Robust identification-based state derivative estimation for nonlinear systems

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***Abstract*—A robust identification-based state derivative estimation method for uncertain nonlinear systems is developed. The identifier architecture consists of a recurrent multi-layer dynamic neural network which approximates the system dynamics online, and a continuous robust feedback RISE (Robust Integral of the Sign of the Error) term which accounts for modeling errors and exogenous disturbances. Numerical simulations provide comparisons with existing robust derivative estimation methods including: a high gain observer, a 2-sliding mode robust exact differentiator, and numerical differentiation methods, such as backward difference and central difference.**

I. Introduction[1]

Estimation of the state derivative is useful for many applications including: disturbance and parameter estimation [1], fault detection in dynamical systems [2], digital differentiation in signal processing, acceleration feedback in robot contact transition control [3], DC motor control [4] and active vibration control [5]. The problem of computing the state derivative becomes trivial if the state is fully measurable and the system dynamics are exactly known. The presence of uncertainties (parametric and non-parametric) and exogenous disturbances, however, make the problem challenging and motivate the state derivative estimation method for uncertain nonlinear systems developed in the paper.

The most common approach to estimate derivatives is by using numerical differentiation methods. The Euler backward difference approach is one of the simplest and the most common numerical methods to differentiate a signal; however, this ad hoc approach yields erroneous results in the presence of sensor noise. The central difference algorithm performs better than backward difference; however, the central difference algorithm is non-causal since it requires future state values to estimate the current derivative. Noise attenuation in numerical differentiators may be achieved by using a low-pass filter, at the cost of introducing a phase delay in the system. A more analytically rigorous approach is to cast the problem of state derivative estimation as an observer design problem by augmenting the state with its derivative, where the state is fully measurable and the state derivative is not, thereby, reducing the problem to designing an observer for the unmeasurable state derivative. Previous approaches to solve the problem use pure robust feedback methods requiring infinite gain or infinite frequency [6]–[8]. A high gain observer is presented in [7] to estimate the output derivatives, and asymptotic convergence to the derivative is achieved as the gain tends to infinity, which is problematic in general and especially in the presence of noise. In [8], a robust exact differentiator using a 2-sliding mode algorithm is developed which assumes a known upper bound for a Lipschitz constant of the derivative.

All the above mentioned methods are robust non model-based approaches. In contrast to purely robust feedback methods, an identification-based robust adaptive approach is considered in this paper. The proposed identifier consists of a dynamic neural network (DNN) [9]–[12] and a RISE (Robust Integral of the Sign of the Error) term [13], [14], where the DNN adaptively identifies the unknown system dynamics online, while RISE, a continuous robust feedback term, is used to guarantee asymptotic convergence to the state derivative in the presence of uncertainties and exogenous disturbances. The DNN with its recurrent feedback connections has been shown to learn dynamics of high dimensional uncertain nonlinear systems with arbitrary accuracy [12], [15], motivating their use in the proposed identifier. Unlike most previous results on DNN-based system identification [10], [11], [16]–[18], which only guarantee bounded stability of the identification error system in the presence of DNN approximation errors and exogenous disturbances, the addition of RISE to the DNN identifier guarantees asymptotic identification.

The RISE structure combines the features of the high gain observer and higher order sliding mode methods, in the sense that it consists of high gain proportional and integral state feedback terms (similar to a high gain observer), and the integral of a signum term, allowing it to implicitly learn and cancel the effects of DNN approximation errors and exogenous disturbances in the Lyapunov stability analysis, guaranteeing asymptotic convergence. Simulation results in presence of noise show the effectiveness of the proposed method as compared to a high gain observer and a 2-sliding mode approach. Numerical differentiation results with backward and central difference are also provided for comparison.

II. Robust Identification-Based State Derivative Estimation

Consider a control-affine uncertain nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i + d, \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the measurable system state, $f(x) \in \mathbb{R}^n$ and $g_i(x) \in \mathbb{R}^n$, $i = 1, ..., m$ are unknown functions, $u_i(t) \in$

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$\mathbb{R}$, $i=1,...,m$ is the control input, and $d(t)\in\mathbb{R}^n$ is an exogenous disturbance. The objective is to design an estimator for the state derivative $\dot{x}(t)$ using a robust identification-based approach that adaptively identifies the uncertain dynamics.

Assumption 1 : The functions $f(x)$ and $g_i(x)$, $i=1,...,m$ are second-order differentiable. *Assumption 2* : The system in (1) is bounded input bounded state (BIBS) stable, i.e., $u_i(t), x(t) \in \mathcal{L}_\infty$, $i=1,...,m$. Also, $u_i(t)$ is second order differentiable, and $\dot{u}_i(t), \ddot{u}_i(t) \in \mathcal{L}_\infty$, $i=1,...,m$. *Assumption 3* : The disturbance $d(t)$ is second order differentiable, and $d(t), \dot{d}(t), \ddot{d}(t) \in \mathcal{L}_\infty$.

Remark 1. Assumptions 1-3 indicate that the technique developed in this paper is only applicable for sufficiently smooth systems (i.e. at least second-order differentiable) that are BIBS stable. The requirement that the disturbance is C^2 can be restrictive. For example, random noise does not satisfy this assumption; however, simulations with added noise show robustness to these disturbances as well.

