Network Connectivity Preserving Formation Stabilization and Obstacle Avoidance via a Decentralized Controller

Zhen Kan, Ashwin P. Dani, John M. Shea, and Warren E. Dixon

Abstract—**A decentralized control method is developed to enable a group of agents to achieve a desired global configuration while maintaining global network connectivity and avoiding obstacles, using only local feedback and no radio communication between the agents for navigation. By modeling the interaction among the agents as a graph, and given a connected initial graph with a desired neighborhood between agents, the developed method ensures the desired communication links remain connected for all time. To guide the agents to a desired configuration while avoiding obstacles, a decentralized controller is developed based on the navigation function formalism. By proving that the proposed controller is a qualified navigation function, convergence to the desired formation is guaranteed.**

Index Terms— **Collision avoidance, decentralized control, navigation function, network connectivity.**

I. Introduction

A wide range of applications require or can benefit from collaborative motion of a group of agents. Some applications can adopt a centralized control approach where one algorithm determines and communicates the next required movement for each agent. For some applications, the centralized approach is not practical due to the potential for compromised communication with or demise/corruption of the central controller. Decentralized control is an alternative approach in which each agent makes an independent decision based on either global information communicated through the network or local information from one-hop neighbors. Methods that use global information require each agent to determine the relative trajectory of all other agents at all time by propagating information through the network, resulting in delays in the trajectory information and consumption of network bandwidth, effects that limit the network size. Methods that use local information need only relative trajectories of neighboring agents; however, difficulties arising from achieving a desired formation for the global network using local feedback can cause the network to partition. When the network partitions, communication between groups of agents can be permanently severed leading to mission failure.

In this paper, a decentralized controller is developed that guarantees that a multi-agent system can achieve an arbitrary desired configuration from a connected initial graph (agents are considered as nodes on a graph) with desired neighborhood, while avoiding collisions with other agents and external obstacles and maintaining global network connectivity. Each agent is equipped with a range sensor (e.g., camera) to provide local feedback of the relative trajectory of other agents within a limited sensing region, and a transceiver to broadcast information to immediate neighbors. The developed controller can enable global network coordination with radio silence.

As nodes move to achieve a desired configuration they must avoid obstacles and remain connected. Navigation functions (a particular class of potential functions) were originally developed in the seminal work in [12], [16] to enable a single point-mass agent to move in an environment with spherical obstacles. The navigation function developed in [12] is a real-valued function that is designed so that the negated gradient field does not have a local minima and converges to a desired destination. The navigation function framework is extended to multi-agent systems for obstacle avoidance in results such as [3], [5], [8], [13]; however, agents within these results acted independently and were not required to achieve a network objective. In contrast, results in [1], [17], [18] use potential fields/navigation functions to achieve obstacle avoidance while the agents are also required to achieve a cooperative network objective (e.g., formation control or consensus); however, these results assume the agents can always communicate (i.e., the graph nodes are assumed to remain connected).

Results such as [2], [6], [7], [9], [10], [15], [20]–[22] are motivated by the need to prevent the graph from partitioning. In [20] and [15], a potential field based centralized control approach is developed to ensure the connectivity of a group of agents using the graph Laplacian matrix. However, global information of the underlying graph is required to compute the graph Laplacian. In [21], connectivity control is performed in the discrete space of graphs to verify link deletions with respect to connectivity, and motion control is performed in the continuous configuration space using a potential field. In [22], a potential field-based neighbor control law is designed to achieve velocity alignment and network connectivity among different topologies. In [7] and [2], a repulsive potential is used for a collision avoidance objective, and an attractive potential field is used to drive agents together. Distributed control laws are investigated to ensure edge maintenance in [10] by allowing unbounded potential force whenever pairs of agents are about to break the existing links. In [6], a potential field is designed for a group of mobile agents to perform desired tasks while maintaining network connectivity. It is unclear how the potential field method in [6] can be extended to include static obstacles, since the resulting closed-loop dynamics can not be expressed as a Metzler matrix with zero sums as required in the analysis in [6][1]. Moreover, the work in [6] only proves that all states converge to a common value that can be influenced by the initial states [14], unlike the proposed method. In comparison to the above results, the method developed in this paper achieves convergence to a desired configuration and maintenance of network connectivity using a decentralized navigation function approach which uses only local feedback information. By using a local range sensor (and not requiring knowledge of the complete network structure as in methods that use a graph Laplacian), an advantageous feature of the developed decentralized controller is that no inter-agent communication is required (i.e., communication free global decentralized group behavior). That is, the goal is to maintain connectivity so that radio communication is available when required for various task/mission scenarios, but communication is not required to navigate, enabling stealth modes of operation. Collision avoidance and network connectivity are embedded as constraints in the navigation function. Compared with [6], collision avoidance with both agents and static obstacles are considered; however, the current development is based on the assumption that the probability of more than one simultaneous collision is negligible

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[1]It is not clear how the method in [6] can be applied for static obstacles, since the terms $G_i\nabla_i\gamma_i - (\gamma_i/k)\nabla_i G_i$ can not be expressed in a form such as $\sum_{j\in N_i}\pi_{ij}(q_i - q_j)$ with q_i and q_j on the right side only.

and no obstacles or agents stay within the collision region of a node when the node is close to breaking an existing link. By proving that the distributed control scheme is a valid navigation function, the multi-agent system is guaranteed to converge to and stabilize at the desired configuration.

II. Problem Formulation

Consider a network composed of N agents in the workspace $\mathcal{F}$, where agent i moves according to the following kinematics:

$$\dot{q}_i = u_i, \quad i = 1, \cdots, N \tag{1}$$

where $q_i \in \mathbb{R}^2$ denotes the position of agent i in a two dimensional (2-D) plane, and $u_i \in \mathbb{R}^2$ denotes the velocity of agent i (i.e., the control input). The workspace $\mathcal{F}$ is assumed to be circular and bounded with radius R, and $\partial\mathcal{F}$ denotes the boundary of $\mathcal{F}$. Each agent in $\mathcal{F}$ is represented by a point-mass with a limited communication and sensing capability encoded by a disk area. It is assumed that each agent is equipped with a range sensor and wireless communication capabilities. Two moving agents can communicate with each other if they are within a distance R_c, while the agent can sense stationary obstacles or other agents within a distance R_s. For simplicity and without loss of generality, the following development is based on the assumption that the sensing area coincides with the communication area, i.e., $R_c = R_s$. Further, all agents are assumed to have equal actuation capabilities. A set of fixed points, $p_1, \cdots, p_M$, are defined to represent M stationary obstacles in the workspace $\mathcal{F}$, and the index set of obstacles is denoted as $\mathcal{M} = \{1, \cdots, M\}$. The assumption of point-obstacles is not restrictive, since a large class of shapes can be mapped to single points through a series of transformations [19], and this "point-world" topology is a degenerate case of the "sphere-world" topology [16].

