Exponential Stability With RISE Controllers

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***Abstract*—A class of continuous robust controllers termed Robust Integral of the Sign of the Error (RISE) have been published over the past two decades as a means to yield asymptotic tracking error convergence and asymptotic identification of time-varying uncertainties, for classes of nonlinear systems that are subject to sufficiently smooth bounded exogenous disturbances and/or modeling uncertainties. Despite the wide application of RISE-based techniques, an open question that has eluded researchers during this time-span is whether the asymptotic tracking error convergence is also uniform or exponential. This question has remained open due to certain limitations in the traditional construction of a Lyapunov function for RISE-based error systems. In this letter, new insights on the construction of a Lyapunov function are used that result in an exponential stability result for RISE-based controllers. As an outcome of this breakthrough, the inherent learning capability of RISE-based controllers is shown to yield exponential identification of state-dependent disturbances/uncertainty.**

***Index Terms*—Robust control, Lyapunov methods, nonlinear control systems.**

I. Introduction

A CLASS of continuous robust controllers termed Robust Integral of the Sign of the Error (RISE) have been published over the past two decades [1]–[14] as a means to yield asymptotic tracking error convergence and asymptotic identification of time-varying uncertainties, for classes of nonlinear systems that are subject to sufficiently smooth bounded exogenous disturbances and/or modeling uncertainties. RISE-based methods have been used for a wide variety of applications involving control [4]–[20], estimation [2], [21], [22], and optimization [23]. Despite the wide application of RISE-based techniques, an open question that has eluded researchers during this time-span is whether the asymptotic tracking error convergence is also uniform or exponential. This question has remained open due to certain limitations in the traditional construction of a Lyapunov function for RISE-based error systems.

The traditional analysis methodology for a RISE-based error system involves a Lyapunov-based approach, where the candidate Lyapunov function (denoted by V_L) includes a P-function (denoted by P) in addition to a typical sum of norm squared error terms. The P-function is designed by selecting $\dot{P}$ to cancel disturbance-based terms in $\dot{V}_L$ and is the essential analysis and design tool to enable asymptotic convergence (instead of uniformly ultimately bounded tracking) despite the presence of a disturbance term that is only upper bounded by a constant. Previous results, including the current result, determine P as a function of the initial conditions of the system that is proven to be non-negative under certain gain conditions. Evaluating $\dot{V}_L$ along the closed-loop error trajectories yields a negative semi-definite $\dot{V}_L$. Then the extension of the LaSalle-Yoshizawa theorem for nonsmooth systems in [24] is invoked to prove asymptotic tracking error convergence. Since the LaSalle-Yoshizawa theorem is based on Barbalat's lemma, the traditional analysis methodology does not guarantee uniform tracking error convergence, and the non-strictness of V_L precluded exponential stability of the closed-loop error system's origin.

To prove exponential stability, it would be sufficient to select a positive-definite V_L such that $\dot{V}_L \leq -\lambda_L V_L$ for almost all time, with some positive constant λ_L. Then exponential stability can be established using the comparison principle. Such a Lyapunov function is developed in [14], which is the only known RISE-based exponential tracking result. The result in [14] was developed for a specific application under the assumption that the first and second derivatives of the uncertainty are bounded by known constants. However, RISE-based controllers have been applied to a broader set of applications where this assumed bound would not hold. For example, in results like [8], [10] and [11] that involve dynamic compensator-based auxiliary control terms, the first or second derivative of the uncertainty have bounds that are state-dependent. It is not clear how the analysis approach in [14] can be extended for such cases.

In this letter, a novel P-function design is developed that results in a strict Lyapunov function. The new analysis results

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in exponential stability of the closed-loop error system's origin using a comparison theorem-based argument. The novel P-function is shown to be non-negative under certain gain conditions by examining the analytically derived solution to the dynamics in $\dot{P}$. Unlike the analysis approach in [14], the developed P-function can be easily modified for various bounds on the first and second derivatives of uncertainty. To rule out the existence of extra solutions for P that could be potentially negative over some time interval, the derived solution for P is shown to be unique corresponding to a given closed-loop error trajectory. Additionally, solution-dependent arguments are employed to show the sign of the error term is integrable, and $\dot{V}_L \leq -\lambda_L V_L$ for almost all time, which involves showing that the set of time-instants where $\dot{V}_L \leq -\lambda_L V_L$ may not be true have Lebesgue measure zero. Furthermore, the disturbance/uncertainty is shown to be estimated with exponential convergence of the disturbance identification error, while prior results only indicated asymptotic convergence.

