Adaptive Control of Time-Varying Parameter Systems With Asymptotic Tracking

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***Abstract*—A continuous adaptive controller is developed for nonlinear dynamical systems with linearly parameterizable uncertainty involving time-varying uncertain parameters. Through a unique stability analysis strategy, a new adaptive feedforward term is developed along with specialized feedback terms, to yield an asymptotic tracking error convergence result by compensating for the time-varying nature of the uncertain parameters. A Lyapunov-based stability analysis is shown for Euler–Lagrange systems, which ensures asymptotic tracking error convergence and boundedness of the closed-loop signals. Additionally, the time-varying uncertain function approximation error is shown to converge to zero. A simulation example of a two-link manipulator is provided to demonstrate the asymptotic tracking result.**

***Index Terms*—Adaptive control, Lyapunov methods, time-varying systems.**

I. Introduction

Adaptive control of nonlinear dynamical systems with time-varying uncertain parameters is an open and practically relevant problem. It has been well established that traditional gradient-based update laws can compensate for constant unknown parameters yielding asymptotic convergence. Moreover, the development of robust modifications of such adaptive update laws result in uniformly ultimately bounded (UUB) results for slowly varying parametric uncertainty using a Lyapunov-based analysis, under the assumption of bounded parameters and their time derivatives [1]–[3].

More recent results focus on tracking and parameter estimation improvement using various adaptive control approaches for systems with unknown time-varying parameters. One such approach involves a fast adaptation law [4], where a matrix of time-varying learning rates is utilized to improve the tracking and estimation performance under a finite excitation condition. Another approach uses a set-theoretic control architecture [5] to reject the effects of parameter variation, while restricting the system error within a prescribed performance bound. While the aforementioned approaches can potentially yield improved transient response, they yield UUB error systems.

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Motivation exists to obtain asymptotic convergence of the tracking error to zero, despite the time-varying nature of the uncertain parameters. Results such as [6] and [7] yield asymptotic tracking for linear systems with asymptotically vanishing time-varying parameter variations. For nonlinear systems involving periodic time-varying uncertain parameters with known periodicity, repetitive/iterative learning based approaches such as [8] yield asymptotic tracking. However, it is challenging to extend these results to nonlinear systems where the uncertain parameter variation is nonvanishing and aperiodic.

Robust adaptive control approaches such as [9, Sec. IV] yield asymptotic adaptive tracking for systems with time-varying uncertain parameters using an adaptive sliding mode-like design, and [9, Sec. VII] and [10] use a continuous robust design. However, such approaches exploit high-gain or high-frequency feedback without any additional adaptive feedforward term that is specifically designed to target the uncertainty through adaptation. Recent results in [11] yield asymptotic tracking using a method called congelation of variables, where each unknown time-varying parameter is treated as a nominal constant unknown parameter perturbed by a time-varying perturbation, and the control input consists of an adaptive feedforward term to compensate for the nominal constant parameters, while a robust high-gain term is designed to compensate the time-varying perturbation. While the congelation of variables based approach can compensate for fast-varying parameters, it requires the regression matrix to vanish with the state, which might be restrictive for a wide variety of applications.

Results such as [12]–[14] investigate the identification of systems with time-varying parameters. A more recent result in [15] utilizes the dynamic regressor extension and mixing technique to yield finite-time parameter convergence for systems with unknown piecewise linearly time-varying parameters. Note that these results concern only adaptive parameter estimation, without developing an adaptive feedforward control term for closed-loop implementation.

In the field of fault-tolerant control design, system faults are typically modeled as unknown piecewise constant time-varying parameters such as in [16], for which, classical adaptive control techniques are used. In this article, we consider the more challenging problem of continuously time-varying parameters, which necessitates an alternative adaptive update law.

To illustrate the technical challenges associated with developing an adaptive feedforward term for systems with time-varying parametric uncertainty, consider the scalar dynamical system[1]

$$\dot{x}(t) = a(t)x(t) + b(t)\cos(x(t)) + u(t) \tag{1}$$

[1]Note that the system (1) is considered only for illustrative purpose. This article presents the result for a general nonlinear Euler–Lagrange system with a vector state and a linearly parameterizable uncertainty with time-varying parameters.

