

Time-Varying Input and State Delay Compensation for Uncertain Nonlinear Systems

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Abstract—A robust controller is developed for uncertain, second-order nonlinear systems subject to simultaneous unknown, time-varying state delays and known, sufficiently small time-varying input delays in addition to additive, sufficiently smooth disturbances. An integral term composed of previous control values facilitates a delay-free open-loop error system and the development of the feedback control structure. A stability analysis based on Lyapunov-Krasovskii (LK) functionals guarantees uniformly ultimately bounded tracking under the assumption that the delays are bounded and slowly varying. Numerical simulations illustrate robustness of the developed method to various combinations of simultaneous input and state delays.

Index Terms—Delay systems, Lyapunov methods, nonlinear control systems, robust control, uncertain systems.

I. INTRODUCTION

Numerous control techniques exist for linear systems with constant input delays (cf. [1]–[3] and references therein). Many of these results are extensions of classic techniques such as Smith predictors [4], Artstein model reduction [5], or finite spectrum assignment [6]. Results that focus on simultaneous constant state and input delays for linear systems are provided in [7]–[9]. Extensions of linear control techniques to time-varying input delays are also available [10]–[15].

For nonlinear systems, controllers considering constant [16]–[23] and time-varying [20], [24]–[32] state delays have been recently developed. However, linear results considering delayed inputs are far less prevalent, especially for systems with model uncertainties and/or disturbances. Examples of these include constant input delay results in [33]–[45] and time-varying input delay results based on linear matrix inequality conditions [46], [47], backstepping [48], [49] and other robust techniques [50]. Even more unique are results that consider both state and input delays in nonlinear systems. Recently in [49], the predictor-based techniques in [7] were extended to nonlinear systems with time-varying delays in the state and/or the input utilizing a backstepping transformation to construct a predictor-based compensator. The development in [49] requires knowledge of the plant dynamics and assumes that the plant is disturbance-free.

In this technical note, we expand our previous time-varying input delay result [50] in two directions: a) Utilizing techniques for constant input-delayed systems first introduced in [45], we consider

Manuscript received March 10, 2015; accepted June 23, 2015. Date of publication June 30, 2015; date of current version February 25, 2016. This work was supported in part by NSF Awards 1161260 and 1217908, ONR Grant N00014-13-1-0151, and a contract with the AFRL Mathematical Modeling and Optimization Institute. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the sponsoring agency. Recommended by Associate Editor F. Mazenc.

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Digital Object Identifier 10.1109/TAC.2015.2451472

time-varying input delays in a nonlinear plant, and b) we add the ability to compensate for simultaneous arbitrarily large unknown time-varying state delays based on the techniques in [32]. Robust control methods are developed to compensate for the unknown time-varying state delays. To compensate for the input delay, an integral term composed of previous control values is used to yield a delay-free open-loop system. A Lyapunov-based stability analysis motivated by Lyapunov-Krasovskii (LK) functionals demonstrates the ability to achieve uniformly ultimately bounded tracking in the presence of model uncertainty, additive sufficiently smooth disturbances, time-varying state delays and sufficiently small input delays. The result is based on the assumption that the unknown state delay is bounded and slowly varying. Improving on the result in [50], we relax previous sufficient conditions on the control gains that required knowledge of the second derivative of the input delay. Numerical simulations examine the robustness of the method to various combinations of simultaneous input and state delays.

II. DYNAMIC SYSTEM

Consider a class of second-order¹ (Euler-Lagrange-like) nonlinear systems given by²:

$$\ddot{x} = f(x, \dot{x}, t) + g(x(t - \tau_s), \dot{x}(t - \tau_s), t) + d + u_{\tau_i} \quad (1)$$

where $x, \dot{x} \in \mathbb{R}^n$ are the system states, $u \in \mathbb{R}^n$ is the control input, $f: \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^n$ is an unknown function, uniformly bounded in t , $g: \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^n$ is an unknown function with delayed internal state, uniformly bounded in t , $d \in \mathbb{R}^n$ denotes a sufficiently smooth disturbance (e.g., unmodeled effects), and $\tau_i, \tau_s \in [0, \infty)$ denote time-varying, non-negative input and state delays, respectively.

The subsequent development is based on the assumption that x and \dot{x} are measurable outputs. Throughout the technical note, a time-dependent delayed function is denoted as

$$\zeta_\tau \triangleq \begin{cases} \zeta(t - \tau) & t - \tau > t_0 \\ 0 & t - \tau \leq t_0 \end{cases}$$

where t_0 denotes the initial time. Thus, u_{τ_i} is defined by

$$u_{\tau_i} \triangleq \begin{cases} u(t - \tau_i) & t - \tau_i > t_0 \\ 0 & t - \tau_i \leq t_0. \end{cases}$$

Additionally, $\|\cdot\|$ denotes the Euclidean norm of a vector and the following assumptions will be exploited.