The universal approximation property of NNs states that any continuous function $F:\mathbb{S}\to\mathbb{R}^n$, where $\mathbb{S}$ is a compact simply connected set, can be approximated arbitrarily well by a linear combination of sigmoidal functions [19], i.e.

$$F(x) = W^T\sigma(V^Tx) + \varepsilon(x), \tag{2}$$

where $x\in\mathbb{S}$, W and V are the ideal weights, $\sigma(\cdot)$ is the sigmoidal activation function, and $\varepsilon(\cdot)$ is the function reconstruction error. Using Assumption 2 and (2), the dynamic system in (1) can be represented by replacing the unknown functions with multi-layer NNs, as

$$\begin{aligned}\dot{x} &= W_f^T\sigma(V_f^Tx) + \sum_{i=1}^{m}\left[W_{gi}^T\sigma(V_{gi}^Tx) + \varepsilon_{gi}(x)\right]u_i \\ &\quad + \varepsilon_f(x) + d,\end{aligned} \tag{3}$$

where $W_f\in\mathbb{R}^{L_f+1\times n}$, $V_f\in\mathbb{R}^{n\times L_f}$, $W_{gi}\in\mathbb{R}^{L_{gi}+1\times n}$, $V_{gi}\in\mathbb{R}^{n\times L_{gi}}$, $\sigma_f \triangleq \sigma(V_f^Tx)\in\mathbb{R}^{L_f+1}$, $\sigma_{gi}\triangleq\sigma(V_{gi}^Tx)\in\mathbb{R}^{L_{gi}+1}$, $\varepsilon_f(x)\in\mathbb{R}^n$, and $\varepsilon_{gi}(x)\in\mathbb{R}^n$, $i=1,...,m$.

Assumption 4: The ideal weights are bounded by known positive constants [20], i.e. $\|W_f\|_F \le \bar{W}_f$, $\|V_f\|_F\le\bar{V}_f$, $\|W_{gi}\|_F\le\bar{W}_g$ and $\|V_{gi}\|_F\le\bar{V}_g$ $i=1,...,m$. *Assumption 5:* The activation functions $\sigma_f(\cdot)$ and $\sigma_{gi}(\cdot)$, and their derivatives with respect to their arguments, $\sigma_f'(\cdot)$, $\sigma_{gi}'(\cdot)$, $\sigma_f''(\cdot)$, $\sigma_{gi}''(\cdot)$, $i=1,...,m$, are bounded with known bounds. *Assumption 6:* On a compact set, the function reconstruction errors $\varepsilon_f(\cdot)$ and $\varepsilon_{gi}(\cdot)$, and their derivatives with respect to their arguments, $\varepsilon_f'(\cdot)$, $\varepsilon_{gi}'(\cdot)$, $\varepsilon_f''(\cdot)$, $\varepsilon_{gi}''(\cdot)$, $i=1,...,m$, are bounded (with known bounds) [20].

Remark 2. Assumption 6 is a strong assumption that can lead to conservative bounds on the approximation error and the respective partial derivatives. However, in practice, the conservative bound could be decreased by increasing the number of hidden nodes and using some knowledge of how the approximation errors change with increasing the hidden nodes. The number of hidden nodes is only limited by the available computational resources.

The following multi-layer dynamic neural network (MLDNN) identifier is proposed to identify the system in (3) and estimate the state derivative

$$\dot{\hat{x}} = \hat{W}_f^T\hat{\sigma}_f + \sum_{i=1}^{m}\hat{W}_{gi}^T\hat{\sigma}_{gi}u_i + \mu, \tag{4}$$

where $\hat{x}(t)\in\mathbb{R}^n$ is the identifier state, $\hat{W}_f(t)\in\mathbb{R}^{L_f+1\times n}$, $\hat{V}_f(t)\in\mathbb{R}^{n\times L_f}$, $\hat{W}_{gi}(t)\in\mathbb{R}^{L_{gi}+1\times n}$, $\hat{V}_{gi}(t)\in\mathbb{R}^{n\times L_{gi}}$, $i=1,...,m$ are the weight estimates, $\hat{\sigma}_f \triangleq \sigma(\hat{V}_f^T\hat{x})\in\mathbb{R}^{L_f+1}$, $\hat{\sigma}_{gi}\triangleq\sigma(\hat{V}_{gi}^T\hat{x})\in\mathbb{R}^{L_{gi}+1}$, $i=1,...,m$, and $\mu(t)\in\mathbb{R}^n$ denotes the RISE feedback term defined as [13], [14]

$$\mu \triangleq k\tilde{x}(t) - k\tilde{x}(0) + v, \tag{5}$$

where $\tilde{x}(t)\triangleq x(t)-\hat{x}(t)\in\mathbb{R}^n$ is the identification error, and $v(t)\in\mathbb{R}^n$ is the Filippov generalized solution [21] to the differential equation

$$\dot{v} = (k\alpha+\gamma)\tilde{x} + \beta_1 sgn(\tilde{x});\quad v(0)=0, \tag{6}$$

where k, α, γ, $\beta_1\in\mathbb{R}$ are positive constant control gains, and $sgn(\cdot)$ denotes a vector signum function.

Remark 3. The DNN-based system identifiers in literature, [9]–[11], [16], [17], typically do not include a feedback term based on the identification error, except in results such as [18], [22], [23], where a high gain proportional feedback term is used to guarantee bounded stability. The novel use of RISE feedback term, $\mu(t)$ in (4), ensures asymptotic regulation of the identification error in the presence of disturbance and NN function approximation errors.