with the controller $u(t) = -kx(t) - \hat{a}(t)x(t) - \hat{b}(t)\cos(x(t))$, where k is a positive constant gain; $a(t)$ and $b(t)$ are unknown time-varying parameters; $\hat{a}(t)$ and $\hat{b}(t)$ are the parameter estimates of $a(t)$ and $b(t)$, respectively; and the parameter estimation errors $\tilde{a}(t)$ and $\tilde{b}(t)$ are defined as $\tilde{a}(t) \triangleq a(t) - \hat{a}(t)$ and $\tilde{b}(t) \triangleq b(t) - \hat{b}(t)$, respectively. The traditional stability analysis approach for such problems is to consider the candidate Lyapunov function $V(x(t), \tilde{a}(t), \tilde{b}(t)) = \frac{1}{2}x^2(t) + \frac{1}{2\gamma_a}\tilde{a}^2(t) + \frac{1}{2\gamma_b}\tilde{b}^2(t)$, where γ_a and γ_b are positive constant gains. The given definitions and controller yield the following time-derivative of the candidate Lyapunov function: $\dot{V}(t) = -kx^2(t) + \tilde{a}(t)x^2(t) + \tilde{b}(t)x(t)\cos(x(t)) + \frac{\tilde{a}(t)}{\gamma_a}(\dot{a}(t) - \dot{\hat{a}}(t)) + \frac{\tilde{b}(t)}{\gamma_b}(\dot{b}(t) - \dot{\hat{b}}(t))$. For the constant parameter case, i.e., $\dot{a}(t) = \dot{b}(t) = 0$, the well-known adaptive update laws $\dot{\hat{a}}(t) = \gamma_a x^2(t)$ and $\dot{\hat{b}}(t) = \gamma_b x(t)\cos(x(t))$, respectively, will cancel $\tilde{a}(t)x^2(t)$ and $\tilde{b}(t)x(t)\cos(x(t))$ in $\dot{V}(t)$, leading to Lyapunov stability and asymptotic tracking. However, when the parameters are time-varying, it is unclear how to address $\dot{a}(t)$ and $\dot{b}(t)$ via a feedforward adaptive update law, such that $\dot{V}(t)$ becomes at least negative semidefinite. Alternatively to obtain a negative semidefinite derivative of the Lyapunov-like function (which is a contribution of this article), the typical approach to design adaptive controllers for the time-varying parameter case is to consider a robust modification of the update laws and assume some constant upper bounds on $|a(t)|, |b(t)|$, $|\dot{a}(t)|$, and $|\dot{b}(t)|$ to obtain a UUB result. For instance, consider a standard gradient update law with sigma-modification [3], $\dot{\hat{a}}(t) = \gamma_a x^2(t) - \gamma_a \sigma \hat{a}(t)$ and $\dot{\hat{b}}(t) = \gamma_b x(t)\cos(x(t)) - \gamma_b \sigma \hat{b}(t)$, which yields $\dot{V}(t) = -kx^2(t) - \sigma\tilde{a}^2(t) - \sigma\tilde{b}^2(t) + \tilde{a}(t)(\frac{\dot{a}(t)}{\gamma_a} + \sigma a(t)) + \tilde{b}(t)(\frac{\dot{b}(t)}{\gamma_b} + \sigma b(t))$, implying a UUB result when the parameters $a(t)$ and $b(t)$, and their time-derivatives $\dot{a}(t)$ and $\dot{b}(t)$ are bounded. More modern approaches (cf., [4]) provide additional modifications to yield UUB results with improved transient performance.

It would be desirable to have a sliding-mode like term based on $\tilde{a}(t)$ and $\tilde{b}(t)$ (i.e., $\mathrm{sgn}(\tilde{a})$ and $\mathrm{sgn}(\tilde{b})$ in the adaptation law) if only $\tilde{a}(t)$ and $\tilde{b}(t)$ were known. Another approach could be to use a pure robust controller, e.g., $u(t) = -kx(t) - \bar{a}x(t) - \bar{b}\mathrm{sgn}(x(t))$, where $\bar{a}$ and $\bar{b}$ are known constant upper-bounds on the norms of parameters $|a(t)|$ and $|b(t)|$, respectively. If the bounds $\bar{a}$ and $\bar{b}$ are unknown, an adaptation law could be designed to yield their adaptive estimates, i.e., $\hat{\bar{a}}$ and $\hat{\bar{b}}$. Either of these approaches would yield an asymptotic tracking result (cf., [9]), but as stated earlier, these approaches require a discontinuous pure sliding-mode term in the control input, and do not include an adaptive feedforward term to compensate for the uncertainty. The congelation of variable-based approach in [11] may help avoid some of the aforementioned challenges; however, it is not applicable for uncertain terms like $b(t)\cos(x(t))$, which do not vanish with the state.