II. Control Design

A. Preliminaries

A function $y : \mathcal{I}_y \to \mathbb{R}^n$ is called a *Filippov solution* of $\dot{y} = h(y, t)$ on the interval $\mathcal{I}_y \subseteq \mathbb{R}_{\geq 0}$, given some Lebesgue measurable and locally essentially bounded function $h : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, if y is absolutely continuous on $\mathcal{I}_y$, and $\dot{y} \in K[h](y, t)$ for almost all $t \in \mathcal{I}_y$, where $K[\cdot]$ denotes the Filippov set-valued map defined in [25, eq. (2b)]. A solution is called *complete* if $\mathcal{I}_y$ is unbounded. A solution $y_2 : [t_0, t_2) \to \mathbb{R}^n$ to $\dot{y} = h(y, t)$ is called a *proper right extension* of a solution $y_1 : [t_0, t_1) \to \mathbb{R}^n$ to $\dot{y} = h(y, t)$ if $t_2 > t_1$ and $y_2(t) = y_1(t)$, $\forall t \in [t_0, t_1)$. A solution to $\dot{y} = h(y, t)$ is called *maximal* if it does not have a proper right extension which is also a solution to $\dot{y} = h(y, t)$. If a solution is maximal and if the closure of its range, $\overline{\{y(t) \in \mathbb{R}^n | t \in \mathcal{I}_y\}}$, is compact, then the solution is called *precompact*. The space of continuous functions with continuous first m derivatives is denoted by $\mathcal{C}^m$. The space of essentially bounded Lebesgue measurable functions is denoted by $\mathcal{L}_\infty$, and $\|\cdot\|_p$ denotes the p-norm, where the subscript is suppressed when $p = 2$. The notation $[a; b]$ denotes the column vector $[\, a^T \quad b^T \,]^T$.

B. Control Objective

Consider a control affine system with the nonlinear dynamics

$$\dot{x} = d(x, \nu, t) + u, \tag{1}$$

where $t \in \mathbb{R}_{\geq 0}$ denotes time, $x : \mathcal{I} \to \mathbb{R}^n$ denotes a Filippov solution[1] to (1), with the interval of existence $\mathcal{I} = [t_0, t_1)$ for some $t_0, t_1 \in \mathbb{R}_{\geq 0}$ s.t. $t_1 > t_0$, $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ denotes an auxiliary function representing some external dynamic compensator-based terms (e.g., adaptive feedforward terms, observer-based terms), $d : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ represents $\mathcal{C}^2$ modeling uncertainty in the system, and $u : \mathcal{I} \to \mathbb{R}^n$ represents the control input. Let $\dot{d}(x, \dot{x}, \nu, \dot{\nu}, t) \triangleq \nabla d^T(x, \nu, t)[\dot{x}; \dot{\nu}; 1]$ and $\ddot{d}(x, \dot{x}, \ddot{x}, \nu, \dot{\nu}, \ddot{\nu}, t) = [\dot{x}; \dot{\nu}; 1]^T \nabla^2 d(x, \nu, t)[\dot{x}; \dot{\nu}; 1] + \nabla d^T(x, \nu, t)[\ddot{x}; \ddot{\nu}; 0]$, respectively, where ∇ and ∇^2 denote the gradient and Hessian operators, respectively. It is assumed that for each $(a, b, p, \nu, w, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$, the mappings $t \mapsto d(a, \nu, t)$, $t \mapsto \dot{d}(a, b, \nu, w, t)$, and $t \mapsto \ddot{d}(a, b, p, \nu, w, s, t)$ are bounded. The objective is to design a controller such that the state tracks a smooth bounded reference trajectory. The objective is quantified by defining the tracking error

$$e \triangleq x - x_d, \tag{2}$$

where $x_d : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a $\mathcal{C}^2$ reference trajectory such that $x_d, \dot{x}_d \in \mathcal{L}_\infty$.

C. Control Law Development

To facilitate the subsequent analysis, a filtered tracking error $r : \mathcal{I} \to \mathbb{R}^n$ is defined as $r \triangleq d(x, \nu, t) + u - \dot{x}_d + \alpha e$, where $\alpha \in \mathbb{R}_{>0}$ is a constant control gain. To facilitate the subsequent analysis, the dynamics in terms of $\dot{e}$ can be rewritten using (1) and (2) as

$$\dot{e} = r - \alpha e. \tag{3}$$

Let $z : \mathcal{I} \to \mathbb{R}^{2n}$ denote the augmented tracking error, $z \triangleq [\, e^T \quad r^T \,]^T$. From the subsequent stability analysis, a continuous RISE control input is designed as [3]

$$u \triangleq \dot{x}_d - \alpha e - \hat{d}, \tag{4}$$

where $\hat{d} : \mathcal{I} \to \mathbb{R}^n$ is an auxiliary term designed as a Filippov solution[2] to

$$\dot{\hat{d}} = kr + e + \beta \operatorname{sgn}(e), \tag{5}$$

given any user-selected $\hat{d}(t_0) \in \mathbb{R}^n$. In (5), $k, \beta \in \mathbb{R}_{>0}$ are constant control gains. Using (1)-(4) yields

$$r = d(x, \nu, t) - \hat{d}. \tag{6}$$

It follows from (5) and (6) that r is a Filippov solution to the closed-loop error system

$$\dot{r} = \widetilde{N} + N_B - kr - e - \beta \operatorname{sgn}(e), \tag{7}$$

where $\widetilde{N} \triangleq \dot{d}(x, \dot{x}, \nu, \dot{\nu}, t) - \dot{d}(x_d, \dot{x}_d, \nu, \dot{\nu}, t)$ and $N_B \triangleq \dot{d}(x_d, \dot{x}_d, \nu, \dot{\nu}, t)$.

