

Optimal Safety for Constrained Differential Inclusions Using Nonsmooth Control Barrier Functions

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Abstract—For a broad class of nonlinear systems, we formulate the problem of guaranteeing safety with optimality under constraints. Specifically, we define controlled safety for differential inclusions with constraints on the states and the inputs. Through the use of nonsmooth analysis tools, we show that a continuous optimal control law can be selected from a set-valued constraint capturing the system constraints and conditions guaranteeing safety using control barrier functions. Our results guarantee optimality and safety via a continuous state-feedback law designed using nonsmooth control barrier functions. An example pertaining to obstacle avoidance with a target illustrates our results and the associated benefits of using nonsmooth control barrier functions.

Index Terms—Constrained systems, differential inclusions, nonsmooth control barrier functions, optimal control, safe control.

I. INTRODUCTION

A POWERFUL approach to guarantee safety for a dynamical system without computing the solutions consists of using barrier functions. Stemming from optimization theory and seminal work by Nagumo in [1], a barrier function ensures that, when properly initialized, the solutions of the dynamical system do not reach an unsafe set. This approach has been exploited for the study of continuous-time, discrete-time, and hybrid systems; see, e.g., [2]; [3]; and [4], [5], respectively. The extension of the barrier function concept to the case when the system has an input, known as control

barrier function (CBF), which is instrumented to synthesize control laws has been pursued in [6], [7] for continuous-time systems, [3] for discrete-time systems, and in [8], [9] for differential inclusions. To facilitate finding suitable barrier functions needed in control applications, the work in [10], [11] proposed multiple and nonsmooth barrier functions, but does not consider optimality.

Recent developments in combining optimization techniques and safety constraints have led to optimization problems that, when solved numerically, result in a control law that assures both safety and optimality. In [7], the authors proposed a quadratic program to find a minimum norm control law that ensures safety and stability for nonlinear control affine differential equations. Though powerful, continuity of the resulting feedback law is not well characterized. In our recent work, we consider feasibility and continuity of the feedback control law defined by the multiple continuously differentiable barrier functions [12]. Here, we consider differential inclusions with constraints, which are more general than differential equations. Differential inclusions are effective at modeling dynamical systems with uncertainty. Nonsmooth barrier functions emerge naturally in many control problems, such as obstacle avoidance. The work in [8] and [11] consider nonsmooth CBFs for differential inclusions, but do not guarantee optimality or continuity of the control law. Motivated by the need for feedback controllers that assure safety and optimality with good regularity properties, we propose methods to design optimal state-feedback laws using nonsmooth barrier functions that, notably, are continuous. Specifically, for constrained differential inclusions, we propose sufficient conditions for selecting a continuous safe and optimal control law by minimizing a cost function and using a nonsmooth CBF. More precisely, we consider differential inclusions with state and input constraints given as

$$\dot{x} \in F(x, u) \quad (x, u) \in C := C_x \times C_u \quad (1)$$

where $F: \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, $C_x \subset \mathbb{R}^n$, $C_u \subset \mathbb{R}^m$. Let $X_u \subset \mathbb{R}^n$ be unsafe set and $X_o \subset C_x$ indicate the desirable initial set. Using nonsmooth CBFs w.r.t. (X_o, X_u) for constrained differential inclusions as in (1), we provide sufficient conditions for the existence of a continuous safe control law that minimizes a cost function over the set-valued map providing safe inputs for each state. More precisely, the problem we study consists

Manuscript received 16 September 2022; revised 22 November 2022; accepted 8 December 2022. Date of publication 26 December 2022; date of current version 24 January 2023. This work was supported in part by the National Science Foundation under Grant ECS-1710621, Grant CNS-2039054, and Grant CNS-2111688; in part by the Air Force Office of Scientific Research under Grant FA9550-19-1-0053, Grant FA9550-19-1-0169, and Grant FA9550-20-1-0238; and in part by the Army Research Office under Grant W911NF-20-1-0253. Recommended by Senior Editor L. Menini. (Corresponding author: Masoumeh Ghanbarpour.)

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Digital Object Identifier 10.1109/LCSYS.2022.3232219

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of solving

$$\begin{aligned} \kappa^*(x) = \operatorname{argmin} \quad & \mathcal{L}(x, u) \\ \text{s.t.} \quad & u \in D(x) \end{aligned} \quad (2)$$

for each $x \in C_x$ so as to synthesize the optimal state-feedback law κ^* ensuring safety, where $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ denotes the cost function and the set-valued map $D : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ indicates feasible safe control inputs at the current state.

Contributions: This letter makes the following contributions:

- 1) In Theorem 1, we specify sufficient conditions for the existence of the continuous safe control law using a nonsmooth CBF for (1).
- 2) In Lemma 2 and Lemma 3, we specify conditions such that the resulting map κ^* in (2) is single valued and continuous.
- 3) In Theorem 2, we formulate sufficient conditions to obtain a continuous and safe state-feedback law that minimizes the given cost function and meets the safety constraints.