The major challenge in achieving asymptotic tracking is that the time-derivative of the parameter acts like an unknown exogenous disturbance in the parameter estimation dynamics, which is difficult to cancel with an adaptive update law in a Lyapunov-based stability analysis. We address this technical challenge through new insights into the closed-loop error system development and stability analysis, coupled with a new adaptive update law design. Specifically, because of the challenges associated with including the uncertain parameter estimation error in the Lyapunov function, we omit such terms, and include a P-function based on [17], while also formulating the closed-loop error system so that they appear in the Lyapunov-based derivative in a manner that facilitates an adaptive update law. We address the unique challenge associated with incorporating the time-varying parameter estimation error in the analysis by formulating the update law so that it contains a signum function of the tracking error term multiplied by a desired regressor. The update law also involves a projection algorithm to ensure that the parameter estimates stay within a known bounded set. However, the projection algorithm introduces a potentially destabilizing term in the time-derivative of the candidate Lyapunov function, leading to an additional technical obstacle to obtain asymptotic tracking. This challenge is resolved by using an additional term in the control input, which compensates for terms that result from using a projection operator. The developed Lyapunov-based stability analysis yields semiglobal asymptotic tracking and boundedness of the closed-loop signals. Additionally, the time-varying uncertain function approximation error is shown to converge to zero. The dynamics of a two-link manipulator are used in a simulation to demonstrate the asymptotic tracking and function approximation error convergence result, and the tracking performance is compared with a robust e-modification update law [18] based controller.

II. Dynamic Model

The subsequent development is based on the general uncertain nonlinear Euler–Lagrange (EL) dynamics given by [20, Sec. 2.2]

$$\begin{aligned} &M(q(t),t)\ddot{q}(t) + V_m(q(t),\dot{q}(t),t)\dot{q}(t) \\ &+G(q(t),t) + F(\dot{q}(t),t) + \tau_d(t) = \tau(t) \end{aligned} \tag{2}$$

where $t \in [t_0, \infty)$ denotes time, $t_0 \in \mathbb{R}_{\geq 0}$ denotes the initial time, $q : [t_0, \infty) \to \mathbb{R}^n$ denotes a vector of generalized positions, $M : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^{n\times n}$ denotes a generalized inertia matrix, $V_m : \mathbb{R}^n \times \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^{n\times n}$ denotes the Coriolis and centrifugal forces matrix, $G : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^n$ denotes a generalized vector of potential forces, $F : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^n$ denotes a generalized vector of dissipation, $\tau_d : [t_0, \infty) \to \mathbb{R}^n$ represents an exogenous disturbance acting on the system, and $\tau : [t_0, \infty) \to \mathbb{R}^n$ represents a generalized control input vector [20, Ch. 2]. The subsequent development is based on the assumption that only $q(t)$ and $\dot{q}(t)$ are measurable. The following assumptions about the EL system are made in the subsequent development [20, Sec. 2.3].

Assumption 1: The inertia matrix satisfies $m_1\|\xi\|^2 \leq \xi^T M(q(t),t)\xi \leq \bar{m}(q)\|\xi\|^2 \,\forall\, \xi \in \mathbb{R}^n$, where $m_1 \in \mathbb{R}_{>0}$ is a known bounding constant, $\bar{m} : \mathbb{R}^n \to \mathbb{R}_{>0}$ is a known bounding function, and $\|\cdot\|$ denotes the Euclidean norm for a vector argument or the spectral norm for a matrix argument.

Assumption 2: The functions $M(q(t),t)$, $V_m(q(t),\dot{q}(t),t)$, $G(q(t),t)$, and $F(\dot{q}(t),t)$ are second order differentiable such that their second time derivatives are bounded if $q^{(i)} \in \mathcal{L}_\infty \,\forall\, i = 0,1,2,3$, where $\mathcal{L}_\infty$ denotes the space of essentially bounded Lebesgue-measurable functions.

Assumption 3: The dynamics in (2) can be linearly parameterized[2] as

$$\begin{aligned} Y_p(q(t),\dot{q}(t),\ddot{q}(t),t)\theta_p(t) = {} & M(q(t),t)\ddot{q}(t) \\ & + F(\dot{q}(t),t) + G(q(t),t) \\ & + V_m(q(t),\dot{q}(t),t)\dot{q}(t) \end{aligned} \tag{3}$$

where $Y_p : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^{n\times m}$ is a known regression matrix and $\theta_p : [t_0, \infty) \to \mathbb{R}^m$ is a vector of time-varying unknown parameters.

[2]A linear parameterization is considered for simplicity. For systems that do not satisfy the linear-in-the-parameters assumption, the parameterization can yet be linearized according to [21, eq. (7)], where the linearization error can be upper bounded using [21, Lemma 1]. Subsequently, the adaptive design approach of this article is then applicable.

