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ON REDUCTION OF DIFFERENTIAL INCLUSIONS AND LYAPUNOV STABILITY*

Rushikesh Kamalapurkar[1,**], Warren E. Dixon[2] and Andrew R. Teel[3]

Abstract. In this paper, locally Lipschitz, regular functions are utilized to identify and remove infeasible directions from set-valued maps that define differential inclusions. The resulting reduced set-valued map is pointwise smaller (in the sense of set containment) than the original set-valued map. The corresponding reduced differential inclusion, defined by the reduced set-valued map, is utilized to develop a generalized notion of a derivative for locally Lipschitz candidate Lyapunov functions in the direction(s) of a set-valued map. The developed generalized derivative yields less conservative statements of Lyapunov stability theorems, invariance theorems, invariance-like results, and Matrosov theorems for differential inclusions. Included illustrative examples demonstrate the utility of the developed theory.

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1. Introduction

Differential inclusions can be used to model and analyze a large variety of practical systems. For example, systems that utilize discontinuous control architectures such as sliding mode control, multiple model and sparse neural network adaptive control, finite state machines, gain scheduling control, etc., are analyzed using the theory of differential inclusions. Differential inclusions are also used to analyze robustness to bounded perturbations and modeling errors, to model physical phenomena such as coulomb friction and impact, and to model differential games [6, 13].

Asymptotic properties of trajectories of differential inclusions are typically analyzed using Lyapunov-like comparison functions. Several generalized notions of the directional derivative are utilized to characterize the change in the value of a candidate Lyapunov function along the trajectories of a differential inclusion. Early results on stability of differential inclusions that utilize nonsmooth candidate Lyapunov functions are based on Dini directional derivatives [20, 23] and contingent derivatives ([1], Chap. 6). For locally Lipschitz, regular

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[1] School of Mechanical and Aerospace Engineering, Oklahoma State University, Stillwater, OK, USA.
[2] Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL, USA.
[3] Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA, USA.

** Corresponding author: rushikesh.kamalapurkar@okstate.edu

candidate Lyapunov functions, stability results based on Clarke's notion of generalized directional derivatives have been developed in results such as [2, 9, 28]. In [28], Shevitz and Paden utilize the Clarke gradient to develop a set-valued generalized derivative along with several Lyapunov-based stability theorems. In [2], Bacciotti and Ceragioli introduce another set-valued generalized derivative that results in sets that are pointwise smaller than those generated by the set-valued derivative in [28]; hence, the Lyapunov theorems in [2] are generally less conservative than their counterparts in [28]. The Lyapunov theorems developed by Bacciotti and Ceragioli have also been shown to be less conservative than those based on Dini and contingent derivatives, provided locally Lipschitz, regular candidate Lyapunov functions are employed (*cf.* [4], Prop. 7).

In this paper, and in the preliminary work in [10], locally Lipschitz, regular functions are utilized to identify and remove infeasible directions from a set-valued map that defines a differential inclusion to yield a pointwise smaller (in the sense of set containment) set-valued map that defines an equivalent reduced differential inclusion. Using the reduced differential inclusion, a novel generalization of the set-valued derivatives in [28] and [2] is introduced for locally Lipschitz candidate Lyapunov functions. The developed technique yields less conservative statements of Lyapunov stability results (*cf.* [2, 17, 19, 20, 23, 28]), invariance results (*cf.* [3, 9, 14, 26]), invariance-like results (*cf.* [8], Thm. 2.5, [7]), and Matrosov results (*cf.* [15, 16, 21, 27, 29]) for differential inclusions.

The paper is organized as follows. Section 2 introduces the notation. Sections 3 and 4 review differential inclusions and Clarke-gradient-based set-valued derivatives from [28] and [2], respectively. In Section 5, locally Lipschitz, regular functions are used to identify the infeasible directions in a set-valued map that defines a differential inclusion. Section 6 develops a novel generalization of the notion of a derivative in the direction(s) of a set-valued map. Section 7 states stability theorems, invariance-like results, and Matrosov theorems for differential inclusions using the developed novel definition of a generalized derivative.[1] Illustrative examples where the developed stability theory is less conservative than results such as [2, 28] are presented. Section 8 summarizes the article and includes concluding remarks.

2. Notation

The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$, μ denotes the Lebesgue measure on $\mathbb{R}^n$, $\mathcal{D}$ denotes an open and connected subset of $\mathbb{R}^n$, and $\Omega \coloneqq \mathcal{D} \times [0, \infty)$. Elements of $\mathbb{R}^n$ are interpreted as column vectors and $(\cdot)^{\mathrm{T}}$ denotes the vector transpose operator. The set of positive integers excluding 0 is denoted by $\mathbb{N}$. For $a \in \mathbb{R}$, $\mathbb{R}_{\geq a}$ denotes the interval $[a, \infty)$ and $\mathbb{R}_{>a}$ denotes the interval (a, ∞). A set-valued map from A to the subsets of B is denoted by $F : A \rightrightarrows B$. For a set A, the convex hull, the closed convex hull, the closure, the interior, and the boundary are denoted by $\operatorname{co} A$, $\overline{\operatorname{co}} A$, $\overline{A}$, $\mathring{A}$, and $\operatorname{bd}(A)$, respectively. If $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ then $[a \ ; \ b]$ denotes the concatenated vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^{m+n}$. For $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$, the set $\{[a \ ; \ b] \mid a \in A, b \in B\}$ is denoted by $\begin{bmatrix} A \\ B \end{bmatrix}$ or $[A \ ; \ B]$. For $A, B \subseteq \mathbb{R}^n$, $A^{\mathrm{T}} B$ denotes the set $\left\{a^{\mathrm{T}} b \mid a \in A, b \in B\right\}$, $A \pm B$ denotes the set $\{a \pm b \in \mathbb{R}^n \mid a \in A, b \in B\}$, and $A\,(\geq) \leq B$ implies $\|a\|\,(\geq) \leq \|b\|$, $\forall a \in A$, and $\forall b \in B$. For $x \in \mathbb{R}^n$ and $r, l > 0$, the sets $\{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$, $\{y \in \mathbb{R}^n \mid \|x - y\| < r\}$, and $\{y \in \mathbb{R}^n \mid r \leq \|y\| \leq l\}$ are denoted by $\overline{\mathrm{B}}(x, r)$, $\mathrm{B}(x, r)$ and $\mathrm{D}(r, l)$, respectively. If $a \in \mathbb{R}$ then $|a|$ denotes the absolute value and if A is a set then $|A|$ denotes its cardinality. For $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $\operatorname{dist}(x, A) \coloneqq \inf_{y \in A} \|x - y\|$. Essentially bounded, $n-$times continuously differentiable, and locally Lipschitz functions with domain A and codomain B are denoted by $\mathcal{L}_\infty(A, B)$, $\mathcal{C}^n(A, B)$, and $\operatorname{Lip}(A, B)$, respectively. The zero element of $\mathbb{R}^n$ is denoted by 0_n, with the subscript n suppressed whenever clear from the context. The notation $\dot{V}$ is reserved for the total derivative of V with respect to time.

[1] An extension of the framework developed in this paper that generalizes LaSalle's invariance principle for time-invariant differential inclusions is available in [11].

3. Differential inclusions

Let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map. Consider the differential inclusion

$$\dot{x} \in F(x,t). \tag{3.1}$$

A locally absolutely continuous function $x : \mathcal{I}_x \to \mathcal{D}$ is called a *solution* to (3.1), with *interval of existence* $\mathcal{I}_x = [t_0, T)$, for some $0 \leq t_0 < T \leq \infty$, if $\dot{x}(t) \in F(x(t), t)$, for almost all $t \in \mathcal{I}_x$ ([6], p. 50). A solution is called *complete* if $\mathcal{I}_x = \mathbb{R}_{\geq t_0}$ and *maximal* if it does not have a proper right extension[2] which is also a solution to (3.1). If a solution is maximal and if the set $\overline{\{x(t) \mid t \in \mathcal{I}_x\}}$ is compact, then the solution is called *precompact*. Similar to ([25], Prop. 1), Zorn's lemma can be used to show that every solution to (3.1) admits a right extension that is also a maximal solution to (3.1). Let $\mathscr{S}(\mathcal{E})$ denote the set of all maximal solutions to (3.1) such that $(x(t_0), t_0) \in \mathcal{E} \subseteq \Omega$. The discussion in this article concerns set-valued maps that define differential inclusions that admit local solutions.

Definition 3.1. Let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map and $\mathcal{E} \subseteq \Omega$. The differential inclusion (3.1) is said to *admit local solutions over* $\mathcal{E}$ if for all $(y, t_0) \in \mathcal{E}$, there exists $T \in \mathbb{R}_{>t_0}$ and a locally absolutely continuous function $x : [t_0, T) \to \mathcal{D}$ such that $x(t_0) = y$ and $\dot{x}(t) \in F(x(t), t)$ for almost all $t \in [t_0, T)$.

Sufficient conditions for the existence of local solutions to differential inclusions can be found in ([6], Sect. 7, Thm. 1) and ([6], Sect. 7, Thm. 5). To assert the existence of complete solutions, the following notions of invariance are utilized in this article.