The interaction of the system is modeled as a graph denoted as $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, where $\mathcal{V} = \{1, \cdots, N\}$ denotes the set of nodes, and $\mathcal{E}(t) = \{(i,j) \in \mathcal{V} \times \mathcal{V} | d_{ij} \leq R_c\}$ denotes the set of time varying edges, where node i and j are located at position q_i and q_j, and $d_{ij} \in \mathbb{R}^+$ is defined as $d_{ij} = \|q_i - q_j\|$. In $\mathcal{G}(t)$, each node i represents an agent, and the edge (i,j) denotes a link between agent i and j when they stay within a distance R_c. Nodes i and j are also called *one-hop neighbors* of each other. The set of one-hop neighbors of node i (i.e., all the agents within the sensing zone of agent i) is given by $\mathcal{N}_i = \{j, j \neq i | j \in \mathcal{V}, (i,j) \in \mathcal{E}\}$. One objective in this work is to have the multi-agent system converge to a desired configuration, determined by a formation matrix $c_{ij} \in \mathbb{R}^2$ representing the desired relative position and orientation of node i with an adjacent node $j \in \mathcal{N}_i^f$, where $\mathcal{N}_i^f \subset \mathcal{N}_i$ denotes the set of nodes required to form a prespecified relative pose with node i in the desired configuration. The neighborhood $\mathcal{N}_i$ is a time varying set since nodes may enter or leave the communication region of node i at any time instant, while $\mathcal{N}_i^f$ is a static set which is specified by the desired configuration. The desired position of node i, denoted by q_{di}, is defined as $q_{di} = \{q_i | \|q_i - q_j - c_{ij}\|^2 = 0, j \in \mathcal{N}_i^f\}$. An edge (i,j) is only established between nodes i and j if $j \in \mathcal{N}_i^f$.

A *collision region*[2] is defined for each agent i as a small disk with radius $\delta_1 < R_c$ around the agent i, such that any other agent $j \in \mathcal{N}_i$, or obstacle p_k, $k \in \mathcal{M}$, inside this region is considered as a potential collision with agent i. To ensure connectivity, an *escape region* for each agent i is defined as the outer ring of the communication area with radius r, $R_c - \delta_2 < r < R_c$, where $\delta_2 \in \mathbb{R}$ is a predetermined buffer distance. Edges formed with any node $j \in \mathcal{N}_i^f$ in the escape region are in danger of breaking.

[2]The potential collision for node i in this work not only refers to the fixed obstacles, but also other moving nodes or the workspace boundary, which are currently located in its collision region.

The objective is to develop a decentralized controller u_i that uses relative position information from the range sensor to regulate a connected initial graph to a desired configuration while maintaining network connectivity and avoiding collisions with other agents and obstacles in radio silence. To achieve this goal, the subsequent development is based on the following assumptions.

Assumption 1: The initial graph $\mathcal{G}$ is connected within a desired neighborhood, (i.e., the desired neighbors of an agent are initially within the agent's sensing zone), and those initial positions do not coincide with some unstable equilibria (i.e., saddle points).

Assumption 2: The desired formation matrix c_{ij} is specified initially and is achievable, which implies that the desired configuration will not lead to a collision or a partitioned graph, (i.e., $\delta_1 < \|c_{ij}\| < R_c - \delta_2$).

III. Control Design

Consider a decentralized navigation function candidate $\varphi_i : \mathcal{F}_i \rightarrow [0,1]$ for each node i as

$$\varphi_i = \frac{\gamma_i}{(\gamma_i^\alpha + \beta_i)^{1/\alpha}} \tag{2}$$

where $\alpha \in \mathbb{R}^+$ is a tuning parameter, $\gamma_i : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ is the goal function, and $\beta_i : \mathbb{R}^2 \rightarrow [0,1]$ is a constraint function for node i. The goal function γ_i in (2) encodes the control objective of node i, specified in terms of the desired relative distance and orientation with respect to the adjacent nodes $\{j \in \mathcal{N}_i^f\}$, and drives the system to a desired configuration[3]. The goal function is designed as

$$\gamma_i(q_i, q_j) = \sum_{j \in \mathcal{N}_i^f} \|q_i - q_j - c_{ij}\|^2. \tag{3}$$

The constraint function β_i in (2) is designed as

$$\beta_i = B_{i0} \prod_{j \in \mathcal{N}_i^f} b_{ij} \prod_{k \in \mathcal{N}_i \cup \mathcal{M}_i} B_{ik} \tag{4}$$

to ensure collision avoidance and network connectivity by only accounting for nodes and obstacles located within its sensing area during each time instant. Specifically, the constraint function in (4) is designed to vanish whenever node i intersects with one of the constraints in the environment, (i.e., if node i touches a fixed obstacle, the workspace boundary, other nodes, or departs away from its adjacent nodes $\{j \in \mathcal{N}_i^f\}$ to a distance of R_c). In (4), $b_{ij} \triangleq b(q_i, q_j) : \mathbb{R}^2 \rightarrow [0,1]$ ensures connectivity of the network graph (i.e., guarantees that nodes $\{j \in \mathcal{N}_i^f\}$ will never leave the communication zone of node i if node j is initially connected to node i) and is designed as shown in (5), at the bottom of the page. Also, in $B_{ik} \triangleq B(q_i, q_k) : \mathbb{R}^2 \rightarrow [0,1]$, for

[3]The formation objective γ_i is developed based on the desire to control the distance and relative bearings between nodes. For some applications, only the relative distance between nodes is important, and the objective could be rewritten as $\gamma_i = \sum_{j \in \mathcal{N}_i^f} (\|q_i - q_j\| - \|c_{ij}\|)^2$; however, this objective can introduce redundant desired configurations. Future efforts could consider this alternative objective, where an approach such as [9] may be explored to address the multiple desired minima.