The identification error dynamics can be written as

$$\begin{aligned}\dot{\tilde{x}} &= W_f^T\sigma_f - \hat{W}_f^T\hat{\sigma}_f + \varepsilon_f(x) + d - \mu \\ &\quad + \sum_{i=1}^{m}\left[(W_{gi}^T\sigma_{gi} - \hat{W}_{gi}^T\hat{\sigma}_{gi}) + \varepsilon_{gi}(x)\right]u_i.\end{aligned} \tag{7}$$

A filtered identification error is defined as

$$r \triangleq \dot{\tilde{x}} + \alpha\tilde{x}. \tag{8}$$

Taking the time derivative of (8) and using (7) yields

$$\begin{aligned}\dot{r} &= W_f^T\sigma_f'V_f^T\dot{x} - \dot{\hat{W}}_f^T\hat{\sigma}_f - \hat{W}_f^T\hat{\sigma}_f'\dot{\hat{V}}_f^T\hat{x} - \hat{W}_f^T\hat{\sigma}_f'\hat{V}_f^T\dot{\hat{x}} \\ &\quad + \sum_{i=1}^{m}\left[(W_{gi}^T\sigma_{gi} - \hat{W}_{gi}^T\hat{\sigma}_{gi})\dot{u}_i + W_{gi}^T\sigma_{gi}'V_{gi}^T\dot{x}u_i\right] \\ &\quad - \sum_{i=1}^{m}\left[\dot{\hat{W}}_{gi}^T\hat{\sigma}_{gi}u_i + \hat{W}_{gi}^T\hat{\sigma}_{gi}'\dot{\hat{V}}_{gi}^T\hat{x}u_i\right] + \dot{\varepsilon}_f(x) \\ &\quad + \sum_{i=1}^{m}\left[-\hat{W}_{gi}^T\hat{\sigma}_{gi}'\hat{V}_{gi}^T\dot{\hat{x}}u_i + \dot{\varepsilon}_{gi}(x)u_i + \varepsilon_{gi}(x)\dot{u}_i\right] \\ &\quad + \dot{d} - kr - \gamma\tilde{x} - \beta_1 sgn(\tilde{x}) + \alpha\dot{\tilde{x}}.\end{aligned} \tag{9}$$

The weight update laws for the DNN in (4) are developed based on the subsequent stability analysis as

$$\begin{aligned}\dot{\hat{W}}_f &= proj(\Gamma_{wf}\hat{\sigma}_f'\hat{V}_f^T\dot{\hat{x}}\tilde{x}^T), \quad \dot{\hat{V}}_f = proj(\Gamma_{vf}\dot{\hat{x}}\tilde{x}^T\hat{W}_f^T\hat{\sigma}_f'), \\ \dot{\hat{W}}_{gi} &= proj(\Gamma_{wgi}\hat{\sigma}_{gi}'\hat{V}_{gi}^T\dot{\hat{x}}u_i\tilde{x}^T), \qquad i=1...m, \\ \dot{\hat{V}}_{gi} &= proj(\Gamma_{vgi}\dot{\hat{x}}u_i\tilde{x}^T\hat{W}_{gi}^T\hat{\sigma}_{gi}'), \qquad i=1...m,\end{aligned} \tag{10}$$

where $proj(\cdot)$ is a smooth projection operator[2], and $\Gamma_{wf} \in \mathbb{R}^{L_f+1\times L_f+1}$, $\Gamma_{vf} \in \mathbb{R}^{n\times n}$, $\Gamma_{wgi} \in \mathbb{R}^{L_{gi}+1\times L_{gi}+1}$, $\Gamma_{vgi} \in \mathbb{R}^{n\times n}$ are constant positive diagonal adaptation gain matrices. Adding and subtracting $\frac{1}{2}W_f^T\hat{\sigma}'_f\hat{V}_f^T\dot{\hat{x}} + \frac{1}{2}\hat{W}_f^T\hat{\sigma}'_fV_f^T\dot{\hat{x}} + \sum_{i=1}^m\left[\frac{1}{2}W_{gi}^T\hat{\sigma}'_{gi}\hat{V}_{gi}^T\dot{\hat{x}}u_i + \frac{1}{2}\hat{W}_{gi}^T\hat{\sigma}'_{gi}V_{gi}^T\dot{\hat{x}}u_i\right]$, and grouping similar terms, the expression in (9) can be rewritten as

$$\dot{r} = \tilde{N} + N_{B1} + \hat{N}_{B2} - kr - \gamma\tilde{x} - \beta_1 sgn(\tilde{x}), \quad (11)$$

where the auxiliary signals, $\tilde{N}(x,\tilde{x},r,\hat{W}_f,\hat{V}_f,\hat{W}_{gi},\hat{V}_{gi},t)$, $N_{B1}(x,\hat{x},\hat{W}_f,\hat{V}_f,\hat{W}_{gi},\hat{V}_{gi},t)$, and $\hat{N}_{B2}(\hat{x},\dot{\hat{x}},\hat{W}_f,\hat{V}_f,\hat{W}_{gi},\hat{V}_{gi},t) \in \mathbb{R}^n$ in (11) are defined as

$$\begin{aligned}\tilde{N} \triangleq\ & \alpha\dot{\tilde{x}} - \dot{\hat{W}}_f^T\hat{\sigma}_f - \hat{W}_f^T\hat{\sigma}'_f\dot{\hat{V}}_f^T\hat{x} + \frac{1}{2}W_f^T\hat{\sigma}'_f\hat{V}_f^T\dot{\tilde{x}} \quad (12)\\ & + \frac{1}{2}\hat{W}_f^T\hat{\sigma}'_fV_f^T\dot{\tilde{x}} - \sum_{i=1}^m\left[\dot{\hat{W}}_{gi}^T\hat{\sigma}_{gi}u_i + \hat{W}_{gi}^T\hat{\sigma}'_{gi}\dot{\hat{V}}_{gi}^T\hat{x}u_i\right]\\ & + \frac{1}{2}\sum_{i=1}^m\left[\hat{W}_{gi}^T\hat{\sigma}'_{gi}V_{gi}^T\dot{\tilde{x}}u_i + W_{gi}^T\hat{\sigma}'_{gi}\hat{V}_{gi}^T\dot{\tilde{x}}u_i\right],\end{aligned}$$