Notation: For $x, y \in \mathbb{R}^n$, $x^\top$ denotes the transpose of x , $|x|$ the Euclidean norm of x , and $\langle x, y \rangle$ denotes the inner product between x and y ; namely, $\langle x, y \rangle := x^\top y$. For a set $K \subset \mathbb{R}^n$, we use $\operatorname{int}(K)$ to denote its interior, ∂K its boundary, $\operatorname{cl}(K)$ its closure, $U(K)$ to denote an open neighborhood around K , and $\operatorname{co}(K)$ to indicate the convex hull of the set K . For $O \subset \mathbb{R}^n$, $K \setminus O$ denotes the subset of elements of K that are not in O . For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\operatorname{dom} f$ denotes the domain of definition of f , $\operatorname{Graph}(f)$ indicates graph of f , and if f is continuously differentiable, ∇f denotes the gradient of f . If f is locally Lipschitz, $\partial_C f$ denotes Clarke generalized gradient of f . Let $\mathbb{B}$ denote the closed unit ball in $\mathbb{R}^n$ centered at the origin. By $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, we denote a set-valued map associating each element $x \in \mathbb{R}^m$ into a subset $F(x) \subset \mathbb{R}^n$.

II. PRELIMINARIES

A. Basic Definitions

Definition 1 (Sublevel Bounded [13, Definition 1.8]): A function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \pm\infty$ is *sublevel bounded* if for every finite $\alpha \in \mathbb{R}$ the set $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is bounded.

Definition 2 (Level Coercive [13, Definition 3.25]): A function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \pm\infty$ is *level coercive* if it is bounded below on bounded sets and satisfies $\lim_{|x| \rightarrow \infty} \inf_{f(x)} \frac{f(x)}{|x|} > 0$.

B. Differential Inclusions

Consider differential inclusions with state constraints

$$\Sigma : \dot{x} \in \tilde{F}(x) \quad x \in C_x \quad (3)$$

where $\tilde{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $C_x \subset \mathbb{R}^n$. Next, solutions to the constrained differential inclusion Σ in (3) are defined.

Definition 3 (Concept of Solution to Σ): The function $x : \operatorname{dom} x \rightarrow \mathbb{R}^n$, where $\operatorname{dom} x \subset [0, \infty)$, is a solution to Σ in (3) if (i) $x(0) \in \operatorname{cl}(C_x)$, (ii) $t \mapsto x(t)$ is locally absolutely continuous, (iii) $x(t) \in C_x$ for all $t \in \operatorname{int}(\operatorname{dom} x)$, and (iv) $\dot{x}(t) \in \tilde{F}(x(t))$ for almost all $t \in \operatorname{dom} x$.

A solution $t \mapsto x(t)$ to Σ in (3) is said to be *complete* if $\operatorname{dom} x$ is unbounded. Furthermore, it is said to be *maximal* if a solution y to Σ does not exist such that $x(t) = y(t)$ for all $t \in \operatorname{dom} x$ with $\operatorname{dom} x$ a proper subset of $\operatorname{dom} y$.

Consider the constrained differential inclusion in (1). Let $\mathcal{U} : C_x \rightrightarrows C_u$ be the set-valued map that provides admissible values in C_u to each $x \in C_x$, namely, $\mathcal{U}(x)$ indicates the feasible control inputs associated with x . Let $\Sigma_{\mathcal{U}}$ indicate the differential inclusion in (1) explicitly constraining u to take values from the set-valued feedback map $\mathcal{U} : C_x \rightrightarrows C_u$,

$$\Sigma_{\mathcal{U}} : \dot{x} \in F(x, u) \quad u \in \mathcal{U}(x), \quad x \in C_x. \quad (4)$$

Given a selection $\kappa(x) \in \mathcal{U}(x)$ for each $x \in C_x$, the (closed-loop) differential inclusion is given by

$$\Sigma_{\kappa} : \dot{x} \in F(x, \kappa(x)) \quad x \in C_x. \quad (5)$$

III. CONTROLLED SAFETY NOTIONS AND CBFs

A. Safety Notions

Given Σ in (3) and a set $K \subset C_x$ following [14], we employ the following forward pre-invariance and controlled pre-invariance notions.

Definition 4 (Forward Pre-Invariance): The set K is said to be *forward pre-invariant* for the differential inclusion Σ in (3) if for each $x_o \in K$, each maximal solution x to Σ starting from x_o satisfies $x(t) \in K$ for all $t \in \operatorname{dom} x$.

Definition 5 (Controlled Pre-Invariance): A set K is *control pre-invariant* for $\Sigma_{\mathcal{U}}$ in (4) if there exists a selection $\kappa(x) \in \mathcal{U}(x)$ for each $x \in C_x$ such that K is forward pre-invariant for the resulting differential inclusion Σ_{κ} in (5).

Suppose that $X_o \cap X_u = \emptyset$. Safety and controlled safety are defined as follows.

Definition 6 (Safety): Given Σ in (3), and $X_o \subset C_x$ and $X_u \subset \mathbb{R}^n$ such that $X_o \cap X_u = \emptyset$, Σ is *safe* with respect to (X_o, X_u) if for each solution x to Σ starting from $x_o \in X_o$, we have $x(t) \in \mathbb{R}^n \setminus X_u$ for all $t \in \operatorname{dom} x$.

Definition 7 (Controlled Safety): Given $\Sigma_{\mathcal{U}}$ in (4), and $X_o \subset \operatorname{dom} \mathcal{U}$ and $X_u \subset \mathbb{R}^n$ such that $X_o \cap X_u = \emptyset$, $\Sigma_{\mathcal{U}}$ is *controlled safe* with respect to (X_o, X_u) , if there exists a selection $\kappa(x) \in \mathcal{U}(x)$ for each $x \in C_x$ such that the resulting differential inclusion Σ_{κ} in (5) is safe with respect to (X_o, X_u) .