Definition 3.2. A set $A \subseteq \mathcal{D}$ is called *weakly forward invariant* with respect to (3.1) if $\forall x_0 \in A$, $\exists x(\cdot) \in \mathscr{S}(\{x_0\} \times \mathbb{R}_{\geq 0})$ such that $x(t) \in A$, $\forall t \in \mathcal{I}_x$. It is called *strongly forward invariant* with respect to (3.1) if every $x(\cdot) \in \mathscr{S}(A \times \mathbb{R}_{\geq 0})$ satisfies $x(t) \in A$, $\forall t \in \mathcal{I}_x$.

Forward invariance of a set $A \subseteq \mathcal{D}$ in the sense of Definition 3.2 does not imply completeness of any $x(\cdot) \in \mathscr{S}(A \times \mathbb{R}_{\geq 0})$ since $x(\cdot)$ can exit $\mathcal{D}$ in finite time, resulting in a finite interval of existence $\mathcal{I}_x$. However, the following Lemma, which is a slight generalization of ([25], Prop. 2), implies that under general conditions on F, if A is also compact then $\mathscr{S}(A \times \mathbb{R}_{\geq 0})$ contains complete solutions, and under strong forward invariance of A, all solutions in $\mathscr{S}(A \times \mathbb{R}_{\geq 0})$ are complete.

Lemma 3.3. *Let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map such that (3.1) admits local solutions over Ω. Let $x(\cdot)$ be a maximal solution to (3.1) such that $\overline{\{x(t) \mid t \in \mathcal{I}_x\}} \subset \mathcal{D}$. If the set $\cup_{t \in \mathcal{J}} F(x(t), t)$ is bounded for every subinterval $\mathcal{J} \subseteq \mathcal{I}_x$ of finite length, then $x(\cdot)$ is complete.*

Proof. For the sake of contradiction, assume that the interval of existence, $\mathcal{I}_x$, is finite. That is, $\mathcal{I}_x = [t_0, T)$ for some $t_0 < T < \infty$. Boundedness of the set $\cup_{t \in [t_0, T)} F(x(t), t)$ implies that $\dot{x}(\cdot) \in \mathcal{L}_\infty([t_0, T), \mathbb{R}^n)$. Since $x(\cdot)$ is locally absolutely continuous on $[t_0, T)$, it can be concluded that $\forall t_1, t_2 \in [t_0, T)$, $\|x(t_2) - x(t_1)\|_2 = \left\| \int_{t_1}^{t_2} \dot{x}(\tau)\, \mathrm{d}\tau \right\|_2$. Furthermore, $\dot{x}(\cdot) \in \mathcal{L}_\infty([t_0, T), \mathbb{R}^n)$ implies that $\left\| \int_{t_1}^{t_2} \dot{x}(\tau)\, \mathrm{d}\tau \right\|_2 \leq \int_{t_1}^{t_2} M\, \mathrm{d}\tau$, where M is a positive constant. Thus, $\|x(t_2) - x(t_1)\|_2 \leq M |t_2 - t_1|$, and hence, $x(\cdot)$ is uniformly continuous on $[t_0, T)$. Therefore, $x(\cdot)$ admits a continuous extension $x' : [t_0, T] \to \mathbb{R}^n$ ([24], Chap. 4, Exer. 13). Since $x'(\cdot)$ is continuous, $\mathcal{D}$ is open, and $\overline{\{x(t) \mid t \in [t_0, T)\}} \subset \mathcal{D}$, it is clear that $x'(T) \in \mathcal{D}$. Since (3.1) admits local solutions over Ω, $x'(\cdot)$ can be extended into a solution to (3.1) on the interval $[t_0, T')$ for some $T' > T$, which contradicts the maximality of $x(\cdot)$. Hence, $x(\cdot)$ is complete. $\square$

Remark 3.4. The hypothesis of Lemma 3.3, that the set $\cup_{t \in \mathcal{J}} F(x(t), t)$ needs to be bounded for every subinterval $\mathcal{J} \subseteq \mathcal{I}_x$ of finite length, is met if, *e.g.*, $(x, t) \mapsto F(x, t)$ is locally bounded over Ω and $x(\cdot)$ is precompact (*cf.* [22], Prop. 5.15).

[2] A solution $y : [t_0, T_y) \to \mathbb{R}^n$ to (3.1) is a *(proper) right extension* of a solution $x : [t_0, T_x) \to \mathbb{R}^n$ to (3.1) if $T_y\,(>) \geq T_x$ and $y(t) = x(t), \forall t \in [t_0, T_x)$.

The following section presents a summary of the relevant Lyapunov methods that utilize Clarke's notion of generalized directional derivatives and gradients ([5], p. 39) for the analysis of differential inclusions.

4. Set-valued derivatives

Clarke gradients are utilized in [28] by Shevitz and Paden to introduce the following set-valued derivative of a locally Lipschitz, positive definite (*i.e.*, locally positive definite in the sense of ([30], Sect. 5.2, Def. 3) at (x,t), for all (x,t) in its domain) candidate Lyapunov function that is regular (*i.e.*, regular at (x,t), in the sense of ([5], Def. 2.3.4), for all (x,t) in its domain).

Definition 4.1. [28] Given a regular function $V \in \operatorname{Lip}(\Omega, \mathbb{R})$, and a set-valued map $F : \Omega \rightrightarrows \mathbb{R}^n$, the *set-valued derivative of V in the direction(s) F* is defined as

$$\dot{\tilde{V}}(x,t) \coloneqq \bigcap_{p \in \partial V(x,t)} p^{\mathrm{T}} \begin{bmatrix} F(x,t) \\ \{1\} \end{bmatrix}, \forall (x,t) \in \Omega,$$

where ∂V denotes the *Clarke gradient* of V, defined as (see also, [5], Thm. 2.5.1)

$$\partial V(x,t) \coloneqq \overline{\mathrm{co}} \left\{ \lim \nabla V(x_i,t_i) \,|\, (x_i,t_i) \to (x,t), (x_i,t_i) \in \Omega \setminus (\Omega_V \cup S) \right\}, \forall (x,t) \in \Omega, \tag{4.1}$$

where Ω_V is the set of Lebesgue measure zero where the gradient ∇V of V is not defined and $S \subset \Omega$ is any other set of Lebesgue measure zero.

Lyapunov stability theorems developed using the set-valued derivative $\dot{\tilde{V}}$ exploit the property that every upper bound of the set $\dot{\tilde{V}}(x(t),t)$ is also an upper bound of $\dot{V}(x(t),t)$, for almost all t where $\dot{V}(x(t),t)$ exists. The aforementioned fact is a consequence of the following proposition.

Proposition 4.2. *[28] Let $x : \mathcal{I}_x \to \mathcal{D}$ be a solution to (3.1). If $V \in \operatorname{Lip}(\Omega,\mathbb{R})$ is a regular function, then $\dot{V}(x(t),t)$ exists for almost all $t \in \mathcal{I}_x$ and $\dot{V}(x(t),t) \in \dot{\tilde{V}}(x(t),t)$, for almost all $t \in \mathcal{I}_x$.*

Proof. See ([28], Thm. 2.2). □

In [2], the notion of a set-valued derivative is further generalized via the following definition.

Definition 4.3. [2] For a regular function $V \in \operatorname{Lip}(\Omega,\mathbb{R})$ and a set-valued map $F : \Omega \rightrightarrows \mathbb{R}^n$, the *set-valued derivative of V in the direction(s) F* is defined as

$$\dot{\overline{V}}(x,t) \coloneqq \left\{ a \in \mathbb{R} \mid \exists q \in F(x,t) \mid p^{\mathrm{T}} [q \;;\; 1] = a, \forall p \in \partial V(x,t) \right\}, \forall (x,t) \in \Omega.$$

The set-valued derivative in Definition 4.3 results in less conservative sufficient conditions for Lyapunov stability than Definition 4.1 since it is contained within the set-valued derivative in Definition 4.1 and, as evidenced by ([2], Exam. 1), the containment can be strict. The Lyapunov stability theorems developed in [2] exploit the property that Proposition 4.2 also holds for $\dot{\overline{V}}$ (see [2], Lem. 1).

Inspired by [2, 28], the following section presents a novel notion of reduced differential inclusions that results in statements of Lyapunov theorems that are less conservative than those available in the literature.

5. Reduced differential inclusions

By definition, $\dot{\overline{V}}(x,t) \subseteq \dot{\tilde{V}}(x,t)$, $\forall (x,t) \in \Omega$, which, assuming compact values, implies $\max \dot{\overline{V}}(x,t) \leq \max \dot{\tilde{V}}(x,t)$, $\forall (x,t) \in \Omega$. In some cases, $\max \dot{\overline{V}}$ can be strictly smaller than $\max \dot{\tilde{V}}$ and Lyapunov theorems based on $\dot{\overline{V}}$ can be less conservative than those based on $\dot{\tilde{V}}$ ([2], Exam. 1). A tighter bound on the evolution of

V along an orbit of (3.1) can be obtained by examining the following equivalent representation of $\max \dot{\overline{V}}$:[3]

$$\max \dot{\overline{V}}(x,t) = \min_{p \in \partial V(x,t)} \max_{q \in G_V^F(x,t)} p^{\mathrm{T}} [q \ ; \ 1], \tag{5.1}$$

where, for any regular function $U \in \mathrm{Lip}(\Omega, \mathbb{R})$, and any set-valued map $H : \Omega \rightrightarrows \mathbb{R}^n$, the reduction $G_U^H : \Omega \rightrightarrows \mathbb{R}^n$ is defined as

$$G_U^H(x,t) := \left\{ q \in H(x,t) \mid \exists a \in \mathbb{R} \mid p^{\mathrm{T}} [q \ ; \ 1] = a, \forall p \in \partial U(x,t) \right\}, \forall (x,t) \in \Omega. \tag{5.2}$$

The representation in (5.1), along with results such as ([2], Thm. 2), suggest that the only directions in F that affect the stability properties of solutions to (3.1) are those included in G_V^F, that is, the directions that map the Clarke gradient of V to a singleton. The key observation in this paper is that *the statement above remains true even when V is replaced by any arbitrary locally Lipschitz, regular function U*. The following proposition formalizes the aforementioned observation. For clarity, the proposition is stated here for autonomous differential inclusions. The analysis of nonautonomous differential inclusions is deferred to Theorem 7.2.