$$b_{ij} = \begin{cases} 1 & d_{ij} \leq R_c - \delta_2 \\ -\frac{1}{\delta_2^2}(d_{ij} + 2\delta_2 - R_c)^2 + \frac{2}{\delta_2}(d_{ij} + 2\delta_2 - R_c) & R_c - \delta_2 < d_{ij} < R_c \\ 0 & d_{ij} \geq R_c \end{cases} \tag{5}$$

point $k \in \mathcal{N}_i \cup \mathcal{M}_i$, where $\mathcal{M}_i$ indicates the set of obstacles within the sensing area of node i at each time instant, ensures that node i is repulsed from other nodes or obstacles to prevent a collision, and is designed as

$$B_{ik} = \begin{cases} -\frac{1}{\delta_1^2} d_{ik}^2 + \frac{2}{\delta_1} d_{ik} & d_{ik} < \delta_1 \\ 1 & d_{ik} \geq \delta_1. \end{cases} \quad (6)$$

Similarly, the function B_{i0} in (4) is used to model the potential collision of node i with the workspace boundary, where the positive scalar $B_{i0} \in \mathbb{R}$ is designed similar to B_{ik} by replacing d_{ik} with d_{i0}, where $d_{i0} \in \mathbb{R}^+$ is the relative distance of the node i to the workspace boundary defined as $d_{i0} = R - \|q_i\|$.

Assumption 2 guarantees that γ_i and β_i will not be zero simultaneously. The navigation function candidate achieves its minimum of 0 when $\gamma_i = 0$ and achieves its maximum of 1 when $\beta_i = 0$. For φ_i to be a navigation function, it has to satisfy the following conditions [16]: 1) smooth on $\mathcal{F}$ (at least a $\mathcal{C}^1$ function [3]); 2) admissible on $\mathcal{F}$, (uniformly maximal on $\partial\mathcal{F}$ and constraint boundary); 3) polar on $\mathcal{F}$, (q_{di} is a unique minimum); 4) a Morse function, (critical points[4] of the navigation function are non-degenerate). If φ_i is a Morse function and q_{di} is a unique minimum of φ_i (i.e., q_{di} is polar on $\mathcal{F}$), then almost all initial positions (except for a set of points of measure zero) asymptotically approach the desired position q_{di} [16]. In addition, the negative gradient of the navigation function is bounded if it is an admissible Morse function with a single minimum at the desired destination [16].

Based on the definition of the navigation function candidate, the decentralized controller for each node is designed as

$$u_i = -K \nabla_{q_i} \varphi_i \quad (7)$$

where K is a positive gain, and $\nabla_{q_i} \varphi_i$ is the gradient of φ_i with respect to q_i. Hence, although the controller can be arbitrarily large due to β_i being arbitrarily small, the controller in (7) is still bounded and yields the desired performance by steering node i along the direction of the negative gradient of φ_i if (2) is a navigation function.

IV. Connectivity and Convergence Analysis

The free configuration workspace $\mathcal{F}_i \subset \mathcal{F}$ is a compact connected analytic manifold for node i, $\mathcal{F}_i \triangleq \{\mathbf{q} | \beta_i(\mathbf{q}) > 0\}$, and $\mathbf{q}$ denotes the stacked position vector of node i. The boundary of $\mathcal{F}_i$ is defined as $\partial\mathcal{F}_i \triangleq \beta_i^{-1}(0)$. The narrow set around a potential collision for node i is defined as $\mathcal{B}_{i,k}^B(\varepsilon) \triangleq \{\mathbf{q} | 0 < B_{ik} < \varepsilon, \varepsilon > 0, k \in \mathcal{N}_i \cup \mathcal{M}_i\}$, and a narrow set around a potential connectivity constraint is defined as $\mathcal{B}_{i,j}^b(\varepsilon) \triangleq \{\mathbf{q} | 0 < b_{ij} < \varepsilon, \varepsilon > 0, j \in \mathcal{N}_i^f\}$. The set $\mathcal{B}_0(\varepsilon) = \{\mathbf{q} | 0 < B_{i0} < \varepsilon, \varepsilon > 0\}$ is used to denote a narrow set around a potential collision of node i with workspace boundary. Inspired by the seminal work in [16], $\mathcal{F}_i$ is partitioned into five subsets $\mathcal{F}_0(\varepsilon), \mathcal{F}_1(\varepsilon), \mathcal{F}_2(\varepsilon), \mathcal{F}_3(\varepsilon)$, and $\mathcal{F}_{di}(\varepsilon)$ as $\mathcal{F}_i = \mathcal{F}_{di} \cup \mathcal{F}_0(\varepsilon) \cup \mathcal{F}_1(\varepsilon) \cup \mathcal{F}_2(\varepsilon) \cup \mathcal{F}_3(\varepsilon)$, where the set of desired configurations for node i is defined as $\mathcal{F}_{di} \triangleq \{\mathbf{q} | \gamma_i(\mathbf{q}) = 0\}$. The sets $\mathcal{F}_0(\varepsilon), \mathcal{F}_1(\varepsilon), \mathcal{F}_2(\varepsilon)$ and $\mathcal{F}_3(\varepsilon)$ describe the regions near the workspace boundary, near the potential collision constraint, near the connectivity constraint and away from all constraints for node i, respectively, and are defined as $\mathcal{F}_0(\varepsilon) \triangleq \mathcal{B}_0(\varepsilon) - \mathcal{F}_{di}$, $\mathcal{F}_1(\varepsilon) \triangleq \bigcup_{k=1}^{\xi_i + \vartheta_i} \mathcal{B}_{i,k}^B(\varepsilon) - \mathcal{F}_{di}$, $\mathcal{F}_2(\varepsilon) \triangleq \bigcup_{j=1}^{\zeta_i} \mathcal{B}_{i,j}^b(\varepsilon) - \mathcal{F}_{di}$, and $\mathcal{F}_3(\varepsilon) \triangleq \mathcal{F}_i - \{\mathcal{F}_{di} \cup \mathcal{F}_0(\varepsilon) \cup \mathcal{F}_1(\varepsilon) \cup \mathcal{F}_2(\varepsilon)\}$, where $\xi_i, \vartheta_i, \zeta_i \in \mathbb{R}^+$ denote the number of nodes in the set $\mathcal{N}_i$, $\mathcal{M}_i$ and $\mathcal{N}_i^f$, respectively. The following Assumptions are used to prove *Proposition 6–8*.