Assumption 1: The function ν is a solution to some external dynamics such that there exist known constants, $\eta_1, \eta_2, \eta_3, \eta_4 \in \mathbb{R}_{\geq 0}$, and a known strictly increasing function, $\rho_{21} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $\|\nu\| \leq \eta_1$, $\|\dot{\nu}\| \leq \eta_2$, and $\|\ddot{\nu}\| \leq \eta_3 + \eta_4\|z\| + \rho_{21}(\|z\|)\|z\|$.

Then, there exist known constants $\gamma_1, \gamma_3, \gamma_4 \in \mathbb{R}_{\geq 0}$ and a known strictly increasing function $\rho_{21} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\|N_B\| \leq \gamma_1$ and $\|\dot{N}_B\| \leq \gamma_3 + \gamma_4\|z\| + \rho_2(\|z\|)\|z\|$, $\forall t \in \mathbb{R}_{\geq 0}$.

[1]We consider Filippov solutions instead of classical solutions to facilitate a nonsmooth control design. Alternatively, Krasovskii solutions can also be considered. Generalized solutions such as Filippov or Krasovskii solutions are guaranteed to exist for nonsmooth systems with Lebesgue measurable and locally essentially bounded right-hand-sides [26, Proposition 3], whereas classical solutions might not exist.

[2]Since r may not be commonly available, $\hat{d}(t)$ is evaluated using $\hat{d}(t) = \hat{d}(t_0) + ke(t) - ke(t_0) + \int_{t_0}^{t} ((k\alpha + 1)e(\tau) + \beta \operatorname{sgn}(e(\tau)))d\tau$ for closed-loop implementation. Note that $\beta \operatorname{sgn}(e(\cdot))$ is Riemann integrable on $[t_0, t]$, $\forall t \in \mathcal{I}$ according to Lemma 2.

Additionally, since ν is bounded and $t \mapsto \dot{d}(a, b, v, w, t)$ is bounded for each $(a, b, v, w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$, $\|\widetilde{N}\| \le \gamma_2\|z\| + \rho_1(\|z\|)\|z\|$, $\forall t \in \mathbb{R}_{\ge 0}$, according to the Mean Value Theorem-based inequality in [27, Lemma 5], where $\gamma_2 \in \mathbb{R}_{\ge 0}$ is a known constant, and $\rho_1 : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is a known strictly increasing function. Note that the type of state-dependent bounds considered in Assumption 1 are general and often required in various applications where the RISE method is used (e.g., [8], [10] and [11]), typically as a consequence of augmenting adaptive feedforward controllers with a RISE term. In the case where ν represents adaptive feedforward terms, the developed approach offers modularity of design in the sense that $\hat{d}$ and ν can be designed independently, as long as ν satisfies Assumption 1. The following example illustrates a type of system satisfying Assumption 1.

Example 1: Consider a dynamic neural network given by

$$\dot{\nu} = \mathrm{proj}\{W^T\sigma(V^Tx), \nu\}, \tag{8}$$

where $\mathrm{proj}\{\cdot,\cdot\}$ denotes the smooth projection operator in [28] that guarantees $\|\nu\| \le \eta_1$, $\sigma : \mathbb{R}^L \to \mathbb{R}^L$ denotes a globally bounded continuous activation function, and $W \in \mathbb{R}^{m\times L}$ and $V \in \mathbb{R}^{L\times n}$ are constant[3] matrices of outer and inner-layer weights, respectively. Using [28, Property 3] and the fact that $\sigma(\cdot)$ is globally bounded, $\dot{\nu}$ can be bounded by a constant, i.e., $\|\dot{\nu}\| \le \eta_2$. Taking the time-derivative of $\dot{\nu}$ yields $\ddot{\nu} = \frac{d}{dt}(\mathrm{proj}\{W^T\sigma(V^Tx), \nu\}) = \frac{\partial}{\partial y}(\mathrm{proj}\{y, \nu\})|_{y=W^T\sigma(V^Tx)}\frac{d}{dt}(W^T\sigma(V^Tx)) + \frac{\partial}{\partial y}(\mathrm{proj}\{W^T\sigma(V^Tx), y\})|_{y=\nu}\dot{\nu}$. Based on the structure of the projection operator in [28, eq. (7)], the terms $\frac{\partial}{\partial y}(\mathrm{proj}\{y, \nu\})|_{y=W^T\sigma(V^Tx)}$ and $\frac{\partial}{\partial y}(\mathrm{proj}\{W^T\sigma(V^Tx), y\})|_{y=\nu}$ can be bounded by some known functions of x. Additionally, based on the right-hand-side of (3), the term $\frac{d}{dt}(W^T\sigma(V^Tx)) = W^T\frac{\partial}{\partial y}\sigma(y)|_{y=V^Tx}V^T\dot{x} = W^T\frac{\partial}{\partial y}\sigma(y)|_{y=V^Tx}V^T(\dot{x}_d + r - \alpha e)$ can be bounded by some known continuous function of z. Therefore, $\ddot{\nu}$ can be bounded as $\|\ddot{\nu}\| \le \eta_3 + \eta_4\|z\| + \rho_{21}(\|z\|)\|z\|$. Thus, the dynamic neural network in (8) satisfies Assumption 1.