Proposition 5.1. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a locally bounded map with compact values such that $\dot{x} \in F(x)$ admits local solutions over $\mathbb{R}^n$. Let $V \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R})$ be a positive definite and regular function and let $U \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R})$ be any other regular function. If*

$$\min_{p \in \partial V(x)} \max_{q \in G_U^F(x)} p^{\mathrm{T}} q \leq 0, \quad \forall x \in \mathbb{R}^n,$$

then $\dot{x} \in F(x)$ is stable at $x = 0$.

Proof. The proposition follows from the more general result stated in Theorem 7.2. □

Proposition 5.1 indicates that locally Lipschitz, regular functions help discover the admissible directions in F. That is, from the point of view of Lyapunov stability, only the directions in G_U^F are relevant, where U can be an arbitrary locally Lipschitz, regular function, possibly different from the candidate Lyapunov function V.

In fact, the differential inclusion $\dot{x} \in G_U^F(x,t)$ is, in a sense, equivalent to the differential inclusion $\dot{x} \in F(x,t)$. To make the equivalence precise, the following definition of a reduced set-valued map is introduced.

Definition 5.2. Let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map and $\mathcal{U} := \{U_i\}_{i \in \mathcal{N}} \subset \mathrm{Lip}(\Omega, \mathbb{R})$ be a countable collection of regular functions, indexed over $\mathcal{N} \subseteq \mathbb{N}$. The set-valued map $\tilde{F}_{\mathcal{U}} : \Omega \rightrightarrows \mathbb{R}^n$, defined as

$$\tilde{F}_{\mathcal{U}}(x,t) := \bigcap_{i \in \mathcal{N}} G_{U_i}^F(x,t) = \bigcap_{i \in \mathcal{N}} \left\{ q \in F(x,t) \mid \exists a \in \mathbb{R} \mid p^{\mathrm{T}} [q \ ; \ 1] = a, \forall p \in \partial U_i(x,t) \right\}, \forall (x,t) \in \Omega,$$

is called the *$\mathcal{U}-$reduced set-valued map for F* and the differential inclusion $\dot{x} \in \tilde{F}_{\mathcal{U}}(x,t)$ is called the *$\mathcal{U}-$reduced differential inclusion for* (3.1). If $\mathcal{U}$ is empty, then $\tilde{F}_{\mathcal{U}} := F$.

In other words, the $\mathcal{U}-$reduced set-valued map collects all directions q in F that, through the inner product $p^{\mathrm{T}} [q \ ; \ 1]$, map the Clarke gradient of all functions in $\mathcal{U}$ to a singleton. The following theorem demonstrates the key utility of the reduction in Definition 5.2, *i.e.*, the reduced differential inclusion is found to be sufficient to characterize the solutions to (3.1).

Theorem 5.3. *If $x : \mathcal{I}_x \to \mathcal{D}$ is a solution to (3.1), then $\dot{x}(t) \in \tilde{F}_{\mathcal{U}}(x(t), t)$ for almost all $t \in \mathcal{I}_x$.*

Proof. The theorem can be proved using techniques similar to ([2], Lem. 1). Consider the set of times $\mathcal{T} \subseteq \mathcal{I}_x$ where $\dot{x}(t)$ is defined, $\dot{U}_i(x(t), t)$ is defined $\forall i \in \mathcal{N}$, and $\dot{x}(t) \in F(x(t), t)$. Since $x(\cdot)$ is a solution to (3.1), $\mathcal{N}$ is countable, and $U_i \in \mathrm{Lip}(\Omega, \mathbb{R})$, it can be concluded that $t \mapsto U_i(x(t), t)$ is absolutely continuous, and hence, $\mu(\mathcal{I}_x \setminus \mathcal{T}) = 0$. The objective is to show that $\dot{x}(t)$ belongs to $\in \tilde{F}_{\mathcal{U}}(x(t), t)$ on $\mathcal{T}$, not just $F(x(t), t)$.

[3] The minimization in (5.1) serves to maintain consistency of notation, but is in fact, redundant.

Since each function U_i is locally Lipschitz, for $t \in \mathcal{T}$ the time derivative of U_i can be expressed as

$$\dot{U}_i\left(x\left(t\right),t\right) = \lim_{h\to 0} \frac{\left(U_i\left(x\left(t\right)+h\dot{x}\left(t\right),t+h\right) - U_i\left(x\left(t\right),t\right)\right)}{h}.$$

Since each U_i is regular, for $i \geq 1$,

$$\begin{aligned}
\dot{U}_i\left(x\left(t\right),t\right) &= U'_{i+}\left(\left[x\left(t\right)\ ;\ t\right],\left[\dot{x}\left(t\right)\ ;\ 1\right]\right) = U_i^o\left(\left[x\left(t\right)\ ;\ t\right],\left[\dot{x}\left(t\right)\ ;\ 1\right]\right) = \max_{p\in\partial U_i\left(x(t),t\right)} p^{\mathrm{T}}\left[\dot{x}\left(t\right)\ ;\ 1\right],\\
\dot{U}_i\left(x\left(t\right),t\right) &= U'_{i-}\left(\left[x\left(t\right)\ ;\ t\right],\left[\dot{x}\left(t\right)\ ;\ 1\right]\right) = U_i^o\left(\left[x\left(t\right)\ ;\ t\right],\left[\dot{x}\left(t\right)\ ;\ 1\right]\right) = \min_{p\in\partial U_i\left(x(t),t\right)} p^{\mathrm{T}}\left[\dot{x}\left(t\right)\ ;\ 1\right],
\end{aligned}$$

where $U'_+\left(x,v\right) \coloneqq \lim_{h\downarrow 0}\frac{U(x+hv)-U(x)}{h}$ and $U'_-\left(x,v\right) \coloneqq \lim_{h\uparrow 0}\frac{U(x+hv)-U(x)}{h}$ denote the right and left directional derivatives, and $U^o\left(x,v\right) \coloneqq \limsup_{y\to x,h\downarrow 0}\frac{U(y+hv)-U(y)}{h}$ denotes the Clarke-generalized derivative of U. Thus, $p^{\mathrm{T}}\left[\dot{x}\left(t\right)\ ;\ 1\right] = \dot{U}_i\left(x\left(t\right),t\right), \forall p \in \partial U_i\left(x\left(t\right),t\right)$, which implies $\dot{x}\left(t\right) \in G^F_{U_i}\left(x\left(t\right),t\right)$, for each i. Therefore, $\dot{x}\left(t\right) \in \tilde{F}_{\mathcal{U}}\left(x\left(t\right),t\right)$, $\forall t \in \mathcal{T}$. Since $\mu\left(\mathcal{I}_x \setminus \mathcal{T}\right) = 0$, $\dot{x}\left(t\right) \in \tilde{F}_{\mathcal{U}}\left(x\left(t\right),t\right)$, for almost all $t \in \mathcal{I}_x$. $\square$

Although not directly related to the current discussion, it is worth mentioning that Theorem 5.3 also expands the class of differential inclusions that admit solutions, as detailed in the following corollary.

Corollary 5.4. *A differential inclusion $\dot{x} \in G\left(x,t\right)$, with $G : \Omega \rightrightarrows \mathbb{R}^n$, admits local solutions over $\mathcal{E} \subseteq \Omega$ if there exists: a set-valued map, $F : \Omega \rightrightarrows \mathbb{R}^n$, such that (3.1) admits local solutions over $\mathcal{E}$; and a countable collection, $\mathcal{U} \subset \mathrm{Lip}\left(\Omega,\mathbb{R}\right)$, of regular functions, such that G is the $\mathcal{U}$-reduced set-valued map for F.*

The following example illustrates the utility of Theorem 5.3.