Assumption 3: No obstacles or other agents are assumed to stay within the collision region of node i, when node i is close to breaking the communication link with a node $j \in \mathcal{N}_i^f$ (i.e., node i and node j belong to the region $\mathcal{B}_{i,j}^b(\varepsilon)$).

Assumption 4: The region $\mathcal{B}_{i,k}^B(\varepsilon)$ for $k \in \mathcal{N}_i \cup \mathcal{M}_i$ is disjoint. This assumption implies a negligible probability of more than one simultaneous collision with node i.

A. Connectivity Analysis

Proposition 1: If the graph $\mathcal{G}$ is connected initially and $j \in \mathcal{N}_i^f$, then (7) ensures nodes i and j will remain connected for all time.

Proof: Consider node i located at a point $q_0 \in \mathcal{F}$ that causes $\prod_{j \in \mathcal{N}_i^f} b_{ij} = 0$. Then two possibilities are considered in the following two cases.

Case 1: There is only one node $j \in \mathcal{N}_i^f$ for which $b_{ij}(q_0, q_j) = 0$ and $b_{il}(q_0, q_l) \neq 0 \ \forall l \in \mathcal{N}_i^f, l \neq j$. The gradient of φ_i with respect to q_i is

$$\nabla_{q_i} \varphi_i = \frac{\alpha \beta_i \nabla_{q_i} \gamma_i - \gamma_i \nabla_{q_i} \beta_i}{\alpha \left(\gamma_i^\alpha + \beta_i\right)^{\frac{1}{\alpha}+1}}. \quad (8)$$

Since $b_{ij} = 0$, the constraint function $\beta_i = 0$ from (4). Thus, the gradient $\nabla_{q_i} \varphi_i$ evaluated at q_0 can be expressed as $\nabla_{q_i} \varphi_i|_{q_0} = -((\gamma_i \nabla_{q_i} \beta_i)/(\alpha \gamma_i^{\alpha+1}))|_{q_0}$. Based on the fact that β_i can be expressed as the product $\beta_i = b_{ij} \bar{b}_{ij}$, where

$$\bar{b}_{ij}(q_0, q_j) = B_{i0} \prod_{l \in \mathcal{N}_i^f, l \neq j} b_{il} \prod_{k \in \mathcal{N}_i \cup \mathcal{M}_i} B_{ik} \quad (9)$$

and $\nabla_{q_i} b_{ij}$ is computed as

$$\nabla_{q_i} b_{ij} = \begin{cases} 0 & d_{ij} < R_c - \delta_2 \text{or} d_{ij} > R_c, \\ -\frac{2(d_{ij} + \delta_2 - R_c)(q_i - q_j)}{\delta_2^2 d_{ij}} & R_c - \delta_2 \leq d_{ij} \leq R_c, \end{cases} \quad (10)$$

the gradient of β_i evaluated at q_0 can be obtained as $\nabla_{q_i} \beta_i|_{q_0} = -((2\bar{b}_{ij})/(\delta R_c))(q_i - q_j)$. Since $K_i, \gamma_i, \alpha, \bar{b}_{ij}$ and δ are all positive terms, (7), and $\nabla_{q_i} \beta_i|_{q_0}$ can be used to determine that the controller (i.e., the negative gradient of $\nabla_{q_i} \varphi_i$) is along the direction of $q_j - q_i$, which implies node i is forced to move toward node j to ensure connectivity. Based on the design of b_{ij} in (5) and its gradient in (10), whenever a node enters the escape region of node i, an attractive force is imposed on node i to ensure connectivity.

Case 2[5]: Consider two nodes $j, l \in \mathcal{N}_i^f$, where $b_{ij} = 0$ and $b_{il} = 0$ (i.e., $\|q_i - q_j\| = R_c$ and $\|q_i - q_l\| = R_c$) simultaneously. In this case, $\beta_i = 0$ and $\nabla_{q_i} \beta_i$ is a zero vector, (8) can be used to determine that q_0 is a critical point (i.e., $\nabla_{q_i} \varphi_i|_{q_0} = 0$), and the navigation function achieves its maximum value at the critical point (i.e., $\varphi_i|_{q_0} = 1$). Since φ_i is maximized at q_0, no open set of initial conditions can be attracted to q_0 under the control law designed in (7).

From the development in Case 1 and Case 2, the control law in (7) ensures that all nodes $j \in \mathcal{N}_i^f$ remain connected with node i for all time. ■

B. Convergence Analysis

Proposition 2: The system in (1) converges to the largest invariant set (i.e., the set of critical points $S = \{q | \nabla_{q_i} \varphi_i|_q = 0\}$) under the controller in (7), provided that the tuning parameter in (2) satisfies $\alpha > \Theta$, where $\Theta = \max\{\sqrt{(|c_1|/c_3)}, (|c_2|/c_3)\}$.

Proof: Consider a Lyapunov candidate $V(\mathbf{q}) = \sum_{i=1}^N \varphi_i$, where $\mathbf{q}$ is the stacked states of all nodes, i.e., $\mathbf{q} = [q_1, \cdots, q_N]^T$. The time derivative of V is computed as $\dot{V} = (\nabla V)^T \dot{q} =$

[4]A point p in the workspace $\mathcal{F}$ is a critical point if $\nabla_{q_i} \varphi_i|_p = 0$.

[5]Case 2 can be extended to more than two nodes without loss of generality.

$-K\sum_{i=1}^{N}\sum_{j=1}^{N}(\nabla_{q_i}\varphi_i)^T(\nabla_{q_i}\varphi_j)$, similar to [4], which can be further separated as

$$\dot{V} = -K\sum_{i:\nabla_{q_i}\varphi_i=0}\left(\|\nabla_{q_i}\varphi_i\|^2+\sum_{j\neq i}(\nabla_{q_i}\varphi_i)^T\nabla_{q_i}\varphi_j\right) -K\sum_{i:\nabla_{q_i}\varphi_i\neq 0}\left(\|\nabla_{q_i}\varphi_i\|^2+\sum_{j\neq i}(\nabla_{q_i}\varphi_i)^T\nabla_{q_i}\varphi_j\right).$$