Remark 1: If we can find a forward pre-invariant (controlled pre-invariant) set $K \subset \mathbb{R}^n$ for Σ (respectively, $\Sigma_{\mathcal{U}}$), such that $X_o \subset K$ and $K \cap X_u = \emptyset$, then safety (respectively, controlled safety) is verified with respect to (X_o, X_u) .

B. CBFs

Following [6], we define control barrier candidates and CBFs.

Definition 8 (Barrier Function Candidate): A function $B : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a barrier function candidate with respect to (X_o, X_u) if

$$B(x) > 0 \quad \forall x \in X_u, \quad B(x) \leq 0 \quad \forall x \in X_o. \quad (6)$$

The zero sublevel set of B is defined as $K_e := \{x \in \mathbb{R}^n : B(x) \leq 0\}$. Consider $\Sigma_{\mathcal{U}}$ in (4). Suppose B is a barrier function candidate with respect to (X_o, X_u) . Let the set K be defined as

$$K := K_e \cap C_x. \quad (7)$$

If K is controlled pre-invariant for $\Sigma_{\mathcal{U}}$, then according to Remark 1, $\Sigma_{\mathcal{U}}$ is controlled safe.

Definition 9 (Control Barrier Function): A locally Lipschitz barrier function candidate B with respect to (X_o, X_u)

is a CBF for $\Sigma_{\mathcal{U}}$ in (4) if there exists a neighborhood of the boundary of the set K , $U(\partial K)$, where K is defined in (7), such that the following condition holds:

$$\inf_{u \in \mathcal{U}(x)} \sup_{\eta \in F(x, u), \zeta \in \partial_C B(x)} \langle \zeta, \eta \rangle \leq 0 \quad \forall x \in (U(\partial K) \setminus \text{int}(K)) \cap C_x. \quad (8)$$

Let the function B be a CBF for $\Sigma_{\mathcal{U}}$ in (4) with respect to (X_o, X_u) , defining the set K in (7). Suppose B is locally Lipschitz. We define the function $g : C_x \times C_u \rightarrow \mathbb{R}$ as

$$g(x, u) := \sup\{\langle \zeta, \eta \rangle : \zeta \in \partial_C B(x), \eta \in F(x, u)\}. \quad (9)$$

For some continuous function $\gamma : C_x \rightarrow \mathbb{R}$, we define the set-valued map $D_\gamma : C_x \rightrightarrows C_u$ as

$$D_\gamma(x) := \{u \in \mathcal{U}(x) : g(x, u) + \gamma(x) < 0\}. \quad (10)$$

The set-valued map D for each x gives the set of all feasible control inputs that makes the function $g + \gamma$ negative; therefore, it provides the possible control inputs for which the CBF decreases along the solutions.

Assumption 1: The set $C = C_x \times C_u$ is closed.

Assumption 2: The map $F : C \rightrightarrows \mathbb{R}^n$ is upper semicontinuous and $F(x, u)$ is nonempty, compact, and convex for all $(x, u) \in C$.

Assumptions 1 and 2 are known as tight requirements in the literature for the existence of solutions and the structural properties for the set of solutions of differential inclusion; see [15], [16], [17]. For Assumption 4, see [18, Sec. 3.4].

Assumption 3: The feedback map $\mathcal{U} : C_x \rightrightarrows C_u$ is lower semicontinuous with nonempty, closed, and convex values.

Assumption 4: The map F is convex in u .

Based on Michael's Theorem [18, Th. 2.18], Assumption 3 guarantees the essential requirement to find a continuous selection from $\mathcal{U}$. In the following lemma, we indicate the regularity of the maps g and D_γ under the said assumptions.

Lemma 1: Consider $\Sigma_{\mathcal{U}}$ in (4) such that Assumptions 1-3 hold. Let $B : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz and $\gamma : C_x \rightarrow \mathbb{R}$ be a continuous function. Suppose the set-valued map $D_\gamma : C_x \rightrightarrows C_u$ is defined in (10) and $g : C_x \times C_u \rightarrow \mathbb{R}$ is defined in (9). Then, the following hold:

- 1.a The function g is upper semicontinuous,
- 1.b The map D_γ is lower semicontinuous,
- 1.c When Assumption 4 holds, for each $x \in C_x$ the function $u \mapsto g(x, u)$ is convex and lower semicontinuous.

Proof: To prove 1.a, using Assumption 2, we obtain that F has nonempty compact values, and using [19, Proposition 2.6.2], we conclude that $\partial_C B$ is nonempty and has convex compact values. Therefore, g is well-defined. Since B is locally Lipschitz, from [20, Lemma 4.6] we conclude that g is upper semicontinuous.

To prove 1.b, from 1.a, we have that g is upper semicontinuous and since γ is continuous, we conclude that $g + \gamma$ is upper semicontinuous. Since, by Assumption 3, $\mathcal{U}$ is a lower semicontinuous, [18, Corollary 2.13] implies that D_γ is lower semicontinuous.