¹The result in this technical note can be extended to n th-order nonlinear systems following a similar development to that presented in [31].

²For brevity of notation, unless otherwise specified, the domain of all the functions is assumed to be $\mathbb{R}_{\geq 0}$. Furthermore, unless otherwise specified, all mathematical quantities are assumed to be time-varying and time-dependence is suppressed in equations and definitions. For example, the trajectory $x: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ is defined by abuse of notation as $x \in \mathbb{R}^n$ and unless otherwise specified, an equation (inequality) of the form $f + h(y, t) = (\leq)g(x)$ is interpreted as $f(t) + h(y(t), t) = (\leq)g(x(t))$ for all $t \in \mathbb{R}_{\geq 0}$.

Assumption 1: Each of the functions f and g , along with their first and second partial derivatives, is bounded on each subset of its domain of the form $\mathcal{X} \times [0, \infty)$, where $\mathcal{X} \subset \mathbb{R}^n \times \mathbb{R}^n$ is compact. Furthermore, given such compact \mathcal{X} , the corresponding bound is known.

Assumption 2: The nonlinear disturbance term and its time derivative are bounded by known constants.³

Assumption 3: The desired trajectory $x_d \in \mathbb{R}^n$ is designed such that $x_d^{(i)} \in \mathbb{R}^n$, $\forall i = 0, 1, \dots, 3$ exist and are bounded by known positive constants, where the superscript (i) denotes the i^{th} time derivative.⁴

Assumption 4: The input and state delays are bounded such that $0 \leq \tau_i \leq \varphi_{i_1}$ and $0 \leq \tau_s \leq \varphi_{s_1}$, and the rate of change of the delays are bounded such that $|\dot{\tau}_i| \leq \varphi_{i_2} < 1$ and $|\dot{\tau}_s| \leq \varphi_{s_2} < 1$ where $\varphi_j \in \mathbb{R}^+$ $\forall j = i_1, i_2, s_1, s_2$ are known constants. Furthermore, the bounds on the input delay satisfy $\varphi_{i_1} + \varphi_{i_2} < 1$. The state delay is assumed to be unknown, while the input delay is assumed to be known.

Remark 1: In Assumption 4, the slowly time-varying constraint (i.e., $|\dot{\tau}_{i,s}| \leq \varphi_{i_2, s_2} < 1$) is common to results which utilize classical LK functionals to compensate for time-varying time-delays [14]. Knowledge of the state delays in the system is not required; however, the input delays present a more significant challenge. Although the controller requires the input delay to be known so that the interval of past control values can be properly sized, numerical simulations illustrate robustness to uncertainties in the input delay.

III. CONTROL OBJECTIVE

The objective is to design a controller to ensure that the state x of the system in (1) tracks a desired state trajectory. To quantify the control objective, a tracking error, denoted by $e_1 \in \mathbb{R}^n$, is defined as

$$e_1 \triangleq x_d - x. \quad (2)$$

To facilitate the subsequent analysis, two auxiliary tracking errors $e_2, r \in \mathbb{R}^n$ are defined as [45]

$$e_2 \triangleq \dot{e}_1 + \alpha_1 e_1 \quad (3)$$

$$r \triangleq \dot{e}_2 + \alpha_2 e_2 + e_u \quad (4)$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ denote constant positive control gains, and $e_u \in \mathbb{R}^n$ denotes the mismatch between the delayed control input and the computed control input, defined as⁵

$$e_u \triangleq u_{\tau_i} - u. \quad (5)$$

The auxiliary signal e_u injects a delay-free control input into the error system development. In contrast to the development in [50], the term in (5) is embedded in a higher order derivative (i.e., r instead of e_2). Functionally, e_u still injects an integral of past control values into the open-loop system; however, the development introduces fewer cross-terms. The auxiliary signal r is introduced to facilitate the stability analysis and is not used in the control design since the expression in (4) depends on the unmeasurable state \ddot{x} . The structure of the error systems is motivated by the need to inject and cancel terms in the subsequent stability analysis as demonstrated in Section IV.

³Many practical disturbance terms are continuous including friction (see [51], [52]), wind disturbances, wave/ocean disturbances, etc.

⁴Many guidance and navigation applications utilize smooth, high-order differentiable desired trajectories. Curve fitting methods can also be used to generate sufficiently smooth time-varying trajectories.