$$\begin{aligned}N_{B1} \triangleq\ & \sum_{i=1}^m\left[W_{gi}^T\sigma_{gi}\dot{u}_i + W_{gi}^T\sigma'_{gi}V_{gi}^T\dot{x}u_i - \hat{W}_{gi}^T\hat{\sigma}_{gi}\dot{u}_i\right]\\ & + \sum_{i=1}^m\left[\dot{\varepsilon}_{gi}(x)u_i + \varepsilon_{gi}(x)\dot{u}_i\right] + \dot{\varepsilon}_f(x) + \dot{d} \quad (13)\\ & - \sum_{i=1}^m\left[\frac{1}{2}\hat{W}_{gi}^T\hat{\sigma}'_{gi}V_{gi}^T\dot{x}u_i + \frac{1}{2}W_{gi}^T\hat{\sigma}'_{gi}\hat{V}_{gi}^T\dot{x}u_i\right]\\ & + W_f^T\sigma'_fV_f^T\dot{x} - \frac{1}{2}W_f^T\hat{\sigma}'_f\hat{V}_f^T\dot{x} - \frac{1}{2}\hat{W}_f^T\hat{\sigma}'_fV_f^T\dot{x},\end{aligned}$$

$$\begin{aligned}\hat{N}_{B2} \triangleq\ & \sum_{i=1}^m\left[\frac{1}{2}\tilde{W}_{gi}^T\hat{\sigma}'_{gi}\hat{V}_{gi}^T\dot{\hat{x}}u_i + \frac{1}{2}\hat{W}_{gi}^T\hat{\sigma}'_{gi}\tilde{V}_{gi}^T\dot{\hat{x}}u_i\right] \quad (14)\\ & + \frac{1}{2}\tilde{W}_f^T\hat{\sigma}'_f\hat{V}_f^T\dot{\hat{x}} + \frac{1}{2}\hat{W}_f^T\hat{\sigma}'_f\tilde{V}_f^T\dot{\hat{x}}.\end{aligned}$$

To facilitate the subsequent stability analysis, an auxiliary term $N_{B2}(\hat{x},\dot{x},\hat{W}_f,\hat{V}_f,\hat{W}_{gi},\hat{V}_{gi},t) \in \mathbb{R}^n$ is defined by replacing $\dot{\hat{x}}(t)$ in $\hat{N}_{B2}(\cdot)$ by $\dot{x}(t)$, and $\tilde{N}_{B2}(\hat{x},\dot{\tilde{x}},\hat{W}_f,\hat{V}_f,\hat{W}_{gi},\hat{V}_{gi},t) \triangleq \hat{N}_{B2}(\cdot) - N_{B2}(\cdot)$. The terms $N_{B1}(\cdot)$ and $N_{B2}(\cdot)$ are grouped as $N_B \triangleq N_{B1} + N_{B2}$. Using Assumptions 2, 4-6, (8) and (10), the following bound can be obtained for (12)

$$\left\|\tilde{N}\right\| \leq \rho_1(\|z\|)\|z\|, \quad (15)$$

where $z \triangleq \left[\tilde{x}^T\ r^T\right]^T \in \mathbb{R}^{2n}$, and $\rho_1(\cdot) \in \mathbb{R}$ is a positive, globally invertible, non-decreasing function. The following bounds can be developed based on (3), Assumptions 2-7, (10), (13) and (14)

$$\|N_{B1}\| \leq \zeta_1, \quad \|N_{B2}\| \leq \zeta_2, \quad \left\|\dot{N}_B\right\| \leq \zeta_3 + \zeta_4\rho_2(\|z\|)\|z\|, \quad (16)$$

$$\left\|\dot{\tilde{x}}^T\tilde{N}_{B2}\right\| \leq \zeta_5\|\tilde{x}\|^2 + \zeta_6\|r\|^2, \quad (17)$$

where $\zeta_i \in \mathbb{R}$, $i = 1,...,6$ are computable positive constants, and $\rho_2(\cdot) \in \mathbb{R}$ is a positive, globally invertible, non-decreasing function.

Remark 4. The segregation of terms in (11)-(14) and their corresponding bounds in (15)-(17) is typical of the RISE strategy [13], [14], and aids in the subsequent stability analysis. The terms $N_{B1}(\cdot)$ and $N_{B2}(\cdot)$ are both bounded by constants; however, in the subsequent stability analysis $N_{B1}(\cdot)$ is canceled by RISE feedback, whereas $N_{B2}(\cdot)$ is partially canceled by RISE feedback and partially by the NN weight update laws.

To facilitate the subsequent stability analysis, let $\mathcal{D} \subset \mathbb{R}^{2n+2}$ be a domain containing $y(t) = 0$, where $y(t) \in \mathbb{R}^{2n+2}$ is defined as

$$y \triangleq \left[\tilde{x}^T\quad r^T\quad \sqrt{P}\quad \sqrt{Q}\right]^T, \quad (18)$$

where the auxiliary function $P(z,t) \in \mathbb{R}$ is the Filippov generalized solution [21] to the differential equation

$$\dot{P} = -L, \quad P(0) = \beta_1\sum_{i=1}^n|\tilde{x}_i(0)| - \tilde{x}^T(0)\,N_B(0), \quad (19)$$