To prove 1.c, first we show that g is convex in u . Using Assumption 4, for each $\theta \in [0, 1]$, $x \in C_x$, and $u_1, u_2 \in C_u$,

we have

$$\begin{aligned} g(x, \theta u_1 + (1 - \theta)u_2) &\leq \theta \sup_{\zeta \in \partial_C B(x), \eta_1 \in F(x, u_1)} \langle \zeta, \eta_1 \rangle \\ &\quad + (1 - \theta) \sup_{\zeta \in \partial_C B(x), \eta_2 \in F(x, u_2)} \langle \zeta, \eta_2 \rangle \\ &\leq \theta g(x, u_1) + (1 - \theta)g(x, u_2) \end{aligned}$$

To prove that g is lower semicontinuous, let $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ be defined as $h(x, u, \zeta) := \sup_{\eta \in F(x, u)} \langle \eta, \zeta \rangle$. For fixed ζ , the map $\eta \mapsto \langle \eta, \zeta \rangle$ is continuous and convex. Furthermore, since F has bounded values and $\partial_C B$ is bounded, we conclude that h is bounded. Then, [21, Th. 9.4] implies that $u \mapsto h(x, u, \zeta)$ is lower semicontinuous. Therefore, since $g(x, u) = \sup_{\zeta \in \partial_C B(x)} h(x, u, \zeta)$, $u \mapsto g(x, u)$ is lower semicontinuous for each $x \in C_x$. ■

Remark 2: In general, when B is not continuously differentiable, D_γ is not necessarily outer semicontinuous, therefore, D_γ is not continuous. Since the Clarke generalized gradient of the locally Lipschitz function B , $\partial_C B$, is upper semicontinuous, with correct regularity of F , the function g can only be upper semicontinuous. In general, D_γ cannot be continuous without continuity of g .

IV. SUFFICIENT CONDITIONS FOR SAFETY

Theorem 1: Consider $\Sigma_{\mathcal{U}}$ in (4) such that Assumptions 1-4 hold. Let $B : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz CBF with respect to $(X_o, X_u) \subset \mathbb{R}^n \times \mathbb{R}^n$ defining the set K in (7). Suppose there exists a neighborhood of ∂K , denoted by $U(\partial K)$, such that the set-valued map $D_\gamma : C_x \rightrightarrows C_u$ defined in (10) for γ identically zero is nonempty on $(U(\partial K) \setminus \text{int}(K)) \cap C_x$. Then, there exists a continuous control law $\kappa : C_x \rightarrow C_u$ that makes $\Sigma_{\mathcal{U}}$ controlled safe with respect to (X_o, X_u) .

Proof: Let $g : C_x \times C_u \rightarrow \mathbb{R}$ be defined in (9), and from (10), let the map $D_0 : C_x \rightrightarrows C_u$ for $x \mapsto \gamma(x) = 0$ be defined as

$$D_0(x) := \{u \in \mathcal{U}(x) : g(x, u) < 0\}. \quad (11)$$

Since B is a CBF for $\Sigma_{\mathcal{U}}$, suppose $U_1(\partial K)$ is a neighborhood such that (8) holds. Let $U_2(\partial K)$ be a neighborhood such that $U_2(\partial K) \subset U_1(\partial K)$ and $U_2(\partial K) \subset U(\partial K)$. From 1.a and 1.c in Lemma 1, we have that g is continuous in u ; therefore, for each x , its sublevel sets are closed. Then, since also for each $x \in C_x$, $\mathcal{U}(x)$ has closed values, we obtain that $\text{cl}(D_0(x)) = \{u \in \mathcal{U}(x) : g(x, u) \leq 0\} =: \bar{D}_0$. Let $S = \text{cl}((U_2(\partial K) \setminus K) \cap C_x)$. To prove that there exists a continuous selection from $\bar{D}_0$ on S , based on [18, Th. 2.18], we show that $\bar{D}_0$ is lower semicontinuous and nonempty on S , with closed convex values. Since Assumptions 1-3 hold, 1.b in Lemma 1 implies that D_0 is lower semicontinuous. Using [22, Proposition 2.3], we conclude that $\bar{D}_0$ is lower semicontinuous. Furthermore, $\bar{D}_0$ is nonempty on S and has closed values. Finally, using Assumption 3, we obtain that $x \mapsto \mathcal{U}(x)$ has convex values and using the assumption that $u \mapsto g(x, u)$ is convex, we conclude that $\bar{D}_0$ has convex values. Therefore, [18, Th. 2.18] implies that there exists a continuous selection $\kappa_1 : S \rightarrow C_u$ such that $\kappa_1(x) \in \bar{D}_0(x)$ for each $x \in S$. Furthermore, since for each $x \in S$, we have $\bar{D}_0(x) \subset \mathcal{U}(x)$, then κ_1 is also a selection from $\mathcal{U}$. Then, using Assumption 3 and [22, Proposition 1.4], κ_1 can be extended continuously to the entire C_x . Let $\kappa : C_x \rightarrow C_u$ be the extension of κ_1 . Finally, to

prove that Σ_K as defined in (5) is safe with respect to (X_o, X_u) , from (8) we have

$$\sup_{\eta \in F(x, \kappa(x)), \zeta \in \partial_C B(x)} \langle \zeta, \eta \rangle \leq 0 \quad \forall x \in U_2(\partial K) \setminus K.$$