⁵Let $h \triangleq \max(t_0, t - \tau_i)$. Then, $h: [0, \infty) \rightarrow [0, \infty)$ is continuous. Further, since $u(t_0) = 0$, $u_{\tau_i} = u(h)$, and $e_u = u(h) - u$. Hence, e_u is a continuous function of time if u is a continuous function of time, and $e_u(t_0) = 0$.

Using (1)–(3), and (5) to eliminate the delayed input term, (4) can be represented as

$$r = S_1 + S_2 - u \quad (6)$$

where the auxiliary functions $S_1 \in \mathbb{R}^n$ and $S_2 \in \mathbb{R}^n$ are defined as

$$S_1 \triangleq f(x_d, \dot{x}_d, t) - f(x, \dot{x}, t) + g(x_{d\tau_s}, \dot{x}_{d\tau_s}, t) - g(x_{\tau_s}, \dot{x}_{\tau_s}, t) + \alpha_1 \dot{e}_1 + \alpha_2 e_2$$

$$S_2 \triangleq \ddot{x}_d - f(x_d, \dot{x}_d, t) - g(x_{d\tau_s}, \dot{x}_{d\tau_s}, t) - d.$$

Based on (6) and the subsequent stability analysis

$$u \triangleq K(e_2 - e_2(t_0)) + v \quad (7)$$

where $v \in \mathbb{R}^n$ is the solution to the following differential equation

$$\dot{v} = K(\alpha_2 e_2 + e_u), \quad v(t_0) = 0 \quad (8)$$

and $K \in \mathbb{R}$ is a positive constant control gain. In contrast to the development in [50], the controller developed in (7) and (8) acts as a proportional-integral-derivative feedback in the absence of delay.

The closed-loop error system can be developed by taking the time derivative of (6) and substituting for (4) and the time derivative of (7) to yield

$$\dot{r} = \tilde{N} + N_d - e_2 - Kr \quad (9)$$

where $\tilde{N} \in \mathbb{R}^n$ and $N_d \in \mathbb{R}^n$ are defined as

$$\tilde{N} \triangleq \dot{S}_1 + e_2 \quad (10)$$

$$N_d \triangleq \dot{S}_2. \quad (11)$$

The design of the auxiliary signal in (5) is motivated by the desire to eliminate the delayed input, yielding the closed-loop error system in (9). The structure of (9) is advantageous because it facilitates the stability analysis by segregating terms that can be upper bounded by a state-dependent term and terms that can be upper bounded by constants. Based on Assumptions 2 and 3, the following inequalities can be developed from the expression in (11):

$$\|N_d\| \leq \zeta_{N_{d1}} \quad (12)$$

where $\zeta_{N_{d1}} \in \mathbb{R}$, is a known positive constant. The Mean Value Theorem can be utilized to find an upper bound for the expression in (10) as [53, Lemma 5]

$$\|\tilde{N}\| \leq \rho_1 (\|z\|) \|z\| + \rho_2 (\|z_{\tau_s}\|) \|z_{\tau_s}\| \quad (13)$$

where $z \in \mathbb{R}^{4n}$ denotes the vector

$$z \triangleq [e_1^T, e_2^T, r^T, e_u^T]^T \quad (14)$$

and the bounding terms $\rho_1, \rho_2: [0, \infty) \rightarrow [0, \infty)$ are positive, non-decreasing and radially unbounded functions.⁶ The upper bound for the auxiliary function \tilde{N} is segregated into delay-free and delay-dependent bounding functions to eliminate the delayed terms with the use of an LK term in the stability analysis.

To facilitate the subsequent stability analysis, several auxiliary terms are introduced. Let the control gain K be split in two parts as $K = k_s + k_{s1}$. Let $\{\iota_i \in \mathbb{R} | i = 1, \dots, 8\}$ be positive adjustable

⁶For some classes of systems, the bounding functions ρ_1 and ρ_2 could be selected as constants. For these classes of systems, a global uniformly ultimately bounded result can be obtained as described in Remark 2.