Example 5.5. Consider the differential inclusion in (3.1), where $x \in \mathbb{R}$, and $F : \mathbb{R} \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}$ is defined as

$$F\left(x,t\right) \coloneqq \begin{cases} 2\,\mathrm{sgn}\left(x-1\right) & \left|x\right| \neq 1,\\ \left[-2,5\right] & \left|x\right| = 1,\end{cases}$$

where $\mathrm{sgn}\left(x\right)$ denotes the sign of x. The function $U : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, defined as

$$U\left(x,t\right) \coloneqq \begin{cases} \left|x\right| & \left|x\right| \leq 1,\\ 2\left|x\right| - 1 & \left|x\right| > 1,\end{cases}$$

satisfies $U \in \mathrm{Lip}\left(\mathbb{R} \times \mathbb{R}_{\geq 0},\mathbb{R}\right)$. In addition, since U is convex, it is also regular ([5], Prop. 2.3.6). The Clarke gradient of U is given by

$$\partial U\left(x,t\right) = \begin{cases} \left[\left[1,2\right]\ ;\ \left\{0\right\}\right] & x = 1,\\ \left[\left[-2,-1\right]\ ;\ \left\{0\right\}\right] & x = -1,\\ \left[\left\{\mathrm{sgn}\left(x\right)\right\}\ ;\ \left\{0\right\}\right] & 0 < \left|x\right| < 1,\\ \left[\left\{2\,\mathrm{sgn}\left(x\right)\right\}\ ;\ \left\{0\right\}\right] & \left|x\right| > 1,\\ \left[\left[-1,1\right]\ ;\ \left\{0\right\}\right] & x = 0.\end{cases}$$

The set G^F_U is then given by

$$G^F_U\left(x,t\right) = \begin{cases} \left\{0\right\} & \left|x\right| = 1,\\ \emptyset & x = 0,\\ F\left(x,t\right) & \text{otherwise.}\end{cases}$$

Theorem 5.3 can then be invoked to conclude that every solution $x : \mathcal{I}_x \to \mathbb{R}$ to (3.1) satisfies $\dot{x}(t) \in \tilde{F}_{\{U\}}(x(t), t) = G_U^F(x(t), t)$, for almost all $t \in \mathcal{I}_x$.

6. Generalized time derivatives

Given a countable collection $\mathcal{U} \subset \mathrm{Lip}(\Omega, \mathbb{R})$ of regular functions and a set-valued map $F : \Omega \rightrightarrows \mathbb{R}^n$ with compact values, Proposition 5.1 and Theorem 5.3 suggest the following notion of a generalized derivative of V in the direction(s) F.

Definition 6.1. The *$\mathcal{U}$-generalized derivative* of $V \in \mathrm{Lip}(\Omega, \mathbb{R})$ in the direction(s) F, denoted by $\dot{\overline{V}}_{\mathcal{U}} : \Omega \to \mathbb{R}$ is defined, $\forall (x,t) \in \Omega$, as

$$\dot{\overline{V}}_{\mathcal{U}}(x,t) \coloneqq \min_{p \in \partial V(x,t)} \max_{q \in \tilde{F}_{\mathcal{U}}(x,t)} p^{\mathrm{T}} [q \ ; \ 1], \tag{6.1}$$

if V is regular, and

$$\dot{\overline{V}}_{\mathcal{U}}(x,t) \coloneqq \max_{p \in \partial V(x,t)} \max_{q \in \tilde{F}_{\mathcal{U}}(x,t)} p^{\mathrm{T}} [q \ ; \ 1], \tag{6.2}$$

if V is not regular. The $\mathcal{U}$-generalized derivative is understood to be $-\infty$ when $\tilde{F}_{\mathcal{U}}(x,t)$ is empty.

Definition 6.1 facilitates a unified treatment of Lyapunov stability theory using regular as well as nonregular candidate Lyapunov functions. A candidate Lyapunov function will be called a Lyapunov function if the $\mathcal{U}$-generalized derivative is negative.

Definition 6.2. If $V \in \mathrm{Lip}(\Omega, \mathbb{R})$ is positive definite and if $\dot{\overline{V}}_{\mathcal{U}}(x,t) \leq 0, \forall (x,t) \in \Omega$, then V is called a *$\mathcal{U}$-generalized Lyapunov function* for (3.1).

If V is regular, then it can be assumed, without loss of generality, that $V \in \mathcal{U}$. In this case, $\tilde{F}_{\mathcal{U}} \subseteq G_V^F$, and hence, $\dot{\overline{V}}_{\mathcal{U}}(x,t) \leq \max \dot{\overline{V}}(x,t), \forall (x,t) \in \Omega$. Thus, by judicious selection of the functions in $\mathcal{U}$, $\dot{\overline{V}}_{\mathcal{U}}(x,t)$ can be constructed to be less conservative than the set-valued derivatives in [2, 28]. Naturally, if $\mathcal{U} = \{V\}$ then $\dot{\overline{V}}_{\mathcal{U}} = \dot{\overline{V}}$.

In general, the $\mathcal{U}$-generalized derivative does not satisfy the chain rule as stated in Proposition 4.2. However, it satisfies the following *weak chain rule* which turns out to be sufficient for Lyapunov-based analysis of differential inclusions.

Theorem 6.3. *If $V \in \mathrm{Lip}(\Omega, \mathbb{R})$ and $\mathscr{S}(\Omega) \neq \emptyset$, then $\forall x(\cdot) \in \mathscr{S}(\Omega)$,*

$$\dot{V}(x(t), t) \in (\partial V(x(t), t))^{\mathrm{T}} \begin{bmatrix} \tilde{F}_{\mathcal{U}}(x(t), t) \\ \{1\} \end{bmatrix}, \tag{6.3}$$

for almost all $t \in \mathcal{I}_x$. In addition, if there exists a function $W : \Omega \to \mathbb{R}$ such that $\dot{\overline{V}}_{\mathcal{U}}(x,t) \leq W(x,t)$, $\forall (x,t) \in \Omega$, then $\dot{V}(x(t), t) \leq W(x(t), t)$, for almost all $t \in \mathcal{I}_x$.

Proof. Let $x(\cdot) \in \mathscr{S}(\Omega)$. Consider a set of times $\mathcal{T} \subseteq \mathcal{I}_x$ where $\dot{x}(t)$, $\dot{V}(x(t), t)$, and $\dot{U}_i(x(t), t)$ are defined $\forall i \geq 0$ and $\dot{x}(t) \in \tilde{F}_{\mathcal{U}}(x(t), t)$. Using Theorem 5.3 and the facts that $x(\cdot)$ is absolutely continuous and V is locally Lipschitz, it can be concluded that $\mu(\mathcal{I}_x \setminus \mathcal{T}) = 0$.

If V is regular, then arguments similar to the proof of Theorem 5.3 can be used to conclude that $\dot{V}(x(t), t) = p^{\mathrm{T}} [\dot{x}(t) \ ; \ 1], \forall p \in \partial V(x(t), t), \forall t \in \mathcal{T}$. Thus, (6.1) and Theorem 5.3 imply that $\dot{V}(x(t), t) \in (\partial V(x(t), t))^{\mathrm{T}} \left[\tilde{F}_{\mathcal{U}}(x(t), t) \ ; \ \{1\}\right]$ and $\dot{V}(x(t), t) \leq W(x(t), t)$, for almost all $t \in \mathcal{I}_x$.

If V is not regular, then ([4], Prop. 4) (see also, [18], Thm. 2) can be used to conclude that, for almost every $t \in \mathcal{I}_x$, $\exists p_0 \in \partial V(x(t), t)$ such that $\dot{V}(x(t), t) = p_0^{\mathrm{T}}[\dot{x}(t)\ ;\ 1]$. Thus, (6.2) and Theorem 5.3 imply that $\dot{V}(x(t), t) \in (\partial V(x(t), t))^{\mathrm{T}}\left[\tilde{F}_{\mathcal{U}}(x(t), t)\ ;\ \{1\}\right]$ and $\dot{V}(x(t), t) \leq W(x(t), t)$ for almost all $t \in \mathcal{I}_x$. □

The following sections develop relaxed Lyapunov-like stability theorems for differential inclusions based on the properties of the $\mathcal{U}$-generalized derivative hitherto established.

7. Stability

In this section, $\mathcal{U}$-generalized derivatives are used to establish the following forms of uniform and asymptotic stability.

Definition 7.1. The differential inclusion in (3.1) is said to be (strongly)

(a) *uniformly stable* at $x = 0$, if $\forall \epsilon > 0\ \exists \delta > 0$ such that every $x(\cdot) \in \mathscr{S}\left(\overline{\mathrm{B}}(0, \delta) \times \mathbb{R}_{\geq 0}\right)$ is complete and satisfies $x(t) \in \overline{\mathrm{B}}(0, \epsilon)$, $\forall t \in \mathbb{R}_{\geq t_0}$.
(b) *globally uniformly stable* at $x = 0$, if it is uniformly stable at $x = 0$ and $\forall \epsilon > 0\ \exists \Delta > 0$ such that every $x(\cdot) \in \mathscr{S}\left(\overline{\mathrm{B}}(0, \epsilon) \times \mathbb{R}_{\geq 0}\right)$ is complete and satisfies $x(t) \in \overline{\mathrm{B}}(0, \Delta)$, $\forall t \in \mathbb{R}_{\geq t_0}$.
(c) *uniformly asymptotically stable* at $x = 0$ if it is uniformly stable at $x = 0$ and $\exists c > 0$ such that $\forall \epsilon > 0$ $\exists T \geq 0$ such that every $x(\cdot) \in \mathscr{S}\left(\overline{\mathrm{B}}(0, c) \times \mathbb{R}_{\geq 0}\right)$ is complete and satisfies $x(t) \in \overline{\mathrm{B}}(0, \epsilon)$, $\forall t \in \mathbb{R}_{\geq t_0 + T}$.
(d) *globally uniformly asymptotically stable* at $x = 0$ if it is uniformly stable at $x = 0$ and $\forall c, \epsilon > 0\ \exists T \geq 0$ such that every $x(\cdot) \in \mathscr{S}\left(\overline{\mathrm{B}}(0, c) \times \mathbb{R}_{\geq 0}\right)$ is complete and satisfies $x(t) \in \overline{\mathrm{B}}(0, \epsilon)$, $\forall t \in \mathbb{R}_{\geq t_0 + T}$.