The disturbance parameter vector $\tau_d(t)$ can be appended to the $\theta_p(t)$ vector, yielding an augmented parameter vector $\theta : [t_0, \infty) \to \mathbb{R}^{n+m}$ as

$$\theta(t) \triangleq \begin{bmatrix} \theta_p(t) \\ \tau_d(t) \end{bmatrix} \tag{4}$$

and the augmented regressor $Y : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^{n\times(n+m)}$ can be designed as

$$Y(q(t), \dot{q}(t), \ddot{q}(t), t) \triangleq \left[Y_p(q(t), \dot{q}(t), \ddot{q}(t), t) \;\; I_n \right]. \tag{5}$$

Substituting the parameterization in (3)–(5) into (2) yields

$$\begin{aligned} M(q(t), t)\ddot{q}(t) &+ F(\dot{q}(t), t) \\ &+ V_m(q(t), \dot{q}(t), t)\dot{q}(t) \\ &+ G(q(t), t) + \tau_d(t) = Y(q(t), \dot{q}(t), \ddot{q}(t), t)\theta(t) \end{aligned} \tag{6}$$

where $Y(q(t), \dot{q}(t), \ddot{q}(t), t)\theta(t) = \tau(t)$.

Assumption 4: The time-varying augmented parameter $\theta(t)$ and its time derivatives, i.e., $\dot{\theta}(t)$ and $\ddot{\theta}(t)$ are bounded by known constants, i.e., $\|\theta(t)\| \leq \zeta_0$, $\|\dot{\theta}(t)\| \leq \zeta_1$ and $\|\ddot{\theta}(t)\| \leq \zeta_2$, where $\zeta_0, \zeta_1, \zeta_2 \in \mathbb{R}_{>0}$ are known bounding constants.

Remark 1: For practical applications, it is often not difficult to develop sufficiently large bounds on uncertain parameters or their rate of change. For example, variation in a friction coefficient due to wear is difficult to model, but it is not difficult to obtain an upper bound on the friction coefficient. Similarly, it is possible to develop an upper bound on the inertia and drag coefficient parameters of an aircraft. We refer the reader to the result in [22, Sec. 4] for an example of an aerospace system with bounded time-varying parameters. For systems with unknown bounds, robust adaptive control methods such as [9, Sec. IV] may provide insight for a solution, but such an extension is beyond the scope of the contributions of this article.

III. Control Design

A. Control Objective

The objective is to design a controller such that the state tracks a smooth bounded reference trajectory, despite the time-varying nature of the uncertain parameters. The objective is quantified by defining the tracking error $e_1 : [t_0, \infty) \to \mathbb{R}^n$ as[3]

$$e_1 \triangleq q - q_d \tag{7}$$

where $q_d : [t_0, \infty) \to \mathbb{R}^n$ is a desired trajectory. To facilitate the subsequent analysis, filtered tracking errors e_2 and $r : [t_0, \infty) \to \mathbb{R}^n$ are defined as

$$e_2 \triangleq \dot{e}_1 + \alpha_1 e_1 \tag{8}$$

$$r \triangleq \dot{e}_2 + \alpha_2 e_2 \tag{9}$$

respectively, where $\alpha_1, \alpha_2 \in \mathbb{R}_{>0}$ are constant control gains.

Assumption 5: The desired trajectory $q_d(t)$ is bounded and smooth, such that $\|q_d(t)\| \leq \delta_0$, $\|\dot{q}_d(t)\| \leq \delta_1$, and $\|\ddot{q}_d(t)\| \leq \delta_2$, where δ_0, δ_1, and $\delta_2 \in \mathbb{R}_{>0}$ are known bounding constants.

Substituting (7)–(9) into (6) yields the open-loop error system

$$M(q, t)r = \tau + S(t) - Y_d\theta(t) \tag{10}$$

where $S(t) \triangleq V_m(q_d, \dot{q}_d, t)\dot{q}_d - V_m(q, \dot{q}, t)\dot{q} + G(q_d, t) - G(q, t) + F(\dot{q}_d, t) - F(\dot{q}, t) + (M(q_d, t) - M(q, t))\ddot{q}_d + M(q, t)(\alpha_1(e_2 - \alpha_1 e_1) + \alpha_2 e_2)$ and $Y_d \triangleq Y(q_d, \dot{q}_d, \ddot{q}_d, t)$ denotes the desired regression matrix.