The structure of the closed-loop error system in (7) may appear similar to a higher order sliding-mode design (see [29]); however, there are some remarkable differences to highlight. Specifically, the $\beta\mathrm{sgn}(e)$ term in (7) would need to be $\beta\mathrm{sgn}(r)$ to facilitate the analysis for a standard continuous higher-order sliding-mode design. Since sensor measurements for the highest order derivative (e.g., $\dot{e}$ or r) may not be available for feedback, the controller in (4) is designed to depend only on state measurements. Additionally, the closed-loop error system in (3) and (7) is also different from a super-twisting system, since (3) would require an additional $-|e|^{1/2}\mathrm{sgn}(e)$ term to facilitate a super-twisting design, which needs a different analysis approach [30].

We now present some supporting lemmas which facilitate the subsequent analysis. Proofs of all lemmas can be found in the Appendix.

Lemma 1: Given some Filippov solutions, e and r, to (3) and (7), respectively, the set of time-instants $T \triangleq \{t \in \mathcal{I} | \exists i \in \{1, 2, \ldots, n\}\, s.t.\, e_i(t) = 0 \wedge r_i(t) \neq 0\}$ has Lebesgue measure zero, where e_i and r_i denote the i^{th} element of e and r, respectively.

Lemma 2: Given some Filippov solution, e, to (3), $\mathrm{sgn}(e(\cdot))$ is Riemann integrable on $[t_0, t_1]$, $\forall t_1 \in \mathcal{I}$.

III. Stability Analysis

Following the development in Section II, every Filippov solution to (1) and (5) with the controller in (4) corresponds to a Filippov solution of the transformed system in (3) and (7). Additionally, a P-function is introduced to facilitate the construction of a candidate Lyapunov function for analyzing the stability and convergence properties of z. The P-function is denoted by $P : \mathcal{I} \to \mathbb{R}$ and is defined as a Filippov solution to

$$\dot{P} = -\lambda_P P - L, \tag{9}$$

where $\lambda_P \in \mathbb{R}_{>0}$ is an auxiliary constant, and

$$L \triangleq r^T N_B - r^T\beta\mathrm{sgn}(e) - (\gamma_4 + \rho_2(\|z\|))\|z\|\|e\|_1, \tag{10}$$

where $\|\cdot\|_1$ denotes the 1-norm, and

$$P(t_0) = 0. \tag{11}$$

The analytical solution to (9) is derived in Lemma 3. To facilitate the inclusion of the P-function in the candidate Lyapunov function, P is designed to be non-negative under certain gain conditions as described in Lemma 4.

Lemma 3: Given some Filippov solutions, e and r, to (3) and (7), respectively,

$$\begin{aligned} P = {} & \beta\|e\|_1 - e^TN_B + \mathrm{e}^{-\lambda_P t} * \big((\alpha - \lambda_P)\big(\beta\|e\|_1 - e^TN_B\big) + e^T\dot{N}_B\big) \\ & + \mathrm{e}^{-\lambda_P t} * ((\gamma_4 + \rho_2(\|z\|))\|z\|\|e\|_1), \end{aligned} \tag{12}$$

is the unique Filippov solution to the differential equation in (9) initialized according to (11), where '$*$' denotes the convolution operator, i.e., $p(t) * q(t) = \int_{t_0}^{t} p(t-\tau)q(\tau)d\tau$, for any given $p, q : [t_0, \infty) \to \mathbb{R}$.

Lemma 4: Given any pair of Filippov solutions, e and r, to (3) and (7), respectively, provided that P is initialized according to (11), and the gain conditions

$$\alpha > \lambda_P, \tag{13}$$

$$\beta > \gamma_1 + \frac{\gamma_3}{\alpha - \lambda_P}, \tag{14}$$

are satisfied, $P(t) \ge 0$, $\forall t \in \mathcal{I}$, where the gains α and β are introduced in (3) and (5), respectively, and γ_1, γ_4 are introduced in Assumption 1.