While the results in this section are stated in terms of stability of the state at the origin and uniformity with respect to time, they extend in a straightforward manner to partial stability and uniformity with respect to a part of the state (see, *e.g.*, [8], Def. 4.1), and stability of arbitrary compact sets.

7.1. Lyapunov stability

The following fundamental Lyapunov-based stability result demonstrates the utility of $\mathcal{U}$-generalized derivatives.

Theorem 7.2. *Let $0 \in \mathcal{D}$ and let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a locally bounded set-valued map with compact values such that (3.1) admits local solutions over Ω. If there exists a positive definite function $V \in \mathrm{Lip}(\Omega, \mathbb{R})$, a pair of positive definite functions $\overline{W}, \underline{W} \in \mathcal{C}^0(\mathcal{D}, \mathbb{R})$, and a countable collection $\mathcal{U} \subset \mathrm{Lip}(\Omega, \mathbb{R})$ of regular functions, such that*

$$\underline{W}(x) \leq V(x, t) \leq \overline{W}(x), \quad \forall (x, t) \in \Omega,$$
$$\overline{\dot{V}}_{\mathcal{U}}(x, t) \leq 0, \qquad \text{for all } x \in \mathcal{D}, \text{ and almost all } t \in \mathbb{R}_{\geq 0}, \tag{7.1}$$

then (3.1) is uniformly stable at $x = 0$. In addition, if there exists a positive definite function $W \in \mathcal{C}^0(\mathcal{D}, \mathbb{R})$ such that

$$\overline{\dot{V}}_{\mathcal{U}}(x, t) \leq -W(x), \tag{7.2}$$

for all $x \in \mathcal{D}$ and almost all $t \in \mathbb{R}_{\geq 0}$, then (3.1) is uniformly asymptotically stable at $x = 0$. Furthermore, if $\mathcal{D} = \mathbb{R}^n$ and if the sublevel sets $\{x \in \mathbb{R}^n \mid \underline{W}(x) \leq c\}$ are compact $\forall c \in \mathbb{R}_{\geq 0}$, then (3.1) is globally uniformly asymptotically stable at $x = 0$.

Proof. Select $r > 0$ such that $\overline{\mathrm{B}}(0, r) \subset \mathcal{D}$. Let $x(\cdot) \in \mathscr{S}(\Omega_c \times \mathbb{R}_{\geq 0})$ where $\Omega_c \coloneqq \left\{x \in \overline{\mathrm{B}}(0, r) \,|\, \overline{W}(x) \leq c\right\}$ for some $c \in \left[0, \min_{\|x\|_2 = r} \underline{W}(x)\right)$. Using Theorem 6.3 and ([7], Lem. 2),

$$V(x(t_0), t_0) \geq V(x(t), t), \quad \forall t \in \mathcal{I}_x. \tag{7.3}$$

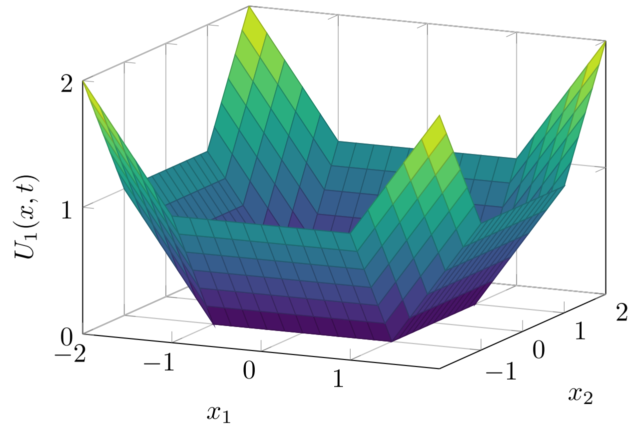

FIGURE 1. A snapshot of the function $U_1 : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$.

Using (7.3) and arguments similar to ([12], Thm. 4.8), it can be shown that every $x(\cdot) \in \mathscr{S}(\Omega_c \times \mathbb{R}_{\geq 0})$ satisfies $x(t) \in \overline{\mathrm{B}}(0,r)$, for all $t \in \mathcal{I}_x$. Therefore, all solutions $x(\cdot) \in \mathscr{S}(\Omega_c \times \mathbb{R}_{\geq 0})$ are precompact, and as a consequence of Lemma 3.3, complete. Since $\overline{W}$ is continuous and positive definite, $\exists \delta > 0$ such that $\overline{\mathrm{B}}(0,\delta) \subset \Omega_c$. Since δ is independent of t_0, uniform stability of (3.1) at $x = 0$ is established. The rest of the proof is identical to ([30], Sect. 5.3.2), and is therefore omitted. □

In the following example, tests based on $\dot{\overline{V}}$ and $\dot{\tilde{V}}$ are inconclusive, but Theorem 7.2 can be invoked to conclude global uniform asymptotic stability of the origin.

Example 7.3. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as

$$H(y) := \begin{cases} \{0\} & |y| \neq 1, \\ \left[-\frac{1}{2}, \frac{1}{2}\right] & |y| = 1, \end{cases}$$

and let $F : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^2$ be defined as

$$F(x,t) := \begin{bmatrix} \{-x_1 + x_2(1 + g(t))\} + H(x_2) \\ \{-x_1 - x_2\} + H(x_1) \end{bmatrix}, \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0},$$

where $g \in \mathcal{C}^1(\mathbb{R}_{\geq 0}, \mathbb{R})$, $0 \leq g(t) \leq 1, \forall t \in \mathbb{R}_{\geq 0}$ and $\dot{g}(t) \leq g(t), \forall t \in \mathbb{R}_{\geq 0}$. Consider the differential inclusion in (3.1) and the candidate Lyapunov function $V : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined as $V(x,t) := x_1^2 + (1 + g(t)) x_2^2$. the candidate Lyapunov function satisfies $\|x\|_2^2 \leq V(x,t) \leq 2\|x\|_2^2, \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. In this case, since $V \in \mathcal{C}^1\left(\mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \mathbb{R}\right)$, similar to ([12], Exam. 4.20), the set-valued derivatives $\dot{\overline{V}}$ in [2] and $\dot{\tilde{V}}$ in [28] satisfy the bound $\dot{\overline{V}}(x,t), \dot{\tilde{V}}(x,t) \leq \left\{-2x_1^2 - 2x_2^2\right\} + 2x_1 H(x_2) + 2x_2 h(t) H(x_1)$, where $h(t) := 1 + g(t)$ and the inequality $2 + 2g(t) - \dot{g}(t) \geq 2$ is utilized. Therefore, neither $\dot{\tilde{V}}(x,t)$ nor $\dot{\overline{V}}(x,t)$ can be shown to be negative semidefinite everywhere.

The function $U_1 : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, defined as (see Fig. 1)

$$U_1(x,t) = \max((x_1 - 1), 0) - \min((x_1 + 1), 0) + \max((x_2 - 1), 0) - \min((x_2 + 1), 0), \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \tag{7.4}$$

satisfies $U_1 \in \mathrm{Lip}\left(\mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \mathbb{R}\right)$. In addition, since U_1 is convex, it is also regular ([5], Prop. 2.3.6). With

$$\operatorname{sgn}1(y) := \begin{cases} 0 & -1 < y < 1, \\ \operatorname{sgn}(y) & \text{otherwise}, \end{cases}$$

the Clarke gradient of U_1 is given by

$$\partial U_1(x,t) = \begin{cases} [\{\operatorname{sgn}1(x_1)\}\ ;\ \{\operatorname{sgn}1(x_2)\}\ ;\ \{0\}] & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ [\overline{\mathrm{co}}\,\{0, \operatorname{sgn}(x_1)\}\ ;\ \{\operatorname{sgn}1(x_2)\}\ ;\ \{0\}] & |x_1| = 1 \wedge |x_2| \neq 1, \\ [\{\operatorname{sgn}1(x_1)\}\ ;\ \overline{\mathrm{co}}\,\{0, \operatorname{sgn}(x_2)\}\ ;\ \{0\}] & |x_1| \neq 1 \wedge |x_2| = 1, \\ [\overline{\mathrm{co}}\,\{0, \operatorname{sgn}(x_1)\}\ ;\ \overline{\mathrm{co}}\,\{0, \operatorname{sgn}(x_2)\}\ ;\ \{0\}] & |x_1| = 1 \wedge |x_2| = 1, \end{cases}$$

The $\{U_1\}$-reduced set-valued map corresponding to F is given by

$$\tilde{F}_{\{U_1\}}(x,t) = \begin{cases} F(x,t) & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ \emptyset & \text{otherwise}. \end{cases}$$

The $\{U_1\}$-generalized derivative of V in the direction(s) F is then given by

$$\begin{aligned} \dot{\overline{V}}_{\{U_1\}}(x,t) &:= \max_{q \in \tilde{F}_{\{U_1\}}(x,t)} \left(\frac{\partial V}{\partial (x,t)}(x,t)\right)^{\mathrm{T}} [q\ ;\ 1], \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \\ &= \begin{cases} \begin{bmatrix} 2x_1 & 2x_2 h(t) & \dot{g}(t) x_2^2 \end{bmatrix} [-x_1 + x_2 h(t)\ ;\ -x_1 - x_2\ ;\ 1] & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ -\infty & \text{otherwise}, \end{cases} \\ &\leq -2\|x\|_2^2, \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}. \end{aligned}$$

Theorem 7.2 can then be invoked to conclude that (3.1) is globally uniformly asymptotically stable at $x = 0$.