Therefore, $\langle \zeta, \eta \rangle \leq 0$ for each $x \in U_2(\partial K) \setminus K$, $\zeta \in \partial_C B(x)$, and for each $\eta \in F(x, \kappa(x))$. Hence, using Assumption 2 and the fact that κ is continuous, we conclude that $x \mapsto F(x, \kappa(x))$ is upper semicontinuous with nonempty, compact, and convex values. Using [23, Lemma 5.15], we conclude that upper semicontinuous maps with compact images are outer semicontinuous and locally bounded set-valued maps. Therefore, based on [5, Th. 4], K is forward pre-invariant for Σ_K . Then, Σ_K is safe with respect to (X_o, X_u) . Therefore, Σ_U is controlled safe with respect to (X_o, X_u) . ■

Remark 3: For a continuously differentiable function B and single-valued function $f : C_x \times C_u \rightarrow \mathbb{R}^n$, the condition

$$\inf_{u \in U} \langle \nabla B(x), f(x, u) \rangle \leq -\alpha(B(x)) \quad (12)$$

has been used in the literature; see, e.g., [8] and [24], where $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is an extended class $\mathcal{K}_\infty$ function; namely, $\alpha(0) = 0$ and α is strictly increasing. Note that condition (8) is more general than (12), in the sense that the inequality in (8) does not need to hold on the entire set C_x ; however, in (12), the safety constraint is imposed globally, though it may get relaxed in the interior of the safe set. Another advantage of (8) is that the barrier function needs only to be locally Lipschitz in a neighborhood of the boundary of K . Furthermore, using the presented framework, a combination of multiple intersecting and non-intersecting barriers can be addressed independently.

V. GUARANTEEING CONTINUITY OF THE OPTIMAL SOLUTION

Given a set-valued map $D : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ indicating all safe and feasible control actions for each x , and the desired cost function $\mathcal{L}$, an optimal control law is given by solving the optimization problem in (2). In the following, we give two sets of conditions concerning the cost function $\mathcal{L}$ and the set-valued map D , such that the optimal control law κ^* is continuous.

Berge's Maximum Theorem [25, Maximum Theorem] provides conditions such that the optimal solution map κ^* in (2) is upper semicontinuous and has compact values. In the following lemma, we specify conditions such that the optimal solution to (2) is single valued and continuous.

Lemma 2: Suppose the function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous and strictly convex in its second argument. Let the set-valued map $D : C_x \rightrightarrows C_u$ be continuous and have nonempty and compact convex values. Then, the function $\kappa^* : C_x \rightarrow C_u$, defined in (2), for each $x \in C_x$ is single valued and continuous.

Proof: Since $\mathcal{L}$ is continuous and the set-valued map D is continuous, and has nonempty compact values, [25, Maximum Theorem] implies that the set-valued map κ^* in (2) is nonempty and upper semicontinuous, with compact values. Since $\mathcal{L}$ is strictly convex in u , [13, Th. 2.6] implies that for each x , $\kappa^*(x)$ in (2) has at most one value. Therefore, κ^* is nonempty and single valued. Using [23, Lemma 5.15], we conclude that upper semicontinuous maps with compact images are outer semicontinuous with locally bounded values. From [13, Corollary 5.20], we conclude that κ^* is continuous. ■

The following lemma is more general than Lemma 2 in the sense that in Lemma 3, $\mathcal{L}$ can be sublevel bounded in u when D does not have bounded values.

Lemma 3: Suppose the function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is proper, lower semicontinuous, convex, and strictly convex in its second argument. Let the set-valued map $D : C_x \rightrightarrows C_u$ be continuous, nonempty and have convex values. Let the function $\kappa^* : C_x \rightarrow C_u$ be defined in (2) for each $x \in C_x$. If one of the following conditions holds,

- 1) $\mathcal{L}$ is sublevel bounded in u (see Definition 1);
- 2) D has bounded values, namely, for each $x \in C_x$, $D(x)$ is bounded.

then κ^* is single valued on C_x and continuous on $\text{int}(C_x)$.

Proof: To prove the lemma, we use [13, Corollary 7.43]. Let the function $\tilde{\delta} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$ be defined as

$$\tilde{\delta}(x, u) := \delta_{D(x)}(u) = \begin{cases} 0 & \text{if } x \in C_x, u \in D(x) \\ \infty & \text{otherwise} \end{cases}$$

for each $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$. Note that δ_S is an indicator function of the set S . Let $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$ be given by $f(x, u) := \mathcal{L}(x, u) + \tilde{\delta}(x, u)$. Then, since D is nonempty, f is proper. The sublevel sets of f for $\alpha \in \mathbb{R}$ are defined as

$$\begin{aligned} \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : f(x, u) \leq \alpha\} \\ = \text{Graph}(D) \cap \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : \mathcal{L}(x, u) \leq \alpha\}. \end{aligned} \quad (13)$$