constants such that $\iota_2 + \iota_3 + \iota_4 = 1$, and $\iota_5 + \iota_6 + \iota_7 + \iota_8 = 1$. Let $\rho : [0, \infty) \rightarrow [0, \infty)$ be an auxiliary bounding function defined as

$$\rho(\|z\|) = \sqrt{(\gamma_1 + 2\gamma_2\varphi_{s_1})\rho_2^2(\|z\|) + \frac{\rho_1^2(\|z\|)}{\iota_3}} \quad (15)$$

where γ_1 and γ_2 are positive adjustable constants, and let $\hat{z} \in \mathbb{R}^{3n}$ be defined as

$$\hat{z} \triangleq [e_1^T, e_2^T, r^T]^T. \quad (16)$$

Auxiliary bounding constants $\sigma, \delta \in \mathbb{R}$ are defined as

$$\sigma \triangleq \frac{1}{2} \min \left\{ \frac{\alpha_1}{2}, \frac{\alpha_2}{2}, k_{s_1}, \frac{\omega\iota_5(1-\varphi_{i_2})}{\varphi_{i_1}} \right\} \quad (17)$$

$$\delta \triangleq \frac{1}{2} \min \left\{ \sigma, \frac{\omega\iota_7(1-\varphi_{i_2})}{\varphi_{i_1}}, \frac{\iota_6(1-\varphi_{i_2})}{\varphi_{i_1}}, \frac{\gamma_2(1-\varphi_{s_2})}{\gamma_1}, \frac{(1-\varphi_{s_2})}{2\varphi_{s_1}} \right\} \quad (18)$$

where $\omega \in \mathbb{R}$ is a known, positive, adjustable constant.

Let

$$\begin{aligned} \mathcal{D} &\triangleq \left\{ \xi \in \mathbb{R}^{3n+4} \mid \|\xi\| < \inf \left\{ \rho^{-1} \left(\left[\sqrt{2k_s\sigma}, \infty \right) \right) \right\} \right\} \\ \mathcal{S}_{\mathcal{D}} &\triangleq \left\{ \xi \in \mathcal{D} \mid \|\xi\| < \sqrt{\frac{1}{2}} \inf \left\{ \rho^{-1} \left(\left[\sqrt{2k_s\sigma}, \infty \right) \right) \right\} \right\} \end{aligned}$$

where, for a set $A \subset \mathbb{R}$, the inverse image $\rho^{-1}(A) \subset \mathbb{R}$ is defined as $\rho^{-1}(A) \triangleq \{a \in \mathbb{R} \mid \rho(a) \in A\}$. Furthermore, let the functions⁷ $P_{LK} : [0, \infty) \rightarrow [0, \infty)$, $Q_{LK} : [0, \infty) \rightarrow [0, \infty)$, $R_{LK} : [0, \infty) \rightarrow [0, \infty)$, and $S_{LK} : [0, \infty) \rightarrow [0, \infty)$ be defined as

$$P_{LK} \triangleq \varphi_{i_1} \int_{t-\tau_i}^t \|\dot{u}(\theta)\|^2 d\theta \quad (19)$$

$$Q_{LK} \triangleq \omega \int_{t-\tau_i}^t \left(\int_s^t \|\dot{u}(\theta)\|^2 d\theta \right) ds \quad (20)$$

$$R_{LK} \triangleq \frac{\gamma_1}{2k_s} \int_{t-\tau_s}^t \rho_2^2(\|z(\sigma)\|) \|z(\sigma)\|^2 d\sigma \quad (21)$$

$$S_{LK} \triangleq \frac{\gamma_2}{k_s} \int_{t-\tau_s}^t \left(\int_s^t \rho_2^2(\|z(\sigma)\|) \|z(\sigma)\|^2 d\sigma \right) ds. \quad (22)$$

Additionally, let $y \in \mathbb{R}^{3n+4}$ be defined as

$$y \triangleq [\hat{z}^T \quad \sqrt{P_{LK}} \quad \sqrt{Q_{LK}} \quad \sqrt{R_{LK}} \quad \sqrt{S_{LK}}]^T. \quad (23)$$

IV. STABILITY ANALYSIS

Theorem 1: Given the dynamics in (1), provided the control gains are selected based on the following sufficient conditions

$$\begin{aligned} \alpha_1 &> 1, \quad \alpha_2 > (1 + \iota_1), \quad \gamma_1 > \frac{1}{(1 - \varphi_{s_2})} \\ \omega &> \frac{\varphi_{i_1}}{2(1 - \varphi_{i_2})\iota_1\iota_8} \end{aligned} \quad (24)$$

⁷The construction of P_{LK} , Q_{LK} , R_{LK} , and S_{LK} is based on LK functionals. However, in this result, they are to be interpreted as time-varying signals that are a part of the system state.

and the input delay is small enough so that there exists a gain k_s that satisfies the sufficient conditions⁸

$$\varphi_{i_1} < \frac{\iota_2 k_s}{2(\omega + 1)K^2} \quad (25)$$

$$\frac{\zeta_{N_{d1}}^2}{\iota_4 k_s \delta} < \left(\inf \left\{ \rho^{-1} \left(\left[\sqrt{2k_s\sigma}, \infty \right) \right) \right\} \right)^2 \quad (26)$$

the controller given in (7) and (8) ensures uniformly ultimately bounded tracking in the sense that $\limsup_{t \rightarrow \infty} \|y\| \leq \sqrt{\zeta_{N_{d1}}^2 / \iota_4 k_s \delta}$, provided $y(t_0 + \varphi_{i_1}) \in \mathcal{S}_{\mathcal{D}}$.

