the update law in (29) was removed, resulting in a standard gradient-based update law (hereinafter denoted as RISE + FF). For the third experiment, the proposed composite adaptive controller in (39)–(40) (hereinafter denoted as RISE + CFF) was implemented. The tracking error is shown in Fig. 1. The control torque is shown in Fig. 2 and the adaptive estimates are depicted in Fig. 3. Each experiment was performed five times and the average RMS error and torque values were calculated. The average RMS tracking error (in deg) for the RISE controller is 0.219, compared to 0.138 and 0.102 for the RISE + FF and RISE + CFF (proposed), respectively. The average RMS torques (in Nm) for the respective controllers is 31.75, 32.99, and 32.49, which indicate that the proposed RISE+CFF controller yields the lowest RMS error with a similar control effort.