When all nodes are located at the critical points, $\dot{V}=0$. To show that the set of critical points are the largest invariant set, it requires that $\dot{V}$ is strictly negative, whenever there exists at least one node i such that $\nabla_{q_i}\varphi_i \neq 0$. Since $\nabla_{q_i}\varphi_i \neq 0$ for at least one node, $\dot{V}$ can be rewritten as

$$\dot{V} = -K\sum_{i:\nabla_{q_i}\varphi_i\neq 0}\left(\|\nabla_{q_i}\varphi_i\|^2+\sum_{j\neq i}(\nabla_{q_i}\varphi_i)^T(\nabla_{q_i}\varphi_j)\right). \quad (11)$$

To ensure that $\dot{V} < 0$ in (11), it is sufficient to require that $\sum_{j\neq i}(\nabla_{q_i}\varphi_i)^T(\nabla_{q_i}\varphi_j) > 0$, which can be expanded by using (8) as

$$\frac{\left(\beta_i\nabla_{q_i}\gamma_i-\frac{\gamma_i}{\alpha}\nabla_{q_i}\beta_i\right)^T}{(\gamma_i^\alpha+\beta_i)^{\frac{1}{\alpha}+1}}\left(\sum_{j\neq i}\frac{\left(\beta_j\nabla_{q_i}\gamma_j-\frac{\gamma_j}{\alpha}\nabla_{q_i}\beta_j\right)}{\left(\gamma_j^\alpha+\beta_j\right)^{\frac{1}{\alpha}+1}}\right) > 0. \quad (12)$$

Since γ_i, β_i are all positive from (3) and (4), and γ_i, β_i can not be zero simultaneously from Assumption 2, the inequality in (12) is valid provided

$$\left(\beta_i\nabla_{q_i}\gamma_i-\frac{\gamma_i}{\alpha}\nabla_{q_i}\beta_i\right)^T\left(\sum_{j\neq i}\left(\beta_j\nabla_{q_i}\gamma_j-\frac{\gamma_j}{\alpha}\nabla_{q_i}\beta_j\right)\right) > 0$$

which can be simplified as

$$\frac{1}{\alpha^2}c_1+\frac{1}{\alpha}c_2+c_3 > 0 \quad (13)$$

where $c_2 = -\beta_i(\nabla_{q_i}\gamma_i)^T\sum_{j\neq i}\gamma_j\nabla_{q_i}\beta_j - \gamma_i(\nabla_{q_i}\beta_i)^T\sum_{j\neq i}\beta_j\nabla_{q_i}\gamma_j$, $c_1 = \gamma_i(\nabla_{q_i}\beta_i)^T\sum_{j\neq i}\gamma_j\nabla_{q_i}\beta_j$, and $c_3 = \beta_i(\nabla_{q_i}\gamma_i)^T\sum_{j\neq i}\beta_j\nabla_{q_i}\gamma_j$. In (13), since β_i and β_j are positive, and node i satisfies $\nabla_{q_i}\varphi_i \neq 0$, c_3 is positive from (3). Using the fact that $c_1 \geq -|c_1|$, $c_2 \geq -|c_2|$, (13) can be written as $-(1/\alpha^2)|c_1| - (1/\alpha)|c_2| > -c_3$, which suffices to show that $\alpha > \max\{\sqrt{(|c_1|/c_3)}\}$ and $\alpha > \max\{(|c_2|/c_3)\}$. Therefore, if $\alpha > \max\{\sqrt{(|c_1|/c_3)}, (|c_2|/c_3)\}$, the system converges to the set of critical points. ∎

Proposition 3: The navigation function is minimized at the desired point q_{di}.

Proof: The navigation function φ_i is minimized at a critical point if the Hessian of φ_i evaluated at that point is positive definite. The gradient expression in (8) is used to determine if q_{di} is a critical point. From the definition of q_{di} and (3), the goal function evaluated at the desired point is $\gamma_i|_{q_{di}} = 0$. Also, the gradient of the goal function evaluated at the desired point q_{di} is $\nabla_{q_i}\gamma_i|_{q_{di}} = \sum_{j\in\mathcal{N}_i^f}2(q_{di}-q_j-c_{ij}) = 0$. Since $\gamma_i|_{q_{di}} = 0$ and $\nabla_{q_i}\gamma_i|_{q_{di}} = 0$, (8) can be used to conclude that $\nabla_{q_i}\varphi_i|_{q_{di}} = 0$. Thus, the desired point q_{di} in the workspace $\mathcal{F}$ is a critical point of φ_i. The Hessian of φ_i is

$$\nabla_{q_i}^2\varphi_i = \frac{1}{\alpha\left(\gamma_i^\alpha+\beta_i\right)^{\frac{1}{\alpha}+2}} \times\Big\{(\gamma_i^\alpha+\beta_i)\Big[\alpha\nabla_{q_i}\beta_i(\nabla_{q_i}\gamma_i)^T -\nabla_{q_i}\gamma_i(\nabla_{q_i}\beta_i)^T+\alpha\beta_i\nabla_{q_i}^2\gamma_i-\gamma_i\nabla_{q_i}^2\beta_i\Big] -\frac{\alpha+1}{\alpha}\left[\alpha\beta_i\nabla_{q_i}\gamma_i-\gamma_i\nabla_{q_i}\beta_i\right] \cdot\left[\alpha\gamma_i^{\alpha-1}\nabla_{q_i}\gamma_i+\nabla_{q_i}\beta_i\right]^T\Big\}. \quad (14)$$

Using the facts that $\gamma_i|_{q_{di}} = 0$ and $\nabla_{q_i}\gamma_i|_{q_{di}} = 0$ and the Hessian of γ_i is

$$\nabla_{q_i}^2\gamma_i = 2\zeta_i I_2 \quad (15)$$

where I_2 is the identity matrix in $\mathbb{R}^{2\times 2}$, the Hessian of φ_i evaluated at q_{di} is given by $\nabla_{q_i}^2\varphi_i|_{q_{di}} = 2\beta_i^{-(1/\alpha)}I_2\zeta_i$. The constraint function $\beta_i > 0$ at the desired configuration by Assumption 2, and ζ_i is a positive number. Hence, the Hessian of φ_i evaluated at that point is positive definite. ∎

Proposition 4: No minima of φ_i are on the boundary of the free workspace $\mathcal{F}_i$.

Proof: Please see proposition 3 in [11] for the proof. ∎

Proposition 5: For every $\varepsilon > 0$, there exists a number $\Gamma(\varepsilon)$ such that if $\alpha > \Gamma(\varepsilon)$ no critical points of φ_i are in $\mathcal{F}_3(\varepsilon)$.