Proof: Let $V : \mathcal{D} \rightarrow \mathbb{R}$ be a candidate Lyapunov function defined as

$$V \triangleq \frac{1}{2} e_1^T e_1 + \frac{1}{2} e_2^T e_2 + \frac{1}{2} r^T r + P_{LK} + Q_{LK} + R_{LK} + S_{LK} \quad (27)$$

which satisfies the following inequalities:

$$\frac{1}{2} \|y\|^2 \leq V(y) \leq \|y\|^2. \quad (28)$$

The time derivative of (27) can be found by applying the Leibniz Rule to (19)–(22), and by substituting (2)–(4), (7), and (9), yielding

$$\begin{aligned} \dot{V} &= e_1^T (e_2 - \alpha_1 e_1) + e_2^T (r - \alpha_2 e_2 - e_u) \\ &\quad + r^T (\dot{N} + N_d - e_2 - Kr) \\ &\quad + (\omega\tau_i + \varphi_{i_1}) K^2 \|r\|^2 - \varphi_{i_1} (1 - \dot{\tau}_i) \|\dot{u}_{\tau_i}\|^2 \\ &\quad - \omega(1 - \dot{\tau}_i) \int_{t-\tau_i}^t \|\dot{u}(\theta)\|^2 d\theta \\ &\quad + \left(\frac{\gamma_1}{2k_s} + \frac{\gamma_2}{k_s} \tau_s \right) \rho_2^2(\|z\|) \|z\|^2 \\ &\quad - \frac{\gamma_1(1 - \dot{\tau}_s)}{2k_s} \rho_2^2(\|z_{\tau_s}\|) \|z_{\tau_s}\|^2 \\ &\quad - \frac{\gamma_2}{k_s} (1 - \dot{\tau}_s) \int_{t-\tau_s}^t \rho_2^2(\|z(\theta)\|) \|z(\theta)\|^2 d\theta. \end{aligned} \quad (29)$$

Using (4), (12), (13), the inequality $\dot{\tau}_s < 1$ and the Young's inequality to show that $\|e_1^T e_2\| \leq (1/2)\|e_1\|^2 + (1/2)\|e_2\|^2$, $\|e_2^T e_u\| \leq (\iota_1/2)\|e_2\|^2 + (1/2\iota_1)\|e_u\|^2$ and $\|r\|\rho_2(\|z_{\tau_s}\|)\|z_{\tau_s}\| \leq (k_s/2)\|r\|^2 + (1/2k_s)\rho_2^2(\|z_{\tau_s}\|)\|z_{\tau_s}\|^2$, the expression in (29) can be upper bounded as

$$\begin{aligned} \dot{V} &\leq -\alpha_1 \|e_1\|^2 - \alpha_2 \|e_2\|^2 - \left(\frac{k_s}{2} + k_{s_1} \right) \|r\|^2 \\ &\quad + \frac{1}{2} \|e_1\|^2 + \left(\frac{1}{2} + \frac{\iota_1}{2} \right) \|e_2\|^2 + \frac{1}{2\iota_1} \|e_u\|^2 \\ &\quad + \frac{1}{2k_s} \rho_2^2(\|z_{\tau_s}\|) \|z_{\tau_s}\|^2 - \frac{\gamma_1(1 - \dot{\tau}_s)}{2k_s} \rho_2^2(\|z_{\tau_s}\|) \|z_{\tau_s}\|^2 \\ &\quad + \varphi_{i_1} (1 + \omega) K^2 \|r\|^2 + \|r\| \zeta_{N_{d1}} \\ &\quad - \omega(1 - \dot{\tau}_i) \int_{t-\tau_i}^t \|\dot{u}(\theta)\|^2 d\theta + \|r\| \rho_1(\|z\|) \|z\| \\ &\quad + \left(\frac{\gamma_1}{2k_s} + \frac{\gamma_2}{k_s} \tau_s \right) \rho_2^2(\|z\|) \|z\|^2 \\ &\quad - \frac{\gamma_2}{k_s} (1 - \dot{\tau}_s) \int_{t-\tau_s}^t \rho_2^2(\|z(\theta)\|) \|z(\theta)\|^2 d\theta. \end{aligned} \quad (30)$$

⁸Since δ increases with increasing k_s , the left-hand side of (26) decreases with increasing k_s . Since ρ is a nondecreasing function, the right-hand side of (26) is nondecreasing with respect to k_s . Hence, (26) can be satisfied for some k_s . Furthermore, for any given k_s , (25) is satisfied if the delay is small enough.