where the auxiliary function $L(z,t) \in \mathbb{R}$ is defined as

$$\begin{aligned}L \triangleq\ & r^T(N_{B1} - \beta_1 sgn(\tilde{x})) + \dot{\tilde{x}}^TN_{B2} \quad (20)\\ & -\beta_2\rho_2(\|z\|)\|z\|\|\tilde{x}\|,\end{aligned}$$

where $\beta_1, \beta_2 \in \mathbb{R}$ are selected according to the following sufficient conditions:

$$\beta_1 > \max(\zeta_1 + \zeta_2,\ \zeta_1 + \frac{\zeta_3}{\alpha}), \qquad \beta_2 > \zeta_4, \quad (21)$$

to ensure that $P(t) \geq 0$ [14]. The auxiliary function $Q(\tilde{W}_f,\tilde{V}_f,\tilde{W}_{gi},\tilde{V}_{gi}) \in \mathbb{R}$ in (18) is defined as

$$\begin{aligned}Q \triangleq\ & \frac{1}{4}\alpha\left[tr(\tilde{W}_f^T\Gamma_{wf}^{-1}\tilde{W}_f) + tr(\tilde{V}_f^T\Gamma_{vf}^{-1}\tilde{V}_f) \quad (22)\right.\\ & \left. + \sum_{i=1}^m(tr(\tilde{W}_{gi}^T\Gamma_{wgi}^{-1}\tilde{W}_{gi}) + tr(\tilde{V}_{gi}^T\Gamma_{vgi}^{-1}\tilde{V}_{gi}))\right],\end{aligned}$$

where $tr(\cdot)$ denotes the trace of a matrix.

Theorem 5. *The identifier developed in (4) along with its weight update laws in (10) ensures asymptotic convergence, in the sense that*

$$\lim_{t\to\infty}\|\tilde{x}(t)\| = 0 \qquad \textit{and} \qquad \lim_{t\to\infty}\left\|\dot{\tilde{x}}(t)\right\| = 0$$

provided the control gains k and γ are selected sufficiently large based on the initial conditions of the states[3] *and satisfy the following sufficient conditions*

$$\gamma > \frac{\zeta_5}{\alpha}, \qquad k > \zeta_6, \quad (23)$$

where ζ_5 and ζ_6 are introduced in (17), and β_1 and β_2 are selected according to the sufficient conditions in (21).

[2]The space of DNN weight estimates is projected onto a compact convex set, constructed using known upper bounds of the ideal weights (Assumption 4). This ensures that the weight estimates are always bounded, which is exploited in the subsequent stability analysis. Any of the several smooth projection algorithms may be used (see [24], [25]).

[3]See subsequent stability analysis.

Proof: Let $V : \mathcal{D} \to \mathbb{R}$ be a locally Lipschitz continuous regular positive definite function defined as

$$V \triangleq \frac{1}{2}r^T r + \frac{1}{2}\gamma \tilde{x}^T \tilde{x} + P + Q, \tag{24}$$

which satisfies the following inequalities:

$$U_1(y) \leq V(y) \leq U_2(y), \tag{25}$$

where $U_1(y)$, $U_2(y) \in \mathbb{R}$ are continuous positive definite functions defined as $U_1 \triangleq \frac{1}{2}min(1,\gamma)\|y\|^2$ and $U_2 \triangleq max(1,\gamma)\|y\|^2$, respectively.

Let $\dot{y} = h(y,t)$ represent the closed-loop differential equations in (7), (10), (11), and (19), where $h(y,t) \in \mathbb{R}^{2n+2}$ denotes the right-hand side of the the closed-loop error signals. Using Filippov's theory of differential inclusions [21], [26], the existence of solutions can be established for $\dot{y} \in K[h](y,t)$, where $K[h] \triangleq \bigcap_{\delta>0} \bigcap_{\mu M=0} \overline{co}h(B(y,\delta) - M, t)$, where $\bigcap_{\mu M=0}$ denotes the intersection over all sets M of Lebesgue measure zero, $\overline{co}$ denotes convex closure, and $B(y,\delta) = \{x \in \mathbb{R}^{2n+2} | \ \|y - x\| < \delta\}$. The right hand side of the differential equation, $h(y,t)$, is continuous except for the Lebesgue measure zero set of times $t \in [t_0, t_f]$ when $\tilde{x}(t) = 0$. Hence, the set of time instances for which $\dot{y}(t)$ is not defined is Lebesgue negligible. The absolutely continuous solution $y(t) = y(t_0) + \int_{t_0}^{t} \dot{y}(t)dt$ does not depend on the value of $\dot{y}(t)$ on a Lebesgue negligible set of time-instances [27]. Under Filippov's framework, a generalized Lyapunov stability theory can be used to establish strong stability of the closed-loop system $\dot{y} = h(y,t)$. The generalized time derivative of (24) exists almost everywhere (a.e.), i.e. for almost all $t \in [t_0, t_f]$, and $\dot{V}(y) \in^{a.e.} \dot{\tilde{V}}(y)$ where

$$\dot{\tilde{V}} = \bigcap_{\xi \in \partial V(y)} \xi^T K \left[\dot{r}^T \ \dot{\tilde{x}}^T \ \frac{1}{2}P^{-\frac{1}{2}}\dot{P} \ \frac{1}{2}Q^{-\frac{1}{2}}\dot{Q} \right]^T, \tag{26}$$

where ∂V is the generalized gradient of $V(y)$ [28]. Since $V(y)$ is locally Lipschitz continuous regular and smooth in y, (26) can be simplified as [29]

$$\begin{aligned} \dot{\tilde{V}} &= \nabla V^T K \left[\dot{r}^T \ \dot{\tilde{x}}^T \ \frac{1}{2}P^{-\frac{1}{2}}\dot{P} \ \frac{1}{2}Q^{-\frac{1}{2}}\dot{Q} \right]^T \\ &= \left[r^T \ \gamma\tilde{x}^T \ 2P^{\frac{1}{2}} \ 2Q^{\frac{1}{2}} \right] K\left[\Psi\right]^T, \end{aligned}$$

where

$$\Psi \triangleq \left[\dot{r}^T \ \dot{\tilde{x}}^T \ \frac{1}{2}P^{-\frac{1}{2}}\dot{P} \ \frac{1}{2}Q^{-\frac{1}{2}}\dot{Q} \right].$$