B. Control and Update Law Development

From the subsequent stability analysis, the continuous control input is designed as

$$\tau \triangleq Y_d\hat{\theta} - ke_2 + \mu \tag{11}$$

where $k \in \mathbb{R}_{>0}$ is a constant control gain, $\mu : [t_0, \infty) \to \mathbb{R}^n$ is a subsequently defined auxiliary control term, and $\hat{\theta} : [t_0, \infty) \to \mathbb{R}^{n+m}$ denotes the parameter estimate of $\theta(t)$. Substituting the control input in (11) into the open-loop error system in (10) yields the following closed-loop error system:

$$M(q, t)r = -Y_d\tilde{\theta}(t) + \mu - ke_2 + S(t) \tag{12}$$

where $\tilde{\theta} : [t_0, \infty) \to \mathbb{R}^{n+m}$ denotes the parameter estimation error, i.e., $\tilde{\theta}(t) \triangleq \theta(t) - \hat{\theta}(t)$. Taking the time-derivative of (12) yields

$$\begin{aligned} M(q, t)\dot{r} = &- \dot{M}(q, t)r - \dot{Y}_d\tilde{\theta}(t) - Y_d\dot{\theta}(t) \\ &+ Y_d\dot{\hat{\theta}} - k\dot{e}_2 + \dot{\mu} + \dot{S}(t). \end{aligned} \tag{13}$$

The control variables $\dot{\hat{\theta}}(t)$ and $\dot{\mu}(t)$ now appear in the higher-order dynamics in (13), and these control variables are designed with the use of a continuous projection algorithm [23, Appendix E]. The projection algorithm constrains $\hat{\theta}(t)$ to lie inside a bounded convex set $\mathcal{B} = \{\sigma \in \mathbb{R}^{(n+m)} | \|\sigma\| \leq \zeta_0\}$ by switching the adaptation law to its component tangential to the boundary of the set $\mathcal{B}$ when $\hat{\theta}(t)$ reaches the boundary. A continuously differentiable convex function $f : \mathbb{R}^{(n+m)} \to \mathbb{R}$ is used to describe the boundaries of the bounded convex set $\mathcal{B}$ such that $f(\sigma) < 0 \; \forall \|\sigma\| < \zeta_0$ and $f(\sigma) = 0 \; \forall \|\sigma\| = \zeta_0$. Based on the subsequent analysis, the continuous adaptation law is designed as

$$\begin{aligned} \dot{\hat{\theta}} &\triangleq \text{proj}(\Lambda_0) \\ &= \begin{cases} \Lambda_0, & ||\hat{\theta}|| < \zeta_0 \vee (\nabla f(\hat{\theta}))^T\Lambda_0 \leq 0 \\ \Lambda_1, & ||\hat{\theta}|| \geq \zeta_0 \wedge (\nabla f(\hat{\theta}))^T\Lambda_0 > 0 \end{cases} \end{aligned} \tag{14}$$

where $||\hat{\theta}(0)|| < \zeta_0$, and $\vee$ and $\wedge$ denote the logical "or," "and" operators, respectively; ∇ represents the gradient operator, i.e., $\nabla f(\hat{\theta}) = \left[\frac{\partial f}{\partial \phi_1} \; \cdots \; \frac{\partial f}{\partial \phi_{n+m}} \right]^T_{\phi=\hat{\theta}}$; and $\Lambda_0, \Lambda_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n+m}$ are designed as[4]

$$\Lambda_0 \triangleq -\Gamma Y_d^T(Y_d\Gamma Y_d^T)^{-1}\left[k\alpha_2 e_2 + \beta\text{sgn}(e_2)\right] \tag{15}$$

$$\Lambda_1 \triangleq \left(I_{m+n} - \frac{(\nabla f(\hat{\theta}))(\nabla f(\hat{\theta}))^T}{||\nabla f(\hat{\theta})||^2} \right)\Lambda_0 \tag{16}$$

respectively. In (15), $\beta \in \mathbb{R}_{>0}$ is a constant control gain, and $\Gamma \in \mathbb{R}^{(n+m)\times(n+m)}$ is a constant, positive-definite, control gain matrix with a block diagonal structure, i.e., $\Gamma \triangleq \begin{bmatrix} \Gamma_1 & 0_{m\times n} \\ 0_{n\times m} & \Gamma_2 \end{bmatrix}$, with $\Gamma_1 \in \mathbb{R}^{m\times m}$, $\Gamma_2 \in \mathbb{R}^{n\times n}$ and $I_{m+n} \in \mathbb{R}^{(n+m)\times(n+m)}$ is an identity matrix. The continuous auxiliary term $\mu(t)$ used in the control input in (11) is designed as a generalized solution to

$$\dot{\mu} \triangleq Y_d\left(\Lambda_0 - \text{proj}(\Lambda_0)\right) \tag{17}$$

[3]Time-dependency is suppressed for the sake of brevity, except where explicit time-dependency adds clarity.