TABLE I
 RMS ERRORS FOR TIME-VARYING TIME-DELAY RATES AND MAGNITUDES

	τ_s (ms)	τ_i (ms)	RMS Error	
			x_1	x_2
Case 1	$5 \sin\left(\frac{t}{8}\right) + 10$	$-50 \sin\left(\frac{t}{10}\right) + 100$	1.42^o	1.36^o
Case 2	$10 \sin\left(\frac{t}{2}\right) + 40$	$-10 \sin\left(\frac{t}{3}\right) + 30$	0.32^o	0.36^o
Case 3	$10 \sin\left(\frac{t}{10}\right) + 40$	$-50 \sin\left(\frac{t}{2}\right) + 100$	1.15^o	1.22^o
Case 4	$10 \sin\left(\frac{t}{10}\right) + 40$	$50 \sin\left(\frac{t}{10}\right) + 100$	1.54^o	1.83^o
Case 5	$50 \sin(t) + 800$	$5 \sin\left(\frac{t}{2}\right) + 30$	0.23^o	0.37^o

Using the fact that $e_u = \int_{t-\tau(t)}^t \dot{u}(\theta) d\theta$, the Cauchy-Schwarz inequality can be used to establish the bound $\|e_u\|^2 \leq \tau_i \int_{t-\tau_i}^t \|\dot{u}(\theta)\|^2 d\theta$. Completing the squares for r , utilizing the inequalities

$$-\int_{t-\tau_i}^t \|\dot{u}(\theta)\|^2 d\theta \leq -\frac{1}{\tau_i} \int_{t-\tau_i}^t \left(\int_s^t \|\dot{u}(\theta)\|^2 d\theta \right) ds = -\frac{Q_{LK}}{\omega \tau_i}$$

and

$$-\int_{t-\tau_s}^t \rho_2^2(\|z(\theta)\|) \|z(\theta)\|^2 d\theta \leq -\frac{1}{\tau_s} \int_{t-\tau_s}^t \left(\int_s^t \rho_2^2(\|z(\theta)\|) \|z(\theta)\|^2 d\theta \right) ds = -\frac{k_s S_{LK}}{\gamma_2 \tau_s}$$

and (19), (21), and (26), (30) can be rewritten as

$$\begin{aligned} \dot{V} \leq & -\frac{\alpha_1}{2} \|e_1\|^2 - \frac{\alpha_2}{2} \|e_2\|^2 - k_{s1} \|r\|^2 - \frac{\omega \iota_5 (1 - \dot{\tau}_i)}{\tau_i} \|e_u\|^2 \\ & - \left(\frac{\alpha_1}{2} - \frac{1}{2} \right) \|e_1\|^2 - \left(\frac{\alpha_2}{2} - \frac{1}{2} - \frac{\iota_1}{2} \right) \|e_2\|^2 \\ & - \left(\frac{\iota_2 k_s}{2} - \varphi_{i1} (1 + \omega) K^2 \right) \|r\|^2 + \frac{\zeta_{N_{d1}}^2}{2 \iota_4 k_s} \\ & - \left(\frac{\omega \iota_8 (1 - \dot{\tau}_i)}{\tau_i} - \frac{1}{2 \iota_1} \right) \|e_u\|^2 - \frac{(1 - \dot{\tau}_s)}{2 \tau_s} S_{LK} \\ & - \left(\frac{\gamma_1 (1 - \dot{\tau}_s)}{2 k_s} - \frac{1}{2 k_s} \right) \rho_2^2(\|z_{\tau_s}\|) \|z_{\tau_s}\|^2 \\ & + \frac{1}{2 k_s} \left(\frac{\rho_1^2(\|z\|)}{\iota_3} + (\gamma_1 + 2 \gamma_2 \varphi_{s1}) \rho_2^2(\|z\|) \right) \|z\|^2 \\ & - \frac{\omega \iota_7 (1 - \dot{\tau}_i)}{\varphi_{i1}} P_{LK} - \frac{\iota_6 (1 - \dot{\tau}_i)}{\tau_i} Q_{LK} - \frac{\gamma_2 (1 - \dot{\tau}_s)}{\gamma_1} R_{LK}. \quad (31) \end{aligned}$$