Using the calculus for $K[\cdot]$ from [30] (Theorem 1, Properties 2,5,7), and substituting the dynamics from (11) and (19), yields

$$\begin{aligned} \dot{\tilde{V}} \subset\ & r^T(\tilde{N} + N_{B1} + \hat{N}_{B2} - kr - \beta_1 K[sgn(\tilde{x})] - \gamma\tilde{x}) \\ & +\gamma\tilde{x}^T(r - \alpha\tilde{x}) - r^T(N_{B1} - \beta_1 K[sgn(\tilde{x})]) \\ & -\dot{\tilde{x}}^T N_{B2} + \beta_2\rho_2(\|z\|)\|z\|\|\tilde{x}\| \\ & -\frac{1}{2}\alpha\left[tr(\tilde{W}_f^T \Gamma_{wf}^{-1}\dot{\hat{W}}_f) + tr(\tilde{V}_f^T \Gamma_{vf}^{-1}\dot{\hat{V}}_f) \right] \\ & -\frac{1}{2}\alpha\sum_{i=1}^{m}\left[tr(\tilde{W}_{gi}^T \Gamma_{wgi}^{-1}\dot{\hat{W}}_{gi}) + tr(\tilde{V}_{gi}^T \Gamma_{vgi}^{-1}\dot{\hat{V}}_{gi}) \right]. \end{aligned} \tag{27}$$

$$\begin{aligned} \dot{V} \overset{a.e.}{\leq}\ & -\alpha\gamma\|\tilde{x}\|^2 - k\|r\|^2 + \rho_1(\|z\|)\|z\|\|r\| + \zeta_5\|\tilde{x}\|^2 \\ & +\zeta_6\|r\|^2 + \beta_2\rho_2(\|z\|)\|z\|\|\tilde{x}\|, \end{aligned} \tag{28}$$

where (10), (15), and (17) are used, $K[sgn(\tilde{x})] = SGN(\tilde{x})$ [30], such that $SGN(\tilde{x}_i) = 1$ if $\tilde{x}_i > 0$, $[-1,1]$ if $\tilde{x}_i = 0$, and -1 if $\tilde{x}_i < 0$ (the subscript i denotes the i^{th} element). The set in (27) reduces to the scalar inequality in (28) since the RHS is continuous a.e., i.e., the RHS is continuous except for the Lebesgue measure zero set of times when $\tilde{x}(t) = 0$ [27]. Substituting for $k \triangleq k_1 + k_2$ and $\gamma \triangleq \gamma_1 + \gamma_2$, and completing the squares, the expression in (28) can be upper bounded as

$$\begin{aligned} \dot{V} \overset{a.e.}{\leq}\ & -(\alpha\gamma_1 - \zeta_5)\|\tilde{x}\|^2 - (k_1 - \zeta_6)\|r\|^2 \\ & +\frac{\rho_1(\|z\|)^2}{4k_2}\|z\|^2 + \frac{\beta_2^2\rho_2(\|z\|)^2}{4\alpha\gamma_2}\|z\|^2. \end{aligned} \tag{29}$$

Provided the sufficient conditions in (23) are satisfied, the expression in (29) can be rewritten as

$$\begin{aligned} \dot{V} &\overset{a.e.}{\leq} -\lambda\|z\|^2 + \frac{\rho(\|z\|)^2}{4\eta}\|z\|^2 \\ &\overset{a.e.}{\leq} -U(y) \quad \forall y \in \mathcal{D} \end{aligned} \tag{30}$$

where $\lambda \triangleq min\{\alpha\gamma_1 - \zeta_5, \ k_1 - \zeta_6\}$, $\eta \triangleq min\{k_2, \ \frac{\alpha\gamma_2}{\beta_2^2}\}$, $\rho(\|z\|)^2 \triangleq \rho_1(\|z\|)^2 + \rho_2(\|z\|)^2$ is a positive, globally invertible, non-decreasing function, and $U(y) = c\|z\|^2$, for some positive constant c, is a continuous, positive semi-definite function defined on the domain $\mathcal{D} \triangleq \left\{ y(t) \in \mathbb{R}^{2n+2} \mid \|y\| \leq \rho^{-1}\left(2\sqrt{\lambda\eta}\right) \right\}$. The size of the domain $\mathcal{D}$ can be increased by increasing the gains k and γ. The inequalities in (25) and (30) can be used to show that $V(y) \in \mathcal{L}_\infty$ in $\mathcal{D}$; hence, $\tilde{x}(t), r(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Using (8), standard linear analysis can be used to show that $\dot{\tilde{x}}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $\dot{x}(t) \in \mathcal{L}_\infty$ from (1) and Assumption 2-3, $\dot{\hat{x}}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. From the use of projection in (10), $\hat{W}_f(t), \hat{W}_{gi}(t) \in \mathcal{L}_\infty$, $i = 1...m$. Using the above bounding arguments, it can be shown from (11) that $\dot{r}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $\tilde{x}(t)$,$r(t) \in \mathcal{L}_\infty$, the definition of $U(y)$ can be used to show that it is uniformly continuous in $\mathcal{D}$. Let $\mathcal{S} \subset \mathcal{D}$ denote a set defined as $\mathcal{S} \triangleq \left\{ y(t) \subset \mathcal{D} \mid U_2(y(t)) < \frac{1}{2}\left(\rho^{-1}\left(2\sqrt{\lambda\eta}\right)\right)^2 \right\}$, where the region of attraction can be made arbitrarily large to include any initial conditions by increasing the control gain η (i.e. a semi-global type of stability result), and hence $c\|z\|^2 \to 0$ as $t \to \infty$ $\forall y(0) \in \mathcal{S}$. Using the definition of $z(t)$, it can be shown that $\|\tilde{x}(t)\|, \|\dot{\tilde{x}}(t)\|, \|r\| \to 0$ as $t \to \infty$ $\forall y(0) \in \mathcal{S}$. ■