Since $\mathcal{L}$ is lower semicontinuous, from [13, Th. 1.6] its sublevel sets are closed. Since $\text{Graph}(D)$ is closed, then the sublevel sets of f in (13) are closed. Thus, from [13, Th. 1.6] f is lower semicontinuous. Since D has convex values and $\mathcal{L}$ is convex, then f is convex. Furthermore, since $\mathcal{L}$ is strictly convex in u , f is strictly convex in u . If either D has bounded values or the sublevel sets of $\mathcal{L}$ in u are bounded, then the level sets of f in u are bounded. Namely, let $\mathcal{B} \subset C_x$ be a bounded set and let $L = \{(x, u) : x \in \mathcal{B}, u \in D(x)\}$, for $\alpha \in \mathbb{R}$. The sublevel sets of f in u are defined by $\{(x, u) \in \mathcal{B} \times \mathbb{R}^m : f(x, u) \leq \alpha\} = L \cap \{(x, u) \in \mathcal{B} \times \mathbb{R}^m : \mathcal{L}(x, u) \leq \alpha\}$. Using [13, Corollary 3.27] and the fact that f is sublevel bounded in u , we conclude that f is level coercive in u . Then, using [13, Th. 3.26], we conclude that $f^\infty(0, u) > 0$ (see the horizon function in [13, Definition 3.17]) for all $u \neq 0$. Then, using [13, Corollary 7.43], we conclude that κ^* is single valued on $\text{dom } \kappa^*$ and it is continuous on the interior of its domain. Since f is proper, lower semicontinuous, and sublevel bounded in u , using [13, Th. 1.9] we conclude that $\text{dom } \kappa^* = C_x$. ■

VI. GUARANTEEING CONTINUITY OF THE OPTIMAL SAFE CONTROL LAW

Building from the results in Sections IV and V, we formulate conditions for synthesizing an optimal, safe, and continuous control law. The constraint map D is induced using CBF (safety constraint) as well as control and state constraints.

A. Continuous, Safe, and Optimal Control Law

As indicated in Theorem 1, to ensure safety, the selection of the control law in the outer neighborhood of the zero sublevel set of the CBF should be restricted appropriately. Let the set-valued map $D : C_x \rightrightarrows C_u$ be defined as

$$D(x) := \begin{cases} \bar{D}_0(x) & \text{if } x \in S_1 \\ \mathcal{U}(x) & \text{otherwise,} \end{cases} \quad (14)$$

where $\bar{D}_0 \subset \mathcal{U}$ is given in the proof of Theorem 1 and indicates all safe feedback control laws on the set S_1 , and $S_1 := \text{cl}(U(K) \setminus K \cap C_x)$ indicates the corresponding outer neighborhood of the set K . Then, the set-valued map D contains all feasible and safe feedback control laws. To select a continuous control law from D by minimizing a cost function, based on Lemma 3, D should be continuous.

As explained in Remark 2, the map $\bar{D}_0$ generally is not continuous. Here, we construct a continuous set-valued map $\tilde{U}$ from D . To do this, we should design $\tilde{U}$ such that it is a subset of $\bar{D}_0$ when $x \in S_1$ and also it blends smoothly with the set-valued map $\mathcal{U}$ on the boundary of S .

Assumption 5: The set-valued map $\mathcal{U}$ is outer semicontinuous.

In the following result, we formulate conditions guaranteeing the selection of a continuous control law by minimizing a cost function and simultaneously ensuring safety.

Theorem 2: Consider $\Sigma_{\mathcal{U}}$ in (4) such that Assumptions 1-5 hold. Let $B : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz CBF with respect to $(X_o, X_u) \subset \mathbb{R}^n \times \mathbb{R}^m$ defining the set K in (7). Suppose

- 1) The function $g : C_x \times C_u \rightarrow \mathbb{R}$ defined in (9) is convex in u ,
- 2) There exists a neighborhood of ∂K such that the set-valued map $D_0 : C_x \rightrightarrows C_u$ defined in (11) for γ identically zero is nonempty.

Let $S_1 = \text{cl}(U(K) \setminus K \cap C_x)$ be a neighborhood of ∂K such that D_0 is nonempty and (8) holds. Suppose $\tilde{U} : C_x \rightrightarrows C_u$ satisfies the following properties:

- 3) $\tilde{U}$ is continuous with nonempty, closed, and convex values,
- 4) For each $x \in S_1$, $\tilde{U}(x) \subset \bar{D}_0(x)$,
- 5) For each $x \in C_x$, $\tilde{U}(x) \subset \mathcal{U}(x)$.

Let $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ be proper, lower semicontinuous, convex in both arguments, and strictly convex in its second argument. Let $\kappa^* : C_x \rightarrow C_u$ be defined by

$$\begin{aligned} \kappa^*(x) &= \argmin \quad \mathcal{L}(x, u) \\ \text{s.t.} \quad &u \in \tilde{U}(x). \end{aligned} \quad (15)$$

If one of the following conditions holds

- 1) $\mathcal{L}$ is sublevel bounded in u ;
- 2) The set-valued map $\tilde{U}$ has bounded values,

then κ^* is continuous on $\text{int}(C_x)$, and the resulting differential inclusion Σ_{κ^*} is safe with respect to (X_o, X_u) , namely, κ^* is the optimal safe control law.