If the conditions in (24) are satisfied, based on the inequalities $\|z\|^2 \geq \|\dot{z}\|^2$ and $\|z\| \leq \|y\|$, the expression in (31) reduces to

$$\begin{aligned} \dot{V} \leq & -\sigma \|\dot{z}\|^2 - \frac{\omega \iota_7 (1 - \varphi_{i2})}{\varphi_{i1}} P_{LK} - \frac{\iota_6 (1 - \varphi_{i2})}{\varphi_{i1}} Q_{LK} \\ & - \frac{\gamma_2 (1 - \varphi_{s2})}{\gamma_1} R_{LK} - \frac{(1 - \varphi_{s2})}{2 \varphi_{s1}} S_{LK} + \frac{\zeta_{N_{d1}}^2}{2 \iota_4 k_s} \\ & \leq -\delta \|y\|^2, \quad \forall \|y\| \geq \sqrt{\frac{\zeta_{N_{d1}}^2}{2 \iota_4 k_s \delta}} \quad (32) \end{aligned}$$

provided $y \in \mathcal{D}$, where $\rho(\|z\|)$, σ , and δ were introduced in (15), (17) and (18). Using (26), (28), and (32), [54, Theorem 4.18] can be invoked to conclude that y is uniformly ultimately bounded in the sense that $\limsup_{t \rightarrow \infty} \|y\| \leq \sqrt{\zeta_{N_{d1}}^2 / \iota_4 k_s \delta}$, provided $y(t_0 + \varphi_{i1}) \in \mathcal{S}_{\mathcal{D}}$.

Since $e_1, e_2, r \in \mathcal{L}_{\infty}$, from (6), $u \in \mathcal{L}_{\infty}$, which implies $u_{\tau_i} \in \mathcal{L}_{\infty}$, and hence, $e_u \in \mathcal{L}_{\infty}$. The closed-loop error system can then be used to conclude that the remaining signals are bounded. ■

Remark 2: If the system dynamics are such that $\|\tilde{N}\|$ is linear in $\|z\|$, then the function ρ can be selected to be a constant, i.e., $\rho(\|z\|) = \bar{\rho}$, $\forall z \in \mathbb{R}^{4n}$ for some known $\bar{\rho} > 0$. In this case, the gain condition in (26) reduces to $k_s > \bar{\rho}/2\sigma$, and the result is global in the sense that $\mathcal{D} = \mathcal{S}_{\mathcal{D}} = \mathbb{R}^{3n+4}$.

V. SIMULATION RESULTS

The controller in (7) and (8) was simulated to examine the performance and robustness to variations in both the state and input delay. Specifically the dynamics⁹ from (1) are utilized where $n = 2$, $f \triangleq \begin{bmatrix} -p_4 s_2 \\ p_5 s_2 \dot{x}_2 \end{bmatrix}$, $g \triangleq \begin{bmatrix} -p_3 s_2 \tau_s \dot{x}_{2\tau_s} & -p_3 s_2 \tau_s (\dot{x}_{1\tau_s} + \dot{x}_{2\tau_s}) \\ p_3 s_2 \tau_s \dot{x}_{1\tau_s} & 0 \end{bmatrix}$, $\begin{bmatrix} \dot{x}_{1\tau_s} \\ \dot{x}_{2\tau_s} \end{bmatrix} + \begin{bmatrix} f_{d1} & 0 \\ 0 & f_{d2} \end{bmatrix} \begin{bmatrix} \dot{x}_{1\tau_s} \\ \dot{x}_{2\tau_s} \end{bmatrix}$, and $x, \dot{x}, \ddot{x} \in \mathbb{R}^2$. A disturbance $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ was applied where $d_1 = 0.2 \sin(t/2)$ and $d_2 = 0.1 \sin(t/4)$. Additionally, $p_1 = 3.473$, $p_2 = 0.196$, $p_3 = 0.242$, $p_4 = 0.238$, $p_5 = 0.146$, $f_{d1} = 5.3$, $f_{d2} = 1$, and $s_2, s_{2\tau_s}$ denote $\sin(x_2)$ and $\sin(x_2(t - \tau_s))$.

The initial conditions for the system are selected as $x_1, x_2 = 0$. The desired trajectories are selected as

$$\begin{aligned} x_{d1} &= (30 \sin(1.5t) + 20) \left(1 - e^{-0.01t^3} \right) \\ x_{d2} &= -(20 \sin(t/2) + 10) \left(1 - e^{-0.01t^3} \right). \end{aligned}$$

To illustrate robustness to the delays, several simulations are completed using various time-varying delays. Various time delay combinations are considered, and for each case the root mean square (RMS) errors are shown in Table I.¹⁰ To illustrate the findings, Fig. 1 depicts the tracking errors, actuation effort and time-varying delays for Case 3 in Table I. Additionally, Case 5 in Table I is provided in Fig. 2. The results indicate that the performance of the system is relatively less sensitive to the delay frequency and more sensitive to the delay magnitude. This outcome agrees with previous input delay results where the tracking performance degrades as larger delays are applied to the system [43], [50]. In general, the simulation results illustrate that the proposed controller is able to achieve better tracking performance as well as tolerate larger input delays (even with added simultaneous state delays) than the time-varying input-delayed work in [50]. The controller is more robust to larger magnitude delays in the state than in the input, as indicated in the stability analysis and as demonstrated in Case 5.