III. SIMULATIONS

The following dynamics of a two-link robot manipulator are used to compare the identification-based state derivative estimator developed in this paper with several other methods:

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + F_d\dot{q} + F_s(\dot{q}) = u(t), \tag{31}$$

where $q(t) = [q_1 \ q_2]^T$ and $\dot{q}(t) = [\dot{q}_1 \ \dot{q}_2]^T$ are the angular positions (rad) and angular velocities (rad/sec) of the two links, respectively, $M(q)$ is the inertia matrix, and $V_m(q,\dot{q})$ is the centripetal-Coriolis matrix, defined as

$$M \triangleq \begin{bmatrix} p_1 + 2p_3c_2 & p_2 + p_3c_2 \\ p_2 + p_3c_2 & p_2 \end{bmatrix}$$
$$V_m \triangleq \begin{bmatrix} -p_3s_2\dot{q}_2 & -p_3s_2(\dot{q}_1 + \dot{q}_2) \\ p_3s_2\dot{q}_1 & 0 \end{bmatrix},$$

where $p_1 = 3.473\ kg \cdot m^2$, $p_2 = 0.196\ kg \cdot m^2$, $p_3 = 0.242\ kg \cdot m^2$, $c_2 = cos(q_2)$, $s_2 = sin(q_2)$, $F_d = diag\{5.3,\ 1.1\}\ Nm \cdot sec$ and $F_s(\dot{q}) = diag\{8.45tanh(\dot{q}_1),\ 2.35tanh(\dot{q}_2)\}\ Nm$ are the models for dynamic and static friction, respectively. The robot model in (31) can be expressed as $\dot{x} = f(x) + g(x)u + d$, where the state $x(t) \in \mathbb{R}^4$ is defined as $x(t) \triangleq [q_1\ \ q_2\ \ \dot{q}_1\ \ \dot{q}_2]^T$, $d(t) \triangleq 0.1sin(10t)[1\ 1\ 1\ 1]^T$ is an exogenous disturbance, and $f(x) \in \mathbb{R}^4$ and $g(x) \in \mathbb{R}^{4\times 2}$ are defined as $f(x) \triangleq \begin{bmatrix} \dot{q}^T & \{M^{-1}(-V_m - F_d)\dot{q} - F_s\}^T \end{bmatrix}^T$ and $g(x) = \begin{bmatrix} 0_{2\times 2} & M^{-1} \end{bmatrix}$, respectively. The control input is designed as a PD controller to track the desired trajectory $q_d(t) = [0.5sin(2t)\quad 0.5cos(2t)]^T$, as $u(t) = -2[q_1(t) - 0.5sin(2t)\quad q_2(t) - 0.5cos(2t)]^T - [\dot{q}_1(t) - cos(2t)\quad \dot{q}_2(t) + sin(2t)]^T$. The objective is to design a state derivative estimator $\hat{\dot{x}}(t)$ to asymptotically converge to $\dot{x}(t)$. The performance of the developed RISE-based DNN identifier in (4) and (10) is compared with the 2-sliding mode robust exact differentiator [8]

$$\hat{\dot{x}} = z_s + \lambda_s\sqrt{|\tilde{x}|}sgn(\tilde{x}), \qquad \dot{z_s} = \alpha_s sgn(\tilde{x}), \quad (32)$$

and the high gain observer [7]

$$\hat{\dot{x}} = z_h + \frac{\alpha_{h1}}{\varepsilon_{h1}}(\tilde{x}), \qquad \dot{z_h} = \frac{\alpha_{h2}}{\varepsilon_{h2}}(\tilde{x}). \quad (33)$$

The gains for the identifier in (4) and (10) are selected as $k = 20$, $\alpha = 5$, $\gamma = 200$, $\beta_1 = 1.25$, and the DNN adaptation gains are selected as $\Gamma_{wf} = 0.1\mathbb{I}_{11\times 11}$, $\Gamma_{vf} = \mathbb{I}_{4\times 4}$, $\Gamma_{wg1} = 0.7\mathbb{I}_{4\times 4}$, $\Gamma_{wg2} = 0.4\mathbb{I}_{4\times 4}$, $\Gamma_{vg1} = \Gamma_{vg2} = \mathbb{I}_{4\times 4}$, where $\mathbb{I}_{n\times n}$ denotes an identity matrix of appropriate dimensions. The neural networks for $f(x)$ and $g(x)$ are designed to have 10 and 3 hidden layer neurons, respectively, and the DNN weights are initialized as uniformly distributed random numbers in the interval $[-1, 1]$. The gains for the 2-sliding mode differentiator in (32) are selected as $\lambda_s = 4.1$, $\alpha_s = 4$, while the gains for the high gain observer in (33) are selected as $\alpha_{h1} = 0.2$, $\varepsilon_{h1} = 0.01$, $\alpha_{h2} = 0.3$, $\varepsilon_{h2} = 0.001$. To ensure a fair comparison, the gains of all the three estimators were tuned for best performance (least RMS error) for the same settling time of approximately 0.4 seconds for the state derivative estimation errors. A white Gaussian noise was added to the state measurements, maintaining a signal to noise ratio of $60\ dB$. The initial conditions of the system and the estimators are chosen as $x(t) = \hat{x}(t) = [1\ 1\ 1\ 1]^T$.