Let $\psi \triangleq [\,e^T \quad r^T \quad P\,]^T$, and $\dot{\psi} = g(\psi, t)$ denote the differential equations in (3), (7) and (9), where $g : \mathbb{R}^{2n+1} \times [t_0, \infty) \to \mathbb{R}^{2n+1}$ is Lebesgue measurable and locally essentially bounded (i.e., bounded on a neighborhood of every point, excluding sets of measure zero), since it is continuous except in the set $\{(\psi, t) \in \mathbb{R}^{2n+1} \times [t_0, \infty) | e = 0\}$. To facilitate the stability analysis, let $V_L : \mathbb{R}^{2n+1} \to \mathbb{R}_{\ge 0}$ be defined as

$$V_L(\psi) \triangleq \frac{1}{2}e^Te + \frac{1}{2}r^Tr + P. \tag{15}$$

Let $c \triangleq \min\{k - \gamma_2 - n\gamma_4, \alpha - \gamma_2 - n\gamma_4, \frac{\lambda_P}{2}\}$, $\rho(\cdot) \triangleq \rho_1(\cdot) + n\rho_2(\cdot)$, and consider the regions, $\mathcal{D} \triangleq \{\sigma \in \mathbb{R}^{2n+1} | V_L(\sigma) <$

[3]See [8] for continuous adaptive weight updates.

$\frac{(\rho^{-1}(c-\lambda_V))^2}{2}\}$ and $\mathcal{S} \triangleq \{\sigma \in \mathbb{R}^{2n} | \|\sigma\| < c - \lambda_V\}$, where $\lambda_V \in \mathbb{R}_{>0}$ is a user-defined constant.

Theorem 1: Given any initial condition $z(t_0) \in \mathbb{R}^{2n}$, every maximal solution to (3), (7), and (9) with $P(t_0)$ initialized according to (11) is complete, and the zero solution to (3) and (7), $(e(t), r(t)) \equiv (0, 0)$, is semi-globally exponentially stable in the sense that $\|z(t)\| \leq \|z(t_0)\| \exp(-\lambda_V(t - t_0))$, $\forall (z(t_0), t) \in \mathbb{R}^{2n} \times [t_0, \infty)$, provided that the gains α, β, k and λ_P are selected according to the gain conditions in (13), (14), and

$$c > \lambda_V + \|z(t_0)\|. \tag{16}$$

Proof: The existence of a Filippov solution, $\psi : \mathcal{I} \to \mathbb{R}^{2n+1}$, to $\dot{\psi} = g(\psi, t)$ is guaranteed[4] by [26, Proposition 3]. The time-derivative of V_L along ψ, starting from the specified initial conditions, exists a.e., and $\dot{V}_L(\psi, t) \overset{a.e.}{\in} \dot{\tilde{V}}_L(\psi, t)$ [31, eqs. (12) and (13)], where the notation $\overset{a.e.}{(\cdot)}$ implies that the relation holds for almost all time $t \in \mathcal{I}$, and

$$\begin{aligned}\dot{\tilde{V}}_L(\psi, t) &= \bigcap_{\xi \in \partial V_L(\psi)} \xi^T K\big[g\big](\psi, t) \\ &= \nabla V_L^T K\big[g\big](\psi, t) \\ &= \big[z^T \ 1\big]^T K\big[g\big](\psi, t).\end{aligned} \tag{17}$$

In (17), $\partial V_L(\psi)$ denotes Clarke's generalized gradient [31, eq. (7)], and $K[\,\cdot\,]$ is defined in [25, eq. (2b)]. Since $\psi \mapsto V_L(\psi)$ is continuously differentiable, $\partial V_L = \{\nabla V_L\}$ using [26, Proposition 6]. Substituting (3), (7), and (9) into (17), utilizing $\|\cdot\|_1 \leq \sqrt{n}\|\cdot\|$, and applying Young's inequality on $\|z\|\|e\|_1$ yields

$$\begin{aligned}\dot{\tilde{V}}_L &= r^T(\tilde{N} + N_B - kr - e - \beta K\big[\mathrm{sgn}\big](e)) \\ &\quad + e^T(r - \alpha e) - \lambda_P P - r^T N_B \\ &\quad + r^T \beta K\big[\mathrm{sgn}\big](e) + (\gamma_4 + \rho_2(\|z\|))\|z\|\|e\|_1 \\ &\leq -k\|r\|^2 - \alpha\|e\|^2 + (\gamma_2 + \rho_1(\|z\|))\|r\|\|z\| \\ &\quad - \lambda_P P + (\gamma_4 + \rho_2(\|z\|))\|z\|\|e\|_1 \\ &\leq -2(c - \rho(\|z\|))V_L \\ &\leq -2\Big(c - \rho\Big(\sqrt{2V_L}\Big)\Big)V_L,\end{aligned} \tag{18}$$