[4]Lemma 1 in the Appendix proves that $Y_d\Gamma Y_d^T$ is invertible.

where $\mu(t_0) = 0$. Substituting (14) and (17) into (13), the closed-loop error system can be obtained as

$$M(q,t)\dot{r} = -\dot{M}(q,t)r - \dot{Y}_d\tilde{\theta}(t) - Y_d\dot{\theta}(t) - \beta\text{sgn}(e_2) - kr + \dot{S}(t) \quad (18)$$

for both cases, i.e., when $||\hat{\theta}|| < \zeta_0 \vee (\nabla f(\hat{\theta}))^T\Lambda_0 \leq 0$ or $||\hat{\theta}|| \geq \zeta_0 \wedge (\nabla f(\hat{\theta}))^T\Lambda_0 > 0$. Let

$$z \triangleq \left[e_1^T \; e_2^T \; r^T \right]^T \in \mathbb{R}^{3n} \quad (19)$$

denote a composite error vector. To facilitate the subsequent analysis, (18) can be rewritten as

$$M(q,t)\dot{r} = -\frac{1}{2}\dot{M}(q,t)r + \widetilde{N}(z,t) + N_B(\tilde{\theta},t) - \beta\text{sgn}(e_2) - kr - e_2 \quad (20)$$

where $\widetilde{N} : \mathbb{R}^{3n} \times [t_0,\infty) \to \mathbb{R}^n$ and $N_B : \mathbb{R}^{n+m} \times [t_0,\infty) \to \mathbb{R}^n$ are defined as $\widetilde{N}(z,t) \triangleq -\frac{1}{2}\dot{M}(q,t)r + \dot{S}(t) + e_2$ and $N_B(\tilde{\theta},t) \triangleq -\dot{Y}_d\tilde{\theta} - Y_d\dot{\theta}(t)$, respectively. The mean value theorem (MVT) can be used to develop the following upper bound on the term $\widetilde{N}(z,t)$:

$$||\widetilde{N}(z,t)|| \leq \rho(||z||)||z|| \quad (21)$$

where $\rho : \mathbb{R}^{3n} \to \mathbb{R}$ is a positive, globally invertible, and non-decreasing function. By Assumptions 4 and 5, Corollary 1 in the Appendix, and the bounding effect of projection algorithm on $\hat{\theta}(t)$, the term $N_B(\tilde{\theta},t)$ and its time-derivative $\dot{N}_B(\tilde{\theta},z,t)$ can be upper bounded using known constants $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}_{>0}$ as

$$||N_B(\tilde{\theta},t)|| \leq \gamma_1, \qquad ||\dot{N}_B(\tilde{\theta},z,t)|| \leq \gamma_2 + \gamma_3 \|e_2\| \quad (22)$$

respectively.

IV. Stability Analysis

To facilitate the subsequent analysis, let $y : [t_0,\infty) \to \mathbb{R}^{3n+1}$ be defined as

$$y \triangleq \left[z^T \; \sqrt{P} \right]^T \quad (23)$$

where $P : [t_0,\infty) \to \mathbb{R}$ is a generalized solution to the differential equation

$$\dot{\mathrm{P}} \triangleq -\mathrm{L}. \quad (24)$$

In (24)

$$P(t_0) \triangleq \beta \|e_2(t_0)\|_1 - e_2(t_0)^T N_B(\tilde{\theta}(t_0), t_0) \quad (25)$$

and

$$L \triangleq r^T(N_B(\tilde{\theta},t) - \beta\text{sgn}(e_2)) - \gamma_3 \|e_2\|^2. \quad (26)$$

In (25), $\|\cdot\|_1$ denotes the 1-norm. Provided that the gain condition

$$\beta > \gamma_1 + \frac{\gamma_2}{\alpha_2} \quad (27)$$

is satisfied, $P(t) \geq 0$,[5] where the bounds γ_1, γ_2, and γ_3 are introduced in (22), and the control gain α_2 is introduced in (9). Therefore, it is valid to use $P(t)$ in the candidate Lyapunov function in the subsequent stability analysis. Furthermore, we introduce the auxiliary constant $\lambda_3 \triangleq \min\{\alpha_1 - \frac{1}{2}, \alpha_2 - \gamma_3 - \frac{1}{2}, \frac{k}{2}\}$, where the control gains α_1 and k are introduced in (8) and (15), respectively. The gains α_1, α_2, and k are selected based on the sufficient gain condition

$$\lambda_3 > \frac{\rho^2\left(\sqrt{\frac{\lambda_2(q(t_0))}{\lambda_1}}\|y(t_0)\|\right)}{2k} \quad (28)$$

with $\lambda_1 \triangleq \frac{1}{2}\min\{1, m_1\}$ and $\lambda_2(q) \triangleq \frac{1}{2}\max\{2, \bar{m}(q)\}$, where m_1 and $\bar{m}(q)$ are introduced in Assumption 1. From (8), (9), (20), (24), and (26), the differential equations describing the closed-loop system are

$$\dot{e}_1 = e_2 - \alpha_1 e_1 \quad (29)$$

$$\dot{e}_2 = r - \alpha_2 e_2 \quad (30)$$

$$\dot{r} = M^{-1}(q,t)(-\frac{1}{2}\dot{M}(q,t)r + \widetilde{N}(z,t) + N_B(\tilde{\theta},t) - \beta\text{sgn}(e_2) - kr - e_2) \quad (31)$$

$$\dot{P} = -r^T(N_B(t) - \beta\text{sgn}(e_2)) + \gamma_3 \|e_2\|^2. \quad (32)$$