7.2. Invariance-like results

In applications such as adaptive control, Lyapunov methods commonly result in semidefinite Lyapunov functions (*i.e.*, candidate Lyapunov functions with time derivatives bounded by a negative semidefinite function of the state). The following theorem establishes the fact that if the function W in (7.2) is positive semidefinite then $t \mapsto W(x(t))$ asymptotically decays to zero. If the differential inclusion is time-invariant, stronger results similar to LaSalle's invariance principle can also be established using $\mathcal{U}$-generalized derivatives (see [11]).

Theorem 7.4. *Let $0 \in \mathcal{D}$, select $r > 0$ such that $\overline{\mathrm{B}}(0,r) \subset \mathcal{D}$, and let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map with compact values that is locally bounded, uniformly in t, over Ω,[4] such that (3.1) admits local solutions over Ω. If there exists a positive definite function $V \in \mathrm{Lip}(\Omega, \mathbb{R})$, a positive semidefinite function $W \in \mathcal{C}^0(\mathcal{D}, \mathbb{R})$, a pair of positive definite functions $\overline{W}, \underline{W} \in \mathcal{C}^0(\mathcal{D}, \mathbb{R})$, and a countable collection $\mathcal{U} \subset \mathrm{Lip}(\Omega, \mathbb{R})$ of regular functions such that (7.1) and (7.2) hold, then every solution $x(\cdot) \in \mathscr{S}(\Omega_c \times \mathbb{R}_{\geq 0})$, with $\Omega_c := \left\{x \in \overline{\mathrm{B}}(0,r) \mid \overline{W}(x) \leq c\right\}$ and $c \in \left[0, \min_{\|x\|_2 = r} \underline{W}(x)\right)$, is complete, bounded, and satisfies $\lim_{t \to \infty} W(x(t)) = 0$.*

Proof. Similar to the proof of ([7], Cor. 1), it is established that the bounds on $\dot{\overline{V}}_F$ in (6.1) and (6.2) imply that V is nonincreasing along all the solutions to (3.1). The nonincreasing property of V is used to establish boundedness of $x(\cdot)$, which is used to prove the existence and uniform continuity of complete solutions. Barbălat's lemma ([12], Lem. 8.2) is then used to conclude the proof.

[4] A set-valued map $F : \mathbb{R}^n \times \mathbb{R} \rightrightarrows \mathbb{R}^n$ is locally bounded, uniformly in t, over $\mathcal{D} \times \mathcal{J}$ for some $\mathcal{D} \subseteq \mathbb{R}^n$ and $\mathcal{J} \subseteq \mathbb{R}$, if for every compact $K \subset \mathcal{D}$, there exists $M > 0$ such that $\forall (x,t,y)$ such that $(x,t) \in K \times \mathcal{J}$, and $y \in F(x,t)$, $\|y\|_2 \leq M$.

Let $x(\cdot) \in \mathscr{S}(\mathit{\Omega}_c \times \mathbb{R}_{\geq 0})$. Using Theorem 6.3 and ([7], Lem. 2), $V(x(t_0), t_0) \geq V(x(t), t), \quad \forall t \in \mathcal{I}_x$. Arguments similar to ([12], Thm. 4.8) can then be used to show that every $x(\cdot) \in \mathscr{S}(\mathit{\Omega}_c \times \mathbb{R}_{\geq 0})$ satisfies $x(t) \in \overline{\mathrm{B}}(0, r), \forall t \in \mathcal{I}_x$. Therefore, all solutions $x(\cdot) \in \mathscr{S}(\mathit{\Omega}_c \times \mathbb{R}_{\geq 0})$ are precompact, and as a consequence of Lemma 3.3, complete.

To establish uniform continuity of the solutions, it is observed that since F is locally bounded, uniformly in t, over Ω, and $x(t) \in \overline{\mathrm{B}}(0, r)$ on $\mathbb{R}_{\geq t_0}$, the map $t \mapsto F(x(t), t)$ is uniformly bounded on $\mathbb{R}_{\geq t_0}$. Hence, $\dot{x} \in \mathcal{L}_\infty(\mathbb{R}_{\geq t_0}, \mathbb{R}^n)$. Since $x(\cdot)$ is locally absolutely continuous, $\forall t_1, t_2 \in \mathbb{R}_{\geq t_0}$, $\|x(t_2) - x(t_1)\|_2 = \left\| \int_{t_1}^{t_2} \dot{x}(\tau)\, \mathrm{d}\tau \right\|_2$. Since $\dot{x} \in \mathcal{L}_\infty(\mathbb{R}_{\geq t_0}, \mathbb{R}^n)$, $\left\| \int_{t_1}^{t_2} \dot{x}(\tau)\, \mathrm{d}\tau \right\|_2 \leq \int_{t_1}^{t_2} M\, \mathrm{d}\tau$, where M is a positive constant. Thus, $\|x(t_2) - x(t_1)\|_2 \leq M |t_2 - t_1|$, and hence, $x(\cdot)$ is uniformly continuous on $\mathbb{R}_{\geq t_0}$.

Since $x \mapsto W(x)$ is continuous and $\overline{\mathrm{B}}(0, r)$ is compact, $x \mapsto W(x)$ is uniformly continuous on $\overline{\mathrm{B}}(0, r)$. Hence, $t \mapsto W(x(t))$ is uniformly continuous on $\mathbb{R}_{\geq t_0}$. Furthermore, $t \mapsto \int_{t_0}^{t} W(x(\tau))\, \mathrm{d}\tau$ is monotonically increasing and from (7.2), $\int_{t_0}^{t} W(x(\tau))\, \mathrm{d}\tau \leq V(x(t_0), t_0) - V(x(t), t) \leq V(x(t_0), t_0)$. Hence, $\lim_{t \to \infty} \int_{t_0}^{t} W(x(\tau))\, \mathrm{d}\tau$ exists and is finite. By Barbălat's Lemma ([12], Lem. 8.2), $\lim_{t \to \infty} W(x(t)) = 0$. $\square$

In the following example $\dot{\overline{V}}$ and $\dot{\tilde{V}}$ do not have a negative semidefinite upper bound, but Theorem 7.4 can be invoked to conclude partial stability.

Example 7.5. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as in Example 7.3 and let $F : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^2$ be defined as

$$F(x, t) = \begin{bmatrix} \{x_2(1 + g(t))\} + H(x_2) \\ \{-x_1 - x_2\} + H(x_1) \end{bmatrix},$$

where $g \in \mathcal{C}^1(\mathbb{R}_{\geq 0}, \mathbb{R})$, $0 \leq g(t) \leq 1, \forall t \in \mathbb{R}_{\geq 0}$ and $\dot{g}(t) \leq g(t), \forall t \in \mathbb{R}_{\geq 0}$. Consider the differential inclusion in (3.1) and the candidate Lyapunov function $V : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined as $V(x, t) \coloneqq x_1^2 + (1 + g(t)) x_2^2$. The candidate Lyapunov function satisfies $\|x\|_2^2 \leq V(x, t) \leq 2\|x\|_2^2, \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. In this case, since $V \in \mathcal{C}^1\left(\mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \mathbb{R}\right)$, the set-valued derivatives $\dot{\overline{V}}$ in [2] and $\dot{\tilde{V}}$ in [28] are bounded by

$$\dot{\overline{V}}(x, t), \dot{\tilde{V}}(x, t) \leq \left\{-2x_2^2\right\} + 2x_2 h(t) H(x_1) + 2x_1 H(x_2), \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0},$$

where $h(t) \coloneqq 1 + g(t)$ and the inequality $2 + 2g(t) - \dot{g}(t) \geq 2$ is utilized. Thus, neither $\dot{\tilde{V}}$ nor $\dot{\overline{V}}$ are negative semidefinite everywhere.

Let U_1 be defined as in (7.4). The $\{U_1\}$-reduced set-valued map corresponding to F is given by

$$\tilde{F}_{\{U_1\}}(x, t) = \begin{cases} F(x, t) & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ \left[\{0\}\ ;\ \left(\{-1\} + \left[-\frac{1}{2}, \frac{1}{2}\right]\right)\right] & x_1 = 1 \wedge x_2 = 0, \\ \left[\{0\}\ ;\ \left(\{1\} + \left[-\frac{1}{2}, \frac{1}{2}\right]\right)\right] & x_1 = -1 \wedge x_2 = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

The $\{U_1\}$-generalized derivative of V in the direction(s) F is then given by

$$\begin{aligned} \dot{\overline{V}}_{\{U_1\}}(x, t) &\coloneqq \max_{q \in \tilde{F}_{\{U_1\}}(x, t)} \left(\frac{\partial V}{\partial(x, t)}(x, t)\right)^{\mathrm{T}} [q\ ;\ 1], \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \\ &= \begin{cases} \begin{bmatrix} 2x_1 & 2x_2 h(t) & \dot{g}(t) x_2^2 \end{bmatrix} \begin{bmatrix} x_2 h(t)\ ;\ -x_1 - x_2\ ;\ 1 \end{bmatrix} & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ \max \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \left[\{0\}\ ;\ \left[\frac{3}{2}, -\frac{1}{2}\right]\ ;\ 1\right] & x_1 = 1 \wedge x_2 = 0, \\ \max \begin{bmatrix} -2 & 0 & 0 \end{bmatrix} \left[\{0\}\ ;\ \left[\frac{1}{2}, \frac{3}{2}\right]\ ;\ 1\right] & x_1 = -1 \wedge x_2 = 0, \\ -\infty & \text{otherwise,} \end{cases} \\ &\leq -2x_2^2, \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}. \end{aligned}$$

Theorem 7.4 can then be invoked to conclude that $t \mapsto x_1(t) \in \mathcal{L}_\infty(\mathbb{R}_{\geq t_0}, \mathbb{R})$ and $\lim_{t\to\infty} x_2(t) = 0$.