for almost all $t \in \mathcal{I}$, where c and ρ are introduced before the theorem statement, the term $t \mapsto r^T(t)\beta K[\mathrm{sgn}](e(t))$ is set-valued only for the set of time instants $T = \{t \in [t_0, \infty) | \exists i \in \{1, 2, \ldots, n\} \text{ s.t. } e_i(t) = 0 \wedge r_i(t) \neq 0\}$. According to Lemma 1, the set T has Lebesgue measure zero. It follows from (18) that V_L is non-increasing along all trajectories initialized such that $\psi(t_0) \in \mathcal{D}$. Selecting c according to (16) and using (11) yields $c > \lambda_V + \|z(t_0)\| = \lambda_V + \rho(\sqrt{2V_L(\psi(t_0))})$. Then, $V_L(\psi(t))$ is non-increasing, implying $\psi(t) \in \mathcal{D}$, $\forall t \in \mathcal{I}$. It follows from (16) and (18) that $\dot{V}_L$ can be upper-bounded as

$$\dot{V}_L \leq -2\lambda_V V_L, \tag{19}$$

for almost all $t \in \mathcal{I}$. Using the comparison principle [32, Lemma 4.4] in (19) yields

$$V_L(\psi(t)) \leq V_L(\psi(t_0)) \exp(-2\lambda_V(t - t_0)), \tag{20}$$

$\forall (\psi(t_0), t) \in \mathcal{D} \times \mathcal{I}$. Since $(\psi, t) \mapsto K[g](\psi, t)$ is locally bounded over $\mathbb{R}^{2n+1} \times [t_0, \infty)$ and (20) implies that ψ is precompact, then [33, Lemma 3.3 and Remark 3.4] can be invoked to show that every maximal solution ψ with $P(t_0)$ initialized according to (11) is complete, i.e., $\mathcal{I} = [t_0, \infty)$. Using the definition of z, (20) and Lemma 4, $V_L(\psi(t)) = \frac{1}{2}\|z(t)\|^2 + P(t) \geq \frac{1}{2}\|z(t)\|^2$, $\forall t \in [t_0, \infty)$. Therefore,

$$\|z(t)\| \leq \sqrt{2V_L(\psi(t))}, \ \forall t \in [t_0, \infty). \tag{21}$$

Using (20) and (21), $\|z(t)\|$ can further be upper-bounded as

$$\|z(t)\| \leq \sqrt{2V_L(\psi(t_0))} \exp(-\lambda_V(t - t_0)), \tag{22}$$

$\forall (\psi(t_0), t) \in \mathcal{D} \times [t_0, \infty)$. Moreover, substituting (11) in the expression for $V_L(\psi(t_0))$ yields $V_L(\psi(t_0)) = \frac{1}{2}\|z(t_0)\|^2$. Consequently, $\psi(t_0) \in \mathcal{D}$ implies $z(t_0) \in \mathcal{S}$. Using (22) yields

$$\|z(t)\| \leq \|z(t_0)\| \exp(-\lambda_V(t - t_0)), \tag{23}$$

$\forall (z(t_0), t) \in \mathcal{S} \times [t_0, \infty)$, implying the zero solution to (3) and (7), $(e(t), r(t)) \equiv (0, 0)$, is semi-globally exponentially stable. Note that the exponential stability result is semi-global (cf., [34, Remark 2]) because the size of the set $\mathcal{S}$ can be arbitrarily increased using (16) to include any $z(t_0) \in \mathbb{R}^{2n}$. Moreover, $x \in \mathcal{L}_\infty$ since $e, x_d \in \mathcal{L}_\infty$. Since $d \in \mathcal{L}_\infty$ by Assumption 1, it follows from (6) that $\mu \in \mathcal{L}_\infty$. Therefore, since all the terms on the right hand side of (4) are bounded and continuous, $u \in \mathcal{L}_\infty$ and is continuous. Moreover, since $e(t) = e(t_0) + \int_{t_0}^{t}(r(\tau) - \alpha e(\tau))d\tau$ for all $t \in [t_0, \infty)$ using (3), the continuity of $r - \alpha e$ implies that the Filippov solution e is also continuously differentiable. ■

Remark 1: The relation in (6) indicates that r is the estimation error between the RISE term $\hat{d}(t)$ and the uncertainty $d(x, \nu, t)$. Therefore, (23) implies that the RISE term is an exponentially convergent estimator of the uncertainty, i.e., $\hat{d}(t) \to d(x, \nu, t)$ with a uniform and exponential convergence as $t \to \infty$.

Remark 2: For the special case when $\dot{d}(t)$ and $\ddot{d}(t)$ are bounded by known constants, the analysis approach in [14] can also be considered.