Proof: From (8), any critical point must satisfy $\alpha\beta_i\nabla_{q_i}\gamma_i = \gamma_i\nabla_{q_i}\beta_i$. If $\alpha > \sup(\gamma_i\|\nabla_{q_i}\beta_i\|/\beta_i\|\nabla_{q_i}\gamma_i\|)$, where sup is taken over $\mathcal{F}_3(\varepsilon)$, then from (8), φ_i will have no critical points in $\mathcal{F}_3(\varepsilon)$. Since $\varepsilon = \inf b_{ij} = \inf B_{ik}$ in $\mathcal{F}_3(\varepsilon)$, $\sup(\gamma_i/(\|\nabla_{q_i}\gamma_i\|))((\|\nabla_{q_i}\beta_i\|)/\beta_i) \leq \Gamma(\varepsilon)$, where

$$\Gamma(\varepsilon) \triangleq \sup\frac{\gamma_i}{\|\nabla_{q_i}\gamma_i\|}\left(\sum_{j=1,j\neq i}^{\zeta_i}\frac{\sup\|\nabla_{q_i}b_{ij}\|}{\varepsilon}+\sum_{k=0,k\neq i}^{\xi_i+\vartheta_i}\frac{\sup\|\nabla_{q_i}B_{ik}\|}{\varepsilon}\right). \quad (16)$$

In (16), $\|\nabla_{q_i}b_{ij}\|$, $\|\nabla_{q_i}B_{ik}\|$ and $(\gamma_i/(\|\nabla_{q_i}\gamma_i\|))$ are bounded terms in $\mathcal{F}_3(\varepsilon)$ from (3), (10) and the fact that

$$\nabla_{q_i}B_{ik} = \begin{cases}\left(-\frac{2}{\delta_1^2}d_{ik}+\frac{2}{\delta_1}\right)\frac{q_i-q_k}{d_{ik}} & d_{ik} < \delta_1\\ 0 & d_{ik} \geq \delta_1.\end{cases} \quad (17)$$

Proposition 6: There exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, then φ_i is a Morse function.

Proof: The development in [12] and [18] proves that for φ_i to be a Morse function, it is sufficient to show that $\hat{u}^T(\nabla_{q_i}^2\varphi_i|_{q_{ci}})\hat{u}$ is positive for some particular vector $\hat{u}$ by choosing a small ε, where q_{ci} is a critical point. To show that $\hat{u}^T(\nabla_{q_i}^2\varphi_i|_{q_{ci}})\hat{u}$ is positive for the unit vector $\hat{u} \triangleq (q_i - q_j/\|q_i - q_j\|)$, (14) is used and the Hessian $\nabla_{q_i}^2\varphi_i$ evaluated at q_{ci} is

$$\frac{\alpha\hat{u}^T\left(\nabla_{q_i}^2\varphi_i|_{q_{ci}}\right)\hat{u}}{(\gamma_i^\alpha+\beta_i)^{-\frac{1}{\alpha}-1}} = \hat{u}^T\left(\alpha\beta_i\nabla_{q_i}^2\gamma_i+\frac{\left(1-\frac{1}{\alpha}\right)\gamma_i}{\beta_i} \cdot\nabla_{q_i}\beta_i\left(\nabla_{q_i}\beta_i\right)^T-\gamma_i\nabla_{q_i}^2\beta_i\right)\hat{u}. \quad (18)$$

To facilitate the subsequent analysis, the set of critical points in $\mathcal{F}_i$ is divided into sets of critical points in regions $\mathcal{F}_0(\varepsilon)$, $\mathcal{F}_1(\varepsilon)$, and $\mathcal{F}_2(\varepsilon)$. For a case where a critical point $q_{ci} \in \mathcal{F}_2(\varepsilon)$, using the fact that $\nabla_{q_i}\beta_i$ and $\nabla_{q_i}^2\beta_i$ can be expressed as

$$\nabla_{q_i}\beta_i = \bar{b}_{ij}\nabla_{q_i}b_{ij}+b_{ij}\nabla_{q_i}\bar{b}_{ij}, \quad (19)$$

$$\nabla_{q_i}^2\beta_i = \bar{b}_{ij}\nabla_{q_i}^2b_{ij}+b_{ij}\nabla_{q_i}^2\bar{b}_{ij} +\left(\nabla_{q_i}\bar{b}_{ij}\nabla_{q_i}^Tb_{ij}+\nabla_{q_i}b_{ij}\nabla_{q_i}^T\bar{b}_{ij}\right) \quad (20)$$

where $\bar{b}_{ij}$ is defined in (9), and the fact that the first term on the right hand side of (18) is always positive from (15), the subsequent expression can be obtained as

$$\alpha\left(\gamma_i^\alpha+\beta_i\right)^{\frac{1}{\alpha}+1}\hat{u}^T\left(\nabla_{q_i}^2\varphi_i|_{q_{ci}}\right)\hat{u} > \gamma_i\Omega \quad (21)$$

where $\Omega = (1/b_{ij})(a_1 b_{ij}^2 + a_2 b_{ij} + a_3)$, with $a_1 = (((\alpha - 1)\|\nabla_{q_i}\bar{b}_{ij}\|^2)/\alpha\bar{b}_{ij}) - \hat{u}^T(\nabla_{q_i}^2\bar{b}_{ij})\hat{u}$, $a_3 = (((\alpha - 1)\bar{b}_{ij})/\alpha)\|\nabla_{q_i}b_{ij}\|^2$ and $a_2 = ((2(\alpha-1)(\nabla_{q_i}\bar{b}_{ij})^T(\nabla_{q_i}b_{ij}))/\alpha) - \bar{b}_{ij}\hat{u}^T(\nabla_{q_i}^2 b_{ij})\hat{u} - \hat{u}^T(\nabla_{q_i}\bar{b}_{ij}\nabla_{q_i}^T b_{ij} + \nabla_{q_i}b_{ij}\nabla_{q_i}^T\bar{b}_{ij})\hat{u}$. Since $b_{ij} > 0$, a necessary condition to show that $\Omega > 0$ is to prove that

$$a_1 b_{ij}^2 + a_2 b_{ij} + a_3 > 0 \tag{22}$$

where $a_3 > 0$ if $\alpha > 1$. To prove the inequality in (22), the following two cases are analyzed.