Theorem 7.4 and its counterparts are widely used in applications such as adaptive control to establish stability (but not asymptotic stability) of the state and convergence of a part of the state (*e.g.*, tracking errors, but not parameter estimation errors) to the origin. Under certain excitation conditions, asymptotic stability (and as a result, convergence of the entire state to the origin) can be established using Matrosov theorems [16].

7.3. Matrosov theorems

In this section, a less conservative generalization of Matrosov results for uniform asymptotic stability of nonautonomous systems is developed. In particular, the nonsmooth version ([29], Thm. 1) of the nested Matrosov theorem ([15], Thm. 1) is generalized. The following definitions of Matrosov functions are inspired by [29].

Definition 7.6. Let $\gamma, \delta, \Delta > 0$ be constants. A finite set of functions $\{Y_j\}_{j=1}^{M} \subset \mathcal{C}^0\left(\overline{\mathrm{B}}(0_m, \gamma) \times \mathrm{D}(\delta, \Delta), \mathbb{R}\right)$ is said to have the *Matrosov property* relative to (γ, δ, Δ) if $\forall j \in \{0, \ldots, M\}$,

$$\left((z, x) \in \overline{\mathrm{B}}(0_m, \gamma) \times \mathrm{D}(\delta, \Delta)\right) \wedge \left(Y_i(z, x) = 0, \forall i \in \{0, \ldots, j\}\right) \implies Y_{j+1}(z, x) \leq 0,$$

where $Y_0(z, x) = 0$ and $Y_{M+1}(z, x) = 1$, $\forall (z, x) \in \overline{\mathrm{B}}(0_m, \gamma) \times \mathrm{D}(\delta, \Delta)$.

Definition 7.7. Let $\delta, \Delta > 0$ be constants such that $\mathrm{D}(\delta, \Delta) \subset \mathcal{D}$. Let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map with compact values. The functions $\{W_j\}_{j=1}^{M} \subset \mathrm{Lip}(\Omega, \mathbb{R})$ are said to be $\mathcal{U}-$*reduced Matrosov functions* for (F, δ, Δ) if $\exists \phi : \Omega \to \mathbb{R}^m$, $\gamma > 0$, and $\{Y_j\}_{j=1}^{M} \subset \mathcal{C}^0\left(\overline{\mathrm{B}}(0_m, \gamma) \times \mathrm{D}(\delta, \Delta), \mathbb{R}\right)$ such that:

(a) the set of functions $\{Y_j\}_{j=1}^{M}$ has the Matrosov property relative to (γ, δ, Δ),
(b) $\forall j \in \{1, \ldots, M\}$ and $\forall (x, t) \in \mathrm{D}(\delta, \Delta) \times \mathbb{R}_{\geq 0}$, $\max\{|W_j(x, t)|, |\phi(x, t)|\} \leq \gamma$, and
(c) $\forall j \in \{1, \ldots, M\}$ there exists a collection of regular functions $\mathcal{U}_j \subset \mathrm{Lip}(\mathrm{D}(\delta, \Delta) \times \mathbb{R}_{\geq 0}, \mathbb{R})$ such that $\forall (x, t) \in \mathrm{D}(\delta, \Delta) \times \mathbb{R}_{\geq 0}$, $\overline{\dot{W}}_{\mathcal{U}_j}(x, t) \leq Y_j(\phi(x, t), x)$.

The following technical Lemmas aid the proof of the Matrosov theorem.

Lemma 7.8. *Given $\delta > 0$, $\exists \epsilon > 0$ such that*

$$\left((z, x) \in \overline{\mathrm{B}}(0_m, \gamma) \times \mathrm{D}(\delta, \Delta)\right) \wedge \left(Y_j(z, x) = 0, \forall j \in \{1, \ldots, M-1\}\right) \implies Y_M(z, x) \leq -\epsilon.$$

Proof. See ([15], Claim 1). □

Lemma 7.9. *Let $l \in \{2, \ldots, M\}$, $\tilde{\epsilon} > 0$, and $\tilde{Y}_l \in \mathcal{C}^0(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$. If*

$$\left((z, x) \in \overline{\mathrm{B}}(0_m, \gamma) \times \mathrm{D}(\delta, \Delta)\right) \wedge \left(Y_j(z, x) = 0, \forall j \in \{1, \ldots, l-1\}\right) \implies \tilde{Y}_l(z, x) \leq -\tilde{\epsilon},$$

then $\exists K_{l-1} > 0$ such that

$$\left((z, x) \in \overline{\mathrm{B}}(0_m, \gamma) \times \mathrm{D}(\delta, \Delta)\right) \wedge \left(Y_j(z, x) = 0, \forall j \in \{1, \ldots, l-2\}\right) \implies K_{l-1} Y_{l-1}(z, x) + \tilde{Y}_l(z, x) \leq -\frac{\tilde{\epsilon}}{2}.$$

Proof. See ([15], Claim 2). □

The Matrosov theorem can now be stated as follows.

Theorem 7.10. *Let $0 \in \mathcal{D}$ and let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map with compact values such that (3.1) admits solutions over Ω and is uniformly stable at $x = 0$. If, for each pair of numbers $\delta, \Delta \in \mathbb{R}$, such that $0 \leq \delta \leq \Delta$ and $\mathrm{D}(\delta, \Delta) \subset \mathcal{D}$, there exist $\mathcal{U}-$reduced Matrosov functions for (F, δ, Δ), then (3.1) is uniformly asymptotically stable at $x = 0$. If $\mathcal{D} = \mathbb{R}^n$ and if (3.1) is uniformly globally stable at $x = 0$ then (3.1) is uniformly globally asymptotically stable at $x = 0$.*

Proof. Select $\Delta > 0$ such that $\overline{\mathrm{B}}(0, \Delta) \subset \mathcal{D}$ and let $r > 0$ be such that

$$x(\cdot) \in \mathscr{S}\left(\overline{\mathrm{B}}(0, r) \times \mathbb{R}_{\geq 0}\right) \implies x(t) \in \overline{\mathrm{B}}(0, \Delta), \forall t \in \mathbb{R}_{\geq t_0}. \tag{7.5}$$

Let $\epsilon \in (0, r)$ and select $\delta > 0$ such that

$$x(\cdot) \in \mathscr{S}\left(\overline{\mathrm{B}}(0, \delta) \times \mathbb{R}_{\geq 0}\right) \implies x(t) \in \overline{\mathrm{B}}(0, \epsilon), \forall t \in \mathbb{R}_{\geq t_0}. \tag{7.6}$$

By repeated application of Lemmas 7.8 and 7.9 it can be shown that $\forall \delta > 0$, $\exists \zeta > 0$ and $K_1, \ldots, K_{M-1} > 0$ such that $\forall (z, x) \in \overline{\mathrm{B}}(0_m, \gamma) \times \mathrm{D}(\delta, \Delta)$,

$$Z(z, x) \coloneqq \sum_{j=1}^{M-1} K_j Y_j(z, x) + Y_M(z, x) \leq -\frac{\zeta}{2^{M-1}}. \tag{7.7}$$

Let $W \in \mathrm{Lip}(\Omega, \mathbb{R})$ be defined as $W(x, t) \coloneqq \sum_{j=1}^{M-1} K_j W_j(x, t) + W_M(x, t)$. From Def. 7.7.b,

$$|V(x, t)| \leq \gamma \left(1 + \sum_{j=1}^{M-1} K_j\right) \eqqcolon \eta. \tag{7.8}$$

Fix $(x_0, t_0) \in \overline{\mathrm{B}}(0, r) \times \mathbb{R}_{\geq 0}$ and $x(\cdot) \in \mathscr{S}(\{(x_0, t_0)\})$. The selection of r in (7.5) implies that the solution $x(\cdot)$ satisfies $x(t) \in \overline{\mathrm{B}}(0, \Delta)$, $\forall t \in \mathbb{R}_{\geq t_0}$. From Definition 7.7.c, $\overline{\dot{V}}_{\mathcal{U}_j}(x, t) \leq Z(\phi(x, t), x)$, $\forall (x, t) \in \mathrm{D}(\delta, \Delta) \times \mathbb{R}_{\geq 0}$, and hence, from Theorem 6.3,

$$\dot{V}(x(t), t) \leq Z(\phi(x(t), t), x(t)), \tag{7.9}$$

for almost all $t \in x^{-1}(\mathrm{D}(\delta, \Delta))$. Using Definition 7.7.b and (7.7),

$$Z(\phi(x(t), t), x(t)) \leq -\frac{\zeta}{2^{M-1}}, \tag{7.10}$$

for almost all $t \in x^{-1}(\mathrm{D}(\delta, \Delta))$.