Theorem 1: Given the EL dynamic system in (2) along with Assumptions 1–5, for any arbitrary initial condition of the states $e_1(t_0)$, $e_2(t_0)$, and $r(t_0)$, selecting $P(t_0)$, $\alpha_1, \alpha_2, \beta$, and k according to (25), (27), and (28) ensures that $e_1, e_2, r, P \in \mathcal{L}_\infty$, and $\|e_1(t)\| \to 0$ as $t \to \infty$.

Proof: Let $\mathcal{D} \subset \mathbb{R}^{3n+1}$ be the open and connected set defined as

$$\mathcal{D} \triangleq \left\{\sigma \in \mathbb{R}^{3n+1} | \, \|\sigma\| < \rho^{-1}\left(\sqrt{2\lambda_3 k}\right)\right\} \quad (33)$$

and $V_L : \mathcal{D} \times [t_0,\infty) \to \mathbb{R}_{\geq 0}$ be a positive-definite candidate Lyapunov function defined as

$$V_L(y,t) \triangleq \frac{1}{2}r^T M(q,t)r + \frac{1}{2}e_2^T e_2 + \frac{1}{2}e_1^T e_1 + P. \quad (34)$$

The candidate Lyapunov function in (34) satisfies

$$\lambda_1 \|y\|^2 \leq V_L \leq \lambda_2(q) \|y\|^2 \quad (35)$$

where λ_1 and $\lambda_2(q)$ are defined after (28). Let $\psi \triangleq [e_1^T \; e_2^T \; r^T \; P]^T$ and $\dot{\psi} \in K[g](\psi,t)$ denote the Filippov differential inclusion corresponding to (29)–(32), where the operator $K[\cdot]$ is defined in [24, (2b)]. Note that $g : \mathbb{R}^{3n+1} \times [t_0,\infty) \to \mathbb{R}^{3n+1}$ is Lebesgue measurable and locally essentially bounded, since it is continuous except in the set with measure zero, $\{(\psi,t) \in \mathbb{R}^{3n+1} \times [0,\infty) | e_2 = 0\}$. Therefore, the existence of an absolutely continuous solution $t \mapsto \psi(t)$ to $\dot{\psi} \in K[g](\psi,t)$ is guaranteed[6] by [25, Proposition 3]. Let $\dot{\tilde{V}}_L(y,t) \triangleq \bigcap_{\xi \in \partial V_L(y,t)} \xi^T[K[g](\psi,t); 1]$ as defined in [26, (13)], where $\partial V_L(y,t)$ denotes Clarke's generalized gradient [26, (7)]. Since $(y,t) \mapsto V_L(y,t)$ is continuously differentiable, Clarke's gradient is the same as the standard gradient, i.e., $\partial V_L = \{\nabla V_L\}$. Using [26, Th. 2.2], $t \mapsto \dot{V}_L(y(t),t)$ exists almost everywhere[7] and $\dot{V}_L(y,t) \in \dot{\tilde{V}}_L(y,t)$ for almost all time (a.a.t.). Evaluating $\dot{\tilde{V}}_L(y,t)$ and (29)–(32) yields

$$\dot{\tilde{V}}_L \overset{a.a.t.}{\leq} r^T(-\frac{1}{2}\dot{M}(q,t)r + \widetilde{N}(z,t) + N_B(\tilde{\theta},t)$$

[5]See the proof approach of [18, Lemma 1] for details.

[6]The existing solution might have a finite escape time. We rule out this possibility by proving the boundedness of Filippov trajectories under the aforementioned sufficient conditions using Lyapunov-based stability theory. Therefore, dom $\psi = [t_0,\infty)$, i.e., the solution is complete. The solution may not be unique; however, the results are applicable to all the trajectories, since we consider a generalized Filippov solution in the analysis.