Proof: Lemma 3 implies that κ^* is single valued on C_x and continuous on $\text{int}(C_x)$. Since κ^* is a continuous selection from a subset of $\bar{D}_0$ on $U(\partial K) \setminus K$, then Σ_{κ^*} satisfies $\langle \zeta, \eta \rangle \leq 0$ for each $x \in (U(\partial K) \setminus K) \cap C_x$, $\zeta \in \partial_C B(x)$, and for each $\eta \in F(x, \kappa^*(x))$. Therefore, [5, Th. 4] implies that K is forward pre-invariant for Σ_{κ^*} . Then, Σ_{κ^*} is safe w.r.t. (X_o, X_u) . ■

B. A Sample Construction of $\tilde{U}$

Because D consists of two set-valued maps $\bar{D}_0$ and $\mathcal{U}$, to construct the continuous set-valued map $\tilde{U}$ from D , we need to apply two types of blending. In general, $\bar{D}_0$ is not continuous, as a result of discontinuity in g with respect to x . In obstacle avoidance, for example, when the barrier function is defined as the minimum or maximum of some hyperplanes, there are discontinuities in g as different constraints are active in the

different regions around the obstacle. First, we should find some continuous map $D_0^s : S_1 \rightarrow C_u$ such that $D_0^s(x) \subset \bar{D}_0(x)$ for each $x \in S$. Second, we must blend continuously two continuous maps, D_0^s and $\mathcal{U}$.

In Example 1, we present an approach to smoothen $\bar{D}_0$ when the unsafe set is defined as a system of linear inequalities, and the barrier function is defined as the minimum of the hyperplanes corresponding to the unsafe set.

Here, we present an approach for continuously blending two continuous maps using the Minkowski sum. Let $S \subset \mathbb{R}^n$ and $\epsilon : S \rightarrow \mathbb{R}_{>0}$ be continuous. The ϵ -neighborhood of S , $U_\epsilon(S)$, is defined as $U_\epsilon(S) := \bigcup_{x \in S} (x + \epsilon(x)\mathbb{B})$.

Lemma 4: Consider a closed set $K \subset \mathbb{R}^n$, and set-valued maps $F_1 : K \rightrightarrows \mathbb{R}^m$ and $F_2 : U_\epsilon(\partial K) \rightrightarrows \mathbb{R}^m$ that are continuous with closed and convex values, where $\epsilon : \partial K \rightarrow \mathbb{R}_{>0}$ is a continuous function such that $\epsilon(x) > \epsilon_1$ for each $x \in \partial K$, where $\epsilon_1 \in \mathbb{R}_{>0}$. Then, the set-valued map $G : K \cup U_\epsilon(\partial K) \rightrightarrows \mathbb{R}^m$ defined as

$$G(x) := \begin{cases} F_1(x) & \text{if } x \in K \setminus U_{\epsilon_1}(\partial K) \\ F_3(x) & \text{if } x \in K \cap U_{\epsilon_1}(\partial K) \\ F_2(x) & \text{if } x \in U_\epsilon(\partial K) \setminus K \end{cases} \quad (16)$$

where $F_3(x) := \frac{d(x, \partial K)}{\epsilon_1} F_1(x) + (1 - \frac{d(x, \partial K)}{\epsilon_1}) F_2(x)$, is lower and outer semicontinuous with closed and convex values.

Proof: Since ∂K is a closed set, then the distance function $x \mapsto d(x, \partial K)$ is continuous [13]. Therefore, F_3 is continuous. Since F_1 and F_2 have closed and convex values, using the Minkowski sum of two closed and convex sets is closed and convex [26], we conclude that F_3 has closed and convex values. For each $x \in \partial K$, $F_3(x) = F_2(x)$ and for each $x \in \partial(K \cap U_{\epsilon_1}(\partial K))$, $F_3(x) = F_1(x)$, therefore, G is continuous with closed and convex values. ■

Example 1: Consider the system $\Sigma_{\mathcal{U}} : \dot{x} = Ax + Bu$ with $x \in \mathbb{R}^2$. Suppose we have a rectangular shape obstacle. Let p_0 be the central point of the obstacle and, for each $i \in \{1, \dots, 4\}$, q_i denotes the middle point of each edge of the obstacle. The unsafe set is defined as the intersection of the half-spaces $(p_0 - q_i)^\top (x - q_i) > 0$ for $i = 1, \dots, 4$. We have $X_u = \{x \in \mathbb{R}^2 : A_u x > b_u\}$, where $A_u = (p_0 - q_1, p_0 - q_2, p_0 - q_3, p_0 - q_4)^\top$, and $b_u = ((p_0 - q_1)^\top q_1, (p_0 - q_2)^\top q_2, (p_0 - q_3)^\top q_3, (p_0 - q_4)^\top q_4)^\top$. Let $\mathcal{U} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined as $\mathcal{U}(x) = [-a, a]^2$, with $a > 0$.