Let $T > \frac{2^M \eta}{\zeta}$. The claim is that $\|x(t)\| \leq \epsilon$, $\forall t \in \mathbb{R}_{\geq t_0 + T}$. If not, then the selection of δ in (7.6) implies that $x(t) \in \mathrm{D}(\delta, \Delta)$, $\forall t \in [t_0, t_0 + T]$. Hence, from (7.9) and (7.10),

$$\dot{V}(x(t), t) \leq -\frac{\zeta}{2^{M-1}}, \tag{7.11}$$

for almost all $t \in [t_0, t_0 + T]$. Integrating (7.11) and using the bound in (7.8), $\frac{T\zeta}{2^{M-1}} \leq 2\eta$, which contradicts $T > \frac{2^M \eta}{\zeta}$. Hence, $\forall \epsilon \in (0, r)$, $\exists T > 0$ such that $x(\cdot) \in \mathscr{S}\left(\overline{\mathrm{B}}(0, r) \times \mathbb{R}_{\geq 0}\right) \implies \|x(t)\| < \epsilon$, $\forall t \in \mathbb{R}_{\geq t_0 + T}$, *i.e.*, (3.1) is uniformly asymptotically stable at $x = 0$.

If $\mathcal{D} = \mathbb{R}^n$ and if (3.1) is uniformly globally stable at $x = 0$ then r can be selected arbitrarily large, and hence, the result is global. ☐

The following example demonstrates an application of the Matrosov theorem.

Example 7.11. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as in Example 7.3 and let $F : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^2$ be defined as in Example 7.5. Let U_1 be defined as in (7.4). Let $W_1 : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ be defined as $W_1(x, t) \coloneqq x_1^2 + (1 + g(t)) x_2^2$. It follows that $\overline{\dot{W}}_{1\{U_1\}}(x, t) \leq -2x_2^2$, $\forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$, and uniform global stability of (3.1) at $x = 0$ can be concluded from Theorem 7.2.

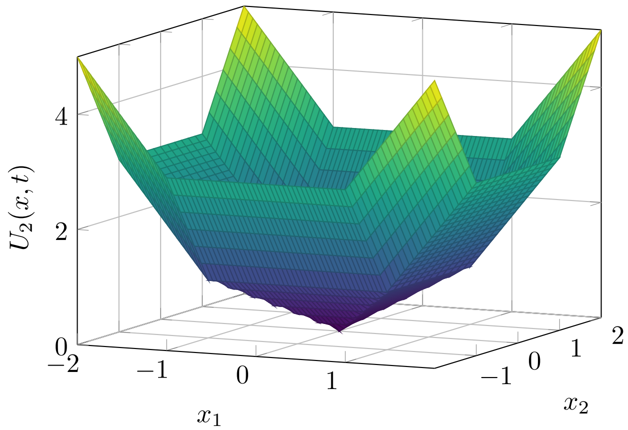

FIGURE 2. A snapshot of the function $U_2 : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$.

Let $\phi(x,t) = 0$, $\forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ and let $Y_1(z,x) \coloneqq -2x_2^2$, $\forall (z,x) \in \mathbb{R} \times \mathbb{R}^2$. Let $W_2(x,t) \coloneqq x_1 x_2$. The function $U_2 : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, defined as (see Fig. 2)

$$U_2(x,t) = \begin{cases} |x_1| & |x_1| > |x_2| \wedge x \in \mathrm{Sq}, \\ |x_2| & |x_1| \leq |x_2| \wedge x \in \mathrm{Sq}, \\ 1 + U^*(x,t) & x \notin \mathrm{Sq}, \end{cases}$$

where

$$U^*(x,t) = \max((2x_1 - 2), 0) - \min((2x_1 + 2), 0) + \max((2x_2 - 2), 0) - \min((2x_2 + 2), 0), \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$$

and 'Sq' denotes the open unit square centered at the origin, satisfies $U_2 \in \mathrm{Lip}\left(\mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \mathbb{R}\right)$. In addition, since U_2 is convex, it is also regular ([5], Prop. 2.3.6). The Clarke gradient of U_2 is given by

$$\partial U_2(x,t) = \begin{cases} [\{2\,\mathrm{sgn}\,1(x_1)\} \;;\; \{2\,\mathrm{sgn}\,1(x_2)\} \;;\; \{0\}] & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ [\overline{\mathrm{co}}\{0, 2\,\mathrm{sgn}(x_1)\} \;;\; \{2\,\mathrm{sgn}\,1(x_2)\} \;;\; \{0\}] & |x_1| = 1 \wedge |x_2| \neq 1, \\ [\{2\,\mathrm{sgn}\,1(x_1)\} \;;\; \overline{\mathrm{co}}\{0, 2\,\mathrm{sgn}(x_2)\} \;;\; \{0\}] & |x_1| \neq 1 \wedge |x_2| = 1, \end{cases}$$

if $x \notin \overline{\mathrm{Sq}}$,

$$\partial U_2(x,t) = \begin{cases} [[-1,1] \;;\; [-1,1] \;;\; \{0\}] & |x_1| = 0 \wedge |x_2| = 0, \\ [\mathrm{sgn}(x_1) \;;\; \{0\} \;;\; \{0\}] & |x_1| > |x_2|, \\ [\{0\} \;;\; \mathrm{sgn}(x_2) \;;\; \{0\}] & |x_1| < |x_2|, \\ [\overline{\mathrm{co}}\{0, \mathrm{sgn}(x_1)\} \;;\; \overline{\mathrm{co}}\{0, \mathrm{sgn}(x_2)\} \;;\; \{0\}] & |x_1| = |x_2| > 0, \end{cases}$$

if $x \in \mathrm{Sq}$, and

$$\partial U_2(x,t) = \begin{cases} [\overline{\mathrm{co}}\{\mathrm{sgn}\,1(x_1), 2\,\mathrm{sgn}\,1(x_1)\} \;;\; \overline{\mathrm{co}}\{\mathrm{sgn}\,1(x_2), 2\,\mathrm{sgn}\,1(x_2)\} \;;\; \{0\}] & |x_1| \neq |x_2|, \\ [\overline{\mathrm{co}}\{\mathrm{sgn}(x_1), 2\,\mathrm{sgn}(x_1), 0\} \;;\; \overline{\mathrm{co}}\{\mathrm{sgn}(x_2), 2\,\mathrm{sgn}(x_2), 0\} \;;\; \{0\}] & |x_1| = |x_2|, \end{cases}$$

if $x \in \operatorname{bd}(\mathrm{Sq})$. The $\{U_2\}$-reduced set-valued map corresponding to F is given by

$$\tilde{F}_{\{U_2\}}(x,t) = \begin{cases} F(x,t) & x \notin \overline{\mathrm{Sq}} \wedge |x_1| \neq 1 \wedge |x_2| \neq 1, \\ F(x,t) & x \in \mathrm{Sq} \wedge |x_1| \neq |x_2|, \\ \{0\} & x \in \mathrm{Sq} \wedge |x_1| = 0 \wedge |x_2| = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

The $\{U_2\}$-generalized derivative of W_2 is then given by

$$\dot{\overline{W}}_{2\{U_2\}}(x,t) = \begin{cases} [x_2 \quad x_1 \quad 0]\,[x_2 h(t)\ ;\ -x_1 - x_2\ ;\ 1] & \begin{array}{l} x \notin \overline{\mathrm{Sq}} \wedge |x_1| \neq 1 \wedge |x_2| \neq 1 \\ \quad \vee x \in \mathrm{Sq} \wedge |x_1| \neq |x_2|, \end{array} \\ 0 & x \in \mathrm{Sq} \wedge |x_1| = 0 \wedge |x_2| = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

That is, $\dot{\overline{W}}_{2\{U_2\}}(x,t) \leq -x_1^2 - x_2 x_1 + 2x_2^2, \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. If $Y_2(z,x) \coloneqq -x_1^2 - x_2 x_1 + 2x_2^2, \forall (z,x) \in \mathbb{R} \times \mathbb{R}^2$, then the functions $\{Y_1, Y_2\}$ have the Matrosov property. Furthermore, since $W_1, W_2 \in \mathcal{C}^0\left(\mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \mathbb{R}\right)$, $\forall 0 < \delta < \Delta$, $\exists \gamma > 0$ such that $|W(x,t)| \leq \gamma$, $\forall (x,t) \in \mathrm{D}(\delta, \Delta) \times \mathbb{R}_{\geq 0}$. Hence, $\{W_1, W_2\}$ are $\mathcal{U}-$reduced Matrosov functions for (F, δ, Δ), $\forall 0 < \delta < \Delta$. Hence, by Theorem 7.10, (3.1) is uniformly globally asymptotically stable at $x = 0$.

8. Conclusion

This paper demonstrates that locally Lipschitz, regular functions can be used to identify infeasible directions in set-valued maps that define differential inclusions. The infeasible directions can then be removed to yield a point-wise smaller (in the sense of set containment) set-valued map that defines an equivalent differential inclusion. The reduction process results in a novel generalization of the set-valued derivative for locally Lipschitz candidate Lyapunov functions. Statements of Lyapunov stability theorems, invariance theorems, invariance-like results, and Matrosov theorems for differential inclusions that are less conservative than those available in the literature are developed using reduced set-valued maps.

The fact that arbitrary locally Lipschitz, regular functions can be used to restrict differential inclusions to smaller sets of admissible directions indicates that there may be a *smallest* set of admissible directions corresponding to each differential inclusion. Further research is needed to establish the existence of such a set and to find a representation of it that facilitates computation.

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