Case 1) For $a_1 < 0$, the inequality in (22) is valid if $b_{ij} < ((-a_2 - \sqrt{a_2^2 - 4a_1a_3})/2a_1)$.

Case 2) For $a_1 \geq 0$, Ω can be rewritten as $\Omega \geq a_2 + (a_3/b_{ij})$, which is positive if $b_{ij} < (a_3/\|a_2\|)$.

Therefore, $\Omega > 0$, and from (21), $\hat{u}^T(\nabla_{q_i}^2\varphi_i|_{q_{ci}})\hat{u} > 0$ for all cases if b_{ij} is chosen as $b_{ij} < \varepsilon_0' \triangleq \min\{(-a_2 - \sqrt{a_2^2 - 4a_1a_3})/2a_1, (a_3/\|a_2\|)\}$. By using the same process of evaluating the Hessian $\nabla_{q_i}^2\varphi_i$ at critical points belonging to $\mathcal{F}_0(\varepsilon)$ and $\mathcal{F}_1(\varepsilon)$, upper bounds ε_0'' and ε_0''' for ε can be obtained for $q_{ci} \in \mathcal{F}_1(\varepsilon)$ and $q_{ci} \in \mathcal{F}_0(\varepsilon)$ respectively. By choosing $\varepsilon < \varepsilon_0 = \min\{\varepsilon_0', \varepsilon_0'', \varepsilon_0'''\}$, the function Ω is guaranteed to be positive which implies all the critical points are non-degenerate critical points of φ_i. ■

Proposition 7: There exists $\varepsilon_1 > 0$, such that φ_i has no local minimum in $\mathcal{F}_2(\varepsilon)$, as long as $\varepsilon < \varepsilon_1$.

Proof: Consider a critical point $q_{ci} \in \mathcal{F}_2(\varepsilon)$. Since φ_i is a Morse function, then if $\nabla_{q_i}^2\varphi_i|_{q_{ci}}$ has at least one negative eigenvalue, φ_i will have no minimum in $\mathcal{F}_2(\varepsilon)$. To show $\nabla_{q_i}^2\varphi_i|_{q_{ci}}$ has at least one negative eigenvalue, a unit vector $\hat{v} \triangleq ((\nabla_{q_i}\beta_i/\|\nabla_{q_i}\beta_i\|))^\perp$ is defined as a test direction to demonstrate that $\hat{v}^T(\nabla_{q_i}^2\varphi_i|_{q_{ci}})\hat{v} < 0$, where $(\chi)^\perp$ denotes a vector that is perpendicular to some vector χ. Using (19) and (20), $\alpha(\gamma_i^\alpha + \beta_i)^{1/\alpha+1}|_{q_{ci}}\hat{v}^T(\nabla_{q_i}^2\varphi_i|_{q_{ci}})\hat{v} = -\gamma_i\Phi + b_{ij}\Psi$, where $\Phi = \hat{v}^T(\nabla_{q_i}\bar{b}_{ij}\nabla_{q_i}^T b_{ij} + \nabla_{q_i}b_{ij}\nabla_{q_i}^T\bar{b}_{ij} - \bar{b}_{ij}\nabla_{q_i}^2 b_{ij})\hat{v}$, $\Psi = \hat{v}^T(\alpha\bar{b}_{ij}\nabla_{q_i}^2\gamma_i - \gamma_i\nabla_{q_i}^2\bar{b}_{ij})\hat{v}$, and

$$\nabla_{q_i}^2 b_{ij} = \begin{cases} 0 & d_{ij} \leq R_c - \delta_2 \\ & \text{or } d_{ij} \geq R_c \\ \frac{2(\delta_2 - R_c)(q_i - q_j)(q_i - q_j)^T}{\delta_2^2 d_{ij}^3} & \\ -\frac{2(d_{ij} + \delta_2 - R_c)I_2}{d_{ij}\delta_2^2} & R_c - \delta_2 < d_{ij} \\ & \text{and } d_{ij} < R_c. \end{cases} \tag{23}$$

Based on Assumption 3 and (5), (6), (10), (23), $\nabla_{q_i}\bar{b}_{ij} = 0$ and $\nabla_{q_i}^2 b_{ij} < 0$. Since the goal function γ_i and $\bar{b}_{ij}$ are positive, $\Phi > 0$. To ensure $\hat{v}^T(\nabla_{q_i}^2\varphi_i\|_{q_{ci}})\hat{v} < 0$, ε must be selected as $\varepsilon < \varepsilon_1$ where $\varepsilon_1 = \inf_{\mathcal{F}_2(\varepsilon)}(|\gamma_i\Phi|/\|\Psi\|)$. ■

Proposition 8: There exists $\varepsilon_2 > 0$, such that φ_i has no local minimum in $\mathcal{F}_1(\varepsilon)$ and $\mathcal{F}_0(\varepsilon)$, as long as $\varepsilon < \varepsilon_2$.

Proof: Consider a critical point $q_{ci} \in \mathcal{F}_1(\varepsilon)$. Similar to the proof for *Proposition 7*, the current proof is based on the fact that if $\hat{w}^T(\nabla_{q_i}^2\varphi_i|_{q_{ci}})\hat{w} < 0$ for some particular vector $\hat{w} \triangleq ((q_i - q_k/\|q_i - q_k\|))^\perp$, then φ_i will have no minimum in $\mathcal{F}_1(\varepsilon)$. To facilitate the subsequent analysis, similar to the definition of $\bar{b}_{ij}$ in (9), β_i can be expressed as the product $\beta_i = B_{ik}\bar{B}_{ik}$ and $\bar{B}_{ik}$ is defined as

$$\bar{B}_{ik}(q_i, q_k) = B_{i0}\prod_{j\in\mathcal{N}_i^f} b_{ij}\prod_{l\in\mathcal{N}_i\cup\mathcal{M}_i, l\neq k} B_{il}. \tag{24}$$