We set the initial set, X_o , to be the complement of the unsafe set with an extra buffer distance, as $X_o = \mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 : A_u x > b_u - d\mathbb{1}\}$, where $\mathbb{1}$ denotes a vector of ones, and d denotes the extra distance. Next, we define $K = X_o$ and the CBF as $B(x) := \min_{i \in \{1, \dots, 4\}} B_i(x)$ for each $x \in \mathbb{R}^2$, where the B_i is given by $B_i(x) := (p_0 - q_i)^\top (x - q_i) + d$ for $i \in \{1, \dots, 4\}$. Based on [19, Proposition 2.3.12], the Clarke generalized gradient of B is given by $\partial_C B(x) = \{\sum_{i \in \mathcal{I}(x)} \theta_i \nabla B_i(x) : \theta_i \geq 0, \sum_{i \in \mathcal{I}(x)} \theta_i = 1\}$, where $\mathcal{I} : \mathbb{R}^2 \rightrightarrows \{1, \dots, 4\}$ indicates the active B_i s and is defined as $\mathcal{I}(x) := \{i : B(x) = B_i(x), i \in \{1, \dots, 4\}\}$. The function g from Theorem 2 is $g(x, u) = \sup\{\langle \sum_{i \in \mathcal{I}(x)} \theta_i (p_0 - q_i), Ax + Bu \rangle : \theta_i \geq 0, i \in \mathcal{I}(x), \sum_{i \in \mathcal{I}(x)} \theta_i = 1\}$. This optimization problem is over θ and of the form has a linear function over the probability simplex. Therefore, $g(x, u) = \max\{\langle \nabla B_i(x), Ax + u \rangle, i \in \mathcal{I}(x)\}$. Since g is the pointwise maximum of affine functions in u , it is convex in u . Then, D_0 in Theorem 2 is given by $D_0(x) = \{u \in$

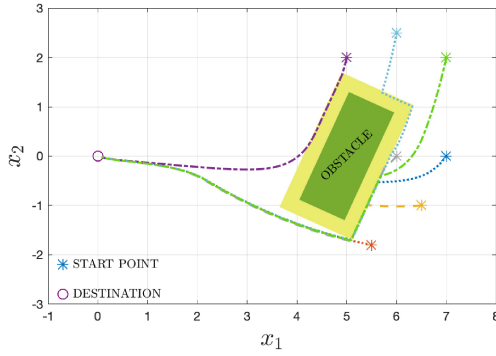


Fig. 1. Trajectories of Example 1. The obstacle is the green rectangle and the initial set is the area outside of the yellow and green regions.

$[-a, a]^2 : \langle p_0 - q_i, Ax + Bu \rangle < 0, \forall i \in \mathcal{I}(x)$. We observe that the map D_0 is nonempty. First, we want to smoothen the discontinuities of $\tilde{D}_0$, which is induced by changing the active constraints. We define $\mathcal{I}_\alpha$ for some $\alpha > 0$ as $\mathcal{I}_\alpha(x) := \{i : |B(x) - B_i(x)| \leq \alpha, i \in \{1, \dots, 4\}\}$. When $\mathcal{I}(x) = \mathcal{I}_\alpha(x)$, the conditions to satisfy are $\nabla B_i(x)^\top (Ax + Bu) \leq 0$, for each $i \in \mathcal{I}(x)$. Let $\phi(x) := \frac{\pi |B(x) - B_j(x)|}{2\alpha}$ for $j \in \mathcal{I}_\alpha(x) \setminus \mathcal{I}(x)$. Then, for each $i \in \mathcal{I}(x), j \in \mathcal{I}_\alpha(x) \setminus \mathcal{I}(x)$ we have $(\sin(\phi(x))\nabla B_i(x) + \cos(\phi(x))\nabla B_j(x))^\top (Ax + Bu) \leq 0$. At a point x , where $\mathcal{I}(x)$ is not equal to $\mathcal{I}_\alpha(x)$, it means that x is near the area of changing the active constraints. Using sin and cos functions, we smoothen these transitions. We define $\tilde{\mathcal{U}}(x) = \{u \in [-a, a]^2 : H(x)^\top u \leq h(x)\}$, where H is defined based on the conditions we explained above as $H(x) = \begin{pmatrix} (p_0 - q_i)^\top B \\ (\sin(\phi)(p_0 - q_i) + \cos(\phi)(p_0 - q_j))^\top B \end{pmatrix}$ for each $i \in \mathcal{I}(x)$, and $j \in \mathcal{I}_\alpha(x) \setminus \mathcal{I}(x)$. The function h is defined as $h(x) = \begin{pmatrix} -(p_0 - q_i)^\top Ax \\ -(\sin(\phi)(p_0 - q_i) + \cos(\phi)(p_0 - q_j))^\top Ax \end{pmatrix}$, and we add $\max\{0, -MB(x)\}$ to every row of h for some sufficiently large positive number M to make the constraint $H(x)^\top u \leq h(x)$ ineffective when x is in the safe region. We use this method instead of the one in Lemma 4 since that can result in a set-valued map with nonconvex values. Considering that all the constraints defining $\tilde{\mathcal{U}}$ are affine with respect to u at each x , and the constraints with strict inequality are nonempty, [13, Example 5.10] implies that $\tilde{\mathcal{U}}$ is continuous. Furthermore, $\tilde{\mathcal{U}}$ has nonempty and compact values. Theorem 2 implies that if the cost function $\mathcal{L}$ is lower semicontinuous, convex and strictly convex in u , then the optimal control law κ^* is continuous and safe. Figure 1 shows trajectories using $\mathcal{L}(x, u) = \frac{1}{2}u^\top u$, $V(x) := \frac{1}{2}x^2$, $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, $B = I$, $\alpha = 0.01$, $M = 100$, and $a = 5$. Simulation files available at <https://github.com/HybridSystemsLab/NonSmoothCBF>.

VII. CONCLUSION

This letter studies controlled safety of constrained differential inclusions using nonsmooth CBFs. We develop sufficient conditions to select a continuous control law using CBFs. Furthermore, we study conditions to find optimal safe control laws while minimizing the cost function. We illustrate the results in an obstacle avoidance example.

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