Remark 3: The exponential stability result is global when the bounds on $\tilde{N}$ and $\dot{N}_B$ are linear in $\|z\|$, i.e., $\rho_1 = \rho_2 = 0$.

IV. Conclusion

In this letter, new insights on the construction of a P-function are used to yield exponential stability with RISE-based controllers. As an outcome of this breakthrough, the inherent learning capability of RISE-based controllers is shown to yield exponential identification of disturbances/uncertainty, as compared to all previous asymptotic results. Future work could involve extension of the proposed stability analysis methodology for RISE-based error systems with sensor noise, and delays in input and state measurements.

[4]The solution ψ may not be unique; however, P is unique according to Lemma 3 for a given pair (e, r). Moreover, the results in this letter are applicable to all the trajectories even when ψ is non-unique, since we consider a generalized Filippov solution in the analysis.

Appendix

Proof of Lemma 1: The set T can also be represented as $T = \{t \in \mathcal{I} | \exists i \in \{1, 2, \ldots, n\} \text{ s.t. } e_i(t) = 0 \ \wedge r_i(t) - \alpha e_i(t) \neq 0\} = \{t \in \mathcal{I} | \exists i \in \{1, 2, \ldots, n\} \text{ s.t. } e_i(t) = 0 \wedge \dot{e}_i(t) \neq 0\}$ to facilitate the subsequent analysis using the dynamics in (7). Select $a \in T$, which yields $e_i(a) = 0$ using the definition of T. Given $\dot{e}_i(a) \neq 0$, we can assume without loss of generality that $\dot{e}_i(a) > 0$; the proof easily extends for $\dot{e}_i(a) < 0$. Since e and r are absolutely continuous, it follows from (3) that $\dot{e}$ is continuous. Given $\dot{e}_i(a) > 0$ and continuity of $\dot{e}$, there exists $\delta > 0$ such that $\dot{e}_i(t) > 0$ for all $t \in (a-\delta, a+\delta)$. Based on $\dot{e}_i(t) > 0$, it follows that $\int_a^t \dot{e}_i(\tau) d\tau > 0$ for all $t \in (a, a+\delta)$. Then, using $e_i(a) = 0$ yields $\int_a^t \dot{e}_i(\tau) d\tau = e_i(t) - e_i(a) = e_i(t) > 0$. Similarly, $-\int_t^a \dot{e}_i(\tau) d\tau = e_i(t) - e_i(a) = e_i(t) < 0$ for all $t \in (a-\delta, a)$, implying $e_i(t) \neq 0$ for all $t \in (a-\delta, a+\delta) \backslash \{a\}$. If more than one component has $e_i(a) = 0$, we select the intersection of each neighborhood found above, represented by $U(a)$. Therefore, there exists a neighborhood, $U(a)$, for any $a \in T$, s.t. $e_i(t) \neq 0$ for all time-instants $t \in U(a) \backslash \{a\}$ and $i \in \{1, 2, \ldots, n\}$. When $t \in T$, $e_i(t) = 0$ for some $i \in \{1, 2, \ldots, n\}$, which implies $U(a) \cap T = \{a\}$ for all $a \in T$; therefore, T is discrete and consequently has measure zero. ■

Proof of Lemma 2: The function $\text{sgn}(e(\cdot))$ is discontinuous only at time-instants where it changes sign, i.e., the set $\{t \in \mathcal{I} | \exists i \in \{1, 2, \ldots, n\} \text{ s.t. } e_i(t) = 0 \ \wedge \dot{e}_i(t) \neq 0\} = \{t \in \mathcal{I} | \exists i \in \{1, 2, \ldots, n\} \text{ s.t. } e_i(t) = 0 \ \wedge r_i(t) - \alpha e_i(t) \neq 0\} = T$. Since T has Lebesgue measure zero according to Lemma 1, $\text{sgn}(e(\cdot))$ is continuous a.e., implying it is Riemann integrable [35, Th. 11.33] on $[t_0, t_1]$, $\forall t_1 \in \mathcal{I}$. ■

Proof of Lemma 3: The right hand side (RHS) of (12) is almost everywhere (a.e.) differentiable with respect to time, because every term on the RHS is absolutely continuous, including $\|e\|_1$, since $\|\cdot\|_1$ is globally Lipschitz and e is absolutely continuous. The time-derivative of $\|e\|_1$, whenever it exists, is $\dot{e}^T \text{sgn}(e)$, using the chain rule. Therefore, taking the time-derivative of both the sides of (12) at points where P is differentiable yields

$$\begin{aligned} \dot{P} \overset{a.e.}{=} & \ \beta \dot{e}^T \text{sgn}(e) - \dot{e}^T N_B - e^T \dot{N}_B \\ & + \frac{d}{dt}(\mathrm{e}^{-\lambda_P t} * ((\alpha - \lambda_P)(\beta \|e\|_1 - e^T N_B) + e^T \dot{N}_B)) \\ & + \frac{d}{dt}\big(\mathrm{e}^{-\lambda_P t} * ((\gamma_4 + \rho_2(\|z\|))\|z\| \|e\|_1)\big). \end{aligned} \tag{24}$$