Using (15), (17) and (24), $\alpha(\gamma_i^\alpha + \beta_i)^{(1/\alpha)+1}|_{q_{ci}}\hat{w}^T(\nabla_{q_i}^2\varphi_i|_{q_{ci}})\hat{w} = \gamma_i\bar{B}_{ik}\Lambda + \gamma_i B_{ik}\Xi$, where $\Lambda = \nabla_{q_i}^T B_{ik}(\nabla_{q_i}\gamma_i/\|\nabla_{q_i}\gamma_i\|)2\zeta_i - (2(\delta_1 - d_{ik})/d_{ik}\delta_1^2)$, $\Xi = \hat{w}^T((\nabla_{q_i}^T\bar{B}_{ik}\nabla_{q_i}\gamma_i/\|\nabla_{q_i}\gamma_i\|)\nabla_{q_i}^2\gamma_i + ((1 - (1/\alpha))/\bar{\beta}_{ij})\nabla_{q_i}\bar{B}_{ik}\nabla_{q_i}^T\bar{B}_{ik} - \nabla_{q_i}^2\bar{B}_{ik})\hat{w}$, and

$$\nabla_{q_i}^2 B_{ik} = \begin{cases} (-(2/\delta_1^2) + (2/d_{ik}\delta_1))I_2 & \\ -((2(q_i - q_k)(q_i - q_k)^T)/\delta_1 d_{ik}^3) & d_{ik} < \delta_1. \\ 0 & d_{ik} \geq \delta_1 \end{cases}$$

Since $d_{ik} < \delta_1$, and $\nabla_{q_i}^T B_{ik}(\nabla_{q_i}\gamma_i/\|\nabla_{q_i}\gamma_i\|)$ can be upper bounded by a positive constant in $\mathcal{F}_1(\varepsilon)$, then if d_{ik} is small enough, Λ is guaranteed to be negative. Hence, there exist a positive scalar $\varepsilon_{21} = B_{ik}(d_{ik})$, which is small enough to ensure $\Lambda < 0$. To ensure $\hat{w}^T(\nabla_{q_i}^2\varphi_i|_{q_{ci}})\hat{w} < 0$, ε must be selected as $\varepsilon < \min\{\varepsilon_{21}, \inf_{\mathcal{F}_1(\varepsilon)}(|\Lambda\bar{B}_{ik}|/|\Xi|)\}$.

Let $\hat{x}$ be an unit vector defined as $\hat{x} \triangleq ((q_i - q_0/\|q_i - q_0\|))^\perp$. The same procedure that was used to show $\hat{w}^T(\nabla_{q_i}^2\varphi_i|_{q_{ci}})\hat{w} < 0$ in $\mathcal{F}_1(\varepsilon)$ can be followed to obtain another upper bound for ε, which ensures $\hat{x}^T(\nabla_{q_i}^2\varphi_i|_{q_{ci}})\hat{x} < 0$ in $\mathcal{F}_0(\varepsilon)$. By choosing ε_2 as the minimum of the upper bound for ε developed for $\mathcal{F}_1(\varepsilon)$ and $\mathcal{F}_0(\varepsilon)$, φ_i is ensured to have no minimum in $\mathcal{F}_1(\varepsilon)$ and $\mathcal{F}_0(\varepsilon)$ as long as $\varepsilon < \varepsilon_2$. ■

Based on *Propositions 2–8*, if ε is chosen such that $\varepsilon \leq \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$ then the minima (a critical point) is not in $\mathcal{F}_0(\varepsilon)$, $\mathcal{F}_1(\varepsilon)$, $\mathcal{F}_2(\varepsilon)$, $\mathcal{F}_3(\varepsilon)$ or the boundary of $\mathcal{F}_i$. Thus, the minima has to be in $\mathcal{F}_{di}(\varepsilon)$ if $\alpha > \max\{1, \Gamma(\varepsilon), \Theta\}$. Hence, nodes starting from any initial positions (except for the unstable equilibria) will converge to the desired formation specified by the formation matrix c_{ij}.

V. Conclusion

Given an initial graph with a desired neighborhood, a navigation function based decentralized controller is developed to ensure the system asymptotically converges to the desired configuration while maintaining network connectivity and avoiding collisions with other agents and obstacles. A distinguishing feature of the developed approach is that the distributed agents achieve a coordinated global configuration without requiring radio communication. Future efforts are focused on enabling radio-silent navigation from an arbitrarily connected distributed network. Moreover, further efforts are required to eliminate Assumption 3 so that other obstacles or agents can be within the collision region of node i when node i is about to break the communication link. Likewise Assumption 4 becomes less practical as a point grows to a sphere in the presence of uncertainty, and as the workspace becomes more crowded. Future work is required to address the pervasive problem of obstacle avoidance in a cluttered workspace with uncertainty.

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Quadratic Stability for Hybrid Systems With Nested Saturations

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***Abstract*—The problems of characterizing quadratic stability and computing an estimation of the domain of attraction for saturated hybrid systems are addressed. Hybrid systems presenting saturations and nested saturations on signals involved in both the continuous-time and the discrete-time dynamics are considered. Geometrical characterizations of local and global quadratic stability are provided. Computation oriented conditions for quadratic stability are given in form of convex constraints.**

***Index Terms*—Domain of attraction, hybrid systems, nested saturations, stability.**

I. Introduction

Hybrid systems are systems with both continuous-time and discrete-time dynamics. Recently, the interest on hybrid systems has been growing, see [4], [6], [7], [14], [19], mainly due to the increasing application of digital devices for the control of real systems, like chemical processes, communications and automotive systems. A proper analysis and control theory has to be developed for hybrid systems. See for instance [13], concerning the design of predictive controllers for hybrid systems, and [16], on the use of hybrid controllers to improve the performance.

In this paper, hybrid systems with nested saturations are handled and both local and global stability are considered. The attention is devoted to quadratic Lyapunov functions and ellipsoidal contractive sets, as estimations of the domain of attraction for hybrid systems with (nested) saturations. Considering ellipsoids entails some conservativeness with respect to other families of sets (as polytopes), but permits to pose the problem in an efficiently solvable form. The issue of estimating the domain of attraction for saturated systems, in continuous-time and discrete-time, has been dealt with considering ellipsoids [1], [8], [11], [12], and polytopes [2].

A first contribution of the paper is the geometrical characterization of saturated functions. It is proved that, given a state, its image through a saturated function is contained in a known state-dependent polytope. The property is also proved for the case of nested saturations. Such results permit to characterize contractiveness of ellipsoids and to determine quadratic Lyapunov functions by means of convex constraints. Some results present in literature for continuous-time, as [1], [11], and discrete-time saturated systems, see [12], are improved or recovered as particular cases of our approach, see also the preliminary version of the work [5]. The results on local and global quadratic stability for hybrid systems with simple and nested saturations are other contributions. We also present how the lower bound on the time interval between jumps

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