Fig. 1 shows the state derivative estimation errors for the 2-sliding mode robust exact differentiator in [8], the high gain observer in [7], and the developed RISE-based DNN estimator. While the maximum overshoot in estimating the state derivative (see Fig. 1) using 2-sliding mode is smaller, the steady state errors are comparatively larger than both the high gain observer and the proposed method. Table I gives a comparison of the transient and steady state RMS state derivative estimation errors for different estimation methods. Results of standard numerical differentiation algorithms - backward difference and central difference (with a step-size of 10^{-4}) are also included; as seen from Table I, they perform significantly worse than the other methods, in presence of noise. Although, simulation results for the high gain observer and the developed method are comparable, as seen from Fig. 1 and Table I, differences exist in the structure of the estimators and proof of convergence of the estimates. The developed identifier includes the RISE structure, which combines the features of the high gain observer with the integral of a signum term, allowing it to implicitly learn and cancel terms in the stability analysis; thus, guaranteeing asymptotic convergence. While singular perturbation methods can be used to prove asymptotic convergence of the high gain observer to the derivative of the output signal ($\dot{x}(t)$ in this case) as the gains tend to infinity [31], Lyapunov-based stability methods are used to prove asymptotic convergence of the proposed identifier (as $t \to \infty$) with finite gains. Further, while both high gain observer and 2-sliding mode robust exact differentiator are purely robust feedback methods, the developed method, in addition to using a robust RISE feedback term, uses a DNN to adaptively identify the system dynamics.

IV. CONCLUSION

A robust identifier is developed for online estimation of the state derivative of uncertain nonlinear systems in the presence of exogenous disturbances. The result differs from existing pure robust methods in that the proposed method combines a DNN system identifier with a robust RISE feedback to ensure asymptotic convergence to the state derivative, which is proven using a Lyapunov-based stability analysis. Simulation results in the presence of noise show an improved transient and steady state performance of the developed identifier in comparison to several other derivative estimation methods. Future efforts will focus on extending the robust identification-based method for output feedback control of nonlinear systems.

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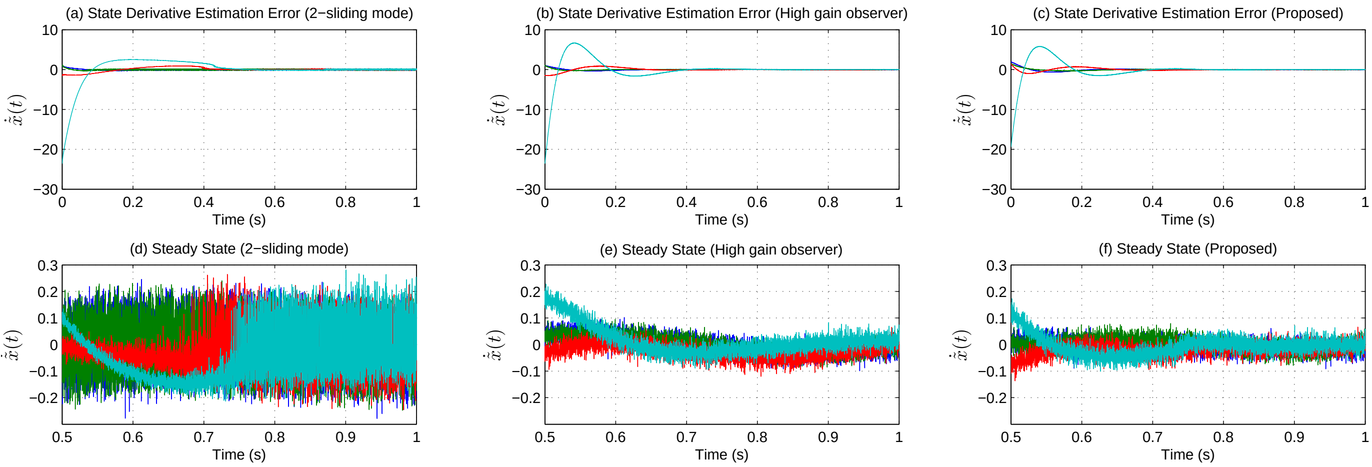

Fig. 1. Comparison of the state derivative estimation errors $\dot{\tilde{x}}(t)$ for (a) 2-sliding mode (b) high gain observer, and (c) proposed methods, in the presence of sensor noise (SNR 60 dB). The bottom figures (d), (e), and (f) indicate the respective performance at steady state.

	Backward difference	Central difference	2-sliding mode	High gain observer	Proposed
Transient RMS Error	14.4443	7.6307	2.3480	2.1326	1.7808
Steady State RMS Error	14.1461	7.0583	0.1095	0.0414	0.0297

TABLE I
COMPARISON OF TRANSIENT ($t = 0 - 5$ SEC.) AND STEADY-STATE ($t = 5 - 10$ SEC.) STATE DERIVATIVE ESTIMATION ERRORS $\dot{\tilde{x}}(t)$ FOR DIFFERENT DERIVATIVE ESTIMATION METHODS IN PRESENCE OF NOISE (60 DB).

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