Based on the Leibniz rule, for any given $q{:}[t_0, \infty) \to \mathbb{R}$, the function $\mathrm{e}^{-\lambda_P t}$ satisfies $\frac{d}{dt}(\mathrm{e}^{-\lambda_P t} * q) = \frac{d}{dt}(\int_{t_0}^t \mathrm{e}^{-\lambda_P (t-\tau)} q(\tau) d\tau) = q(t) - \lambda_P \int_{t_0}^t \mathrm{e}^{-\lambda_P (t-\tau)} q(\tau) d\tau = -\lambda_P \mathrm{e}^{-\lambda_P t} * q + q$. Additionally, $L = \dot{e}^T N_B + \alpha e^T N_B - \beta \dot{e}^T \text{sgn}(e) - \alpha \beta \|e\|_1 - (\gamma_4 + \rho_2(\|z\|))\|z\| \|e\|_1$ is obtained after substituting (3) into (10). Therefore, the expression for $\dot{P}$ in (24) can be rewritten as

$$\begin{aligned} \dot{P} \overset{a.e.}{=} & -\lambda_P \mathrm{e}^{-\lambda_P t} * ((\gamma_4 + \rho_2(\|z\|))\|z\| \|e\|_1) \\ & - \lambda_P \mathrm{e}^{-\lambda_P t} * ((\alpha - \lambda_P)(\beta \|e\|_1 - e^T N_B) + e^T \dot{N}_B) \\ & - \lambda_P \beta \|e\|_1 + \lambda_P e^T N_B + \beta \dot{e}^T \text{sgn}(e) \\ & - \dot{e}^T N_B + \alpha \beta \|e\|_1 - \alpha e^T N_B + (\gamma_4 + \rho_2(\|z\|))\|z\| \|e\|_1 \\ = & -\lambda_P P - L. \end{aligned} \tag{25}$$

Filippov's differential inclusion for (9) is given by

$$\dot{P} \in -\lambda_P P - K[L]. \tag{26}$$

To prove the uniqueness of the solution to (26), consider any two solutions with the same initial conditions, i.e., P_1 and P_2, with $P_1(t_0) = P_2(t_0) = 0$, implying

$$\dot{P}_1 \overset{a.e.}{\in} -\lambda_P P_1 - K[L], \tag{27}$$

$$\dot{P}_2 \overset{a.e.}{\in} -\lambda_P P_2 - K[L]. \tag{28}$$

Based on (10), $t \mapsto K[L](\psi(t))$ is set-valued only when there exists some $i \in \{1, 2, \ldots, n\}$ such that $t \to K[\text{sgn}](e_i(t))$ is set-valued and $r_i(t) \neq 0$. Using Lemma 1, $t \mapsto K[L](\psi(t))$ is set-valued only for a set of time-instants of measure zero. Therefore, defining $\Delta(t) = P_2(t) - P_1(t)$, and using (27) and (28) yields

$$\dot{\Delta} \overset{a.e.}{=} -\lambda_P \Delta, \tag{29}$$

with $\Delta(t_0) = 0$. Since $\Delta \equiv 0$ is an equilibrium point of (29), $\Delta(t_0) = 0$ implies $\|\Delta(t)\| = 0$, $\forall t \in \mathcal{I}$, therefore $P_1(t) = P_2(t)$, $\forall t \in [t_0, \infty)$, i.e., any two solutions are equal, implying the solution is unique. ■

Proof of Lemma 4: Using Holder's inequality and Assumption 1 yields lower bounds on $-e^T N_B$ and $e^T \dot{N}_B$,

$$-e^T N_B \geq -\|e\|_1 \|N_B\|_1 \geq -\gamma_1 \|e\|_1, \tag{30}$$

and

$$\begin{aligned} e^T \dot{N}_B \geq & -\|e\|_1 \big\|\dot{N}_B\big\|_1 \\ \geq & -(\gamma_3 + \gamma_4 \|z\| + \rho_2(\|z\|)\|z\|)\|e\|_1, \end{aligned} \tag{31}$$

$\forall t \in \mathcal{I}$. Substituting the bounds in (30) and (31) into the expression for P in (12) yields

$$\begin{aligned} P \geq & \ \beta \|e\|_1 - \gamma_1 \|e\|_1 \\ & + \mathrm{e}^{-\lambda_P t} * (((\alpha - \lambda_P)(\beta - \gamma_1) - \gamma_3)\|e\|_1). \end{aligned} \tag{32}$$

Selecting $P(t_0)$ according to (11), and α and β according to the gain conditions (13) and (14) yields $P(t) \geq 0$, $\forall t \in \mathcal{I}$ using (32). ■

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