⁹The expressions for f and g are derived from the expression for a two-link revolute, direct drive robot, which is a commonly used example Euler-Lagrange dynamic system.

¹⁰The simulation results indicate that the condition $|\dot{\tau}_i| \leq \varphi_{i2} < 1$ in Assumption 4 is sufficient in the sense that the developed controller can compensate for fast variations in the time delay.

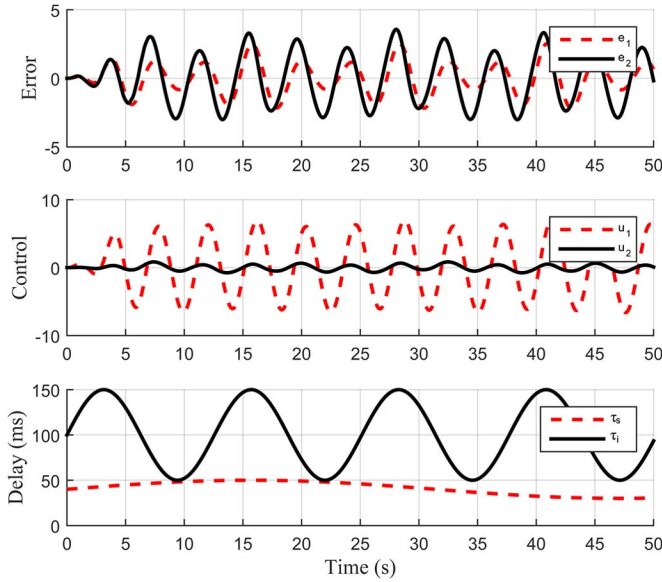


Fig. 1. Tracking errors, actuation effort and time-varying delays vs time for Case 3.

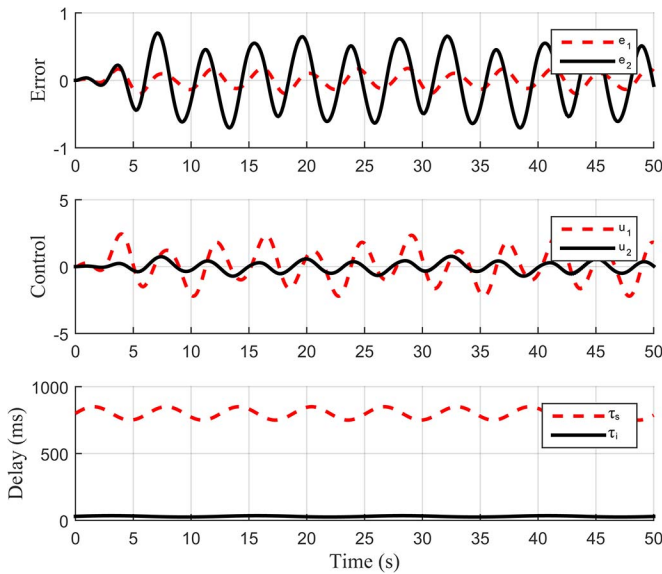


Fig. 2. Tracking errors, actuation effort and time-varying delays vs time for Case 5.

VI. CONCLUSION

This technical note presents a robust controller for uncertain nonlinear systems which include simultaneous time-varying state and input delays, as well as sufficiently smooth additive bounded disturbances. The controller utilizes a robust design approach to compensate for the unknown state delays coupled with an error system structure that provides a delay-free open-loop error system. The controller and LK functionals guarantee uniformly ultimately bounded tracking provided the rates of the delays are sufficiently slow. The controller can be applied when there is uncertainty in the system dynamics and when the state delay is unknown; however, the controller is based on the assumption that the time-varying input delay is known. Numerical simulations examine the robustness of the method to various combinations of simultaneous input and state delays. The simulation results illustrate robustness to uncertainty in the magnitude and frequency of the input

delay and the state delay. These results point to the possibility that different control or analysis methods could be developed to eliminate the assumption that the input delay is known. That is, perhaps the interval of previous control values could somehow be designed big enough to provide predictive properties despite uncertainty in the input delay.

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