[7]Since $\psi = [z^T \; P]^T$ and $y = [z^T \; \sqrt{P}]^T$, $y(t)$ can be evaluated along a Filippov trajectory $\psi(t)$ by a transformation which involves taking the square-root of $P(t)$, which is applicable since $P(t) \geq 0, \forall t \in [t_0,\infty)$.

$$- \beta K\,[\mathrm{sgn}]\,(e_2) - kr - e_2) + e_2^T(r - \alpha_2 e_2)$$
$$+ e_1^T(e_2 - \alpha_1 e_1) - r^T(N_B(t) - \beta K\,[\mathrm{sgn}]\,(e_2))$$
$$+ \gamma_3 \|e_2\|^2 + \frac{1}{2} r^T \dot{M}(q,t) r. \tag{36}$$

Using (21) and applying Young's inequality on $e_1^T e_2$ in (36), $\dot{V}_L$ can be upper bounded[8] as

$$\dot{V}_L \overset{a.a.t.}{\leq} \rho(\|z\|)\,\|z\|\,\|r\| - k\,\|r\|^2 - (\alpha_2 - \gamma_3 - \frac{1}{2})\,\|e_2\|^2$$
$$- (\alpha_1 - \frac{1}{2})\,\|e_1\|^2 .$$

Using Young's Inequality on $\rho(\|z\|)\|z\|\|r\|$ yields $\rho(\|z\|)\|z\|\|r\| \leq \frac{\rho^2(\|z\|)\|z\|^2}{2\,k} + \frac{1}{2}k\|r\|^2$. Therefore,

$$\dot{\tilde{V}}_L \overset{a.a.t.}{\leq} \frac{\rho^2(\|z\|)\,\|z\|^2}{2\,k} - \frac{k}{2}\,\|r\|^2$$
$$- (\alpha_2 - \gamma_3 - \frac{1}{2})\,\|e_2\|^2 - (\alpha_1 - \frac{1}{2})\,\|e_1\|^2$$
$$\leq - \left(\lambda_3 - \frac{\rho^2(\|z\|)}{2\,k}\right) \|z\|^2 . \tag{37}$$

The expression in (37) can be rewritten as

$$\dot{V}_L \overset{a.a.t.}{\leq} -W(y) = -c\,\|z\|^2 , \forall\, y \in \mathcal{D} \tag{38}$$

with some constant $c \in \mathbb{R}_{>0}$, where $W : \mathbb{R}^{3n+1} \to \mathbb{R}$ is a continuous positive semidefinite function.

Whenever $y \in \mathcal{D}$, $\|y(t)\| < \rho^{-1}(\sqrt{2\lambda_3 k})$ by definition of $\mathcal{D}$, which is sufficient to infer $\|z(t)\| < \rho^{-1}(\sqrt{2\lambda_3 k})$ using (23). Therefore, if $y(t) \in \mathcal{D}$, $\lambda_3 > \frac{\rho^2(\|z\|)}{2\,k}$, which implies from (37) that there exists $c \in \mathbb{R}_{>0}$ which satisfies (37), and larger values of λ_3 expand the size of $\mathcal{D}$. Since V_L is nonincreasing, which implies $\|y(t)\| \leq \sqrt{\frac{V_L(t)}{\lambda_1}} \leq \sqrt{\frac{V_L(t_0)}{\lambda_1}}$, it is sufficient to show that $\sqrt{\frac{V_L(t_0)}{\lambda_1}} < \rho^{-1}(\sqrt{2\lambda_3 k})$, to obtain $y(t) \in \mathcal{D}$. Since $V_L(t_0) \leq \lambda_2(q(t_0))\|y(t_0)\|^2$, the result $\sqrt{\frac{V_L(t_0)}{\lambda_1}} < \rho^{-1}(\sqrt{2\lambda_3 k})$ can be sufficiently obtained from $\sqrt{\frac{\lambda_2(q(t_0))}{\lambda_1}}\|y(t_0)\| < \rho^{-1}(\sqrt{2\lambda_3 k})$. Therefore, $\|y(t_0)\| < \sqrt{\frac{\lambda_1}{\lambda_2(q(t_0))}}\rho^{-1}(\sqrt{2\lambda_3 k})$, which implies that $\mathcal{S} \triangleq \{\sigma \in \mathcal{D} | \|\sigma\| < \sqrt{\frac{\lambda_1}{\lambda_2(q(t_0))}}\rho^{-1}(\sqrt{2\lambda_3 k})\}$ is the region where $y(t_0)$ should lie to guarantee that $y(t) \in \mathcal{D}$[9] for all $t \in [t_0, \infty)$. Using (33), (35), and (38), since g is Lebesgue measurable and essentially locally bounded and uniformly in time, the extension of the LaSalle–Yoshizawa corollary in [28, Corollary 1] can be invoked to show that $e_1, e_2, r, P \in \mathcal{L}_\infty$, and $\|z(t)\| \to 0$ as $t \to \infty$. Therefore, using the definition of z in (19), $\|e_1(t)\| \to 0$ as $t \to \infty$. The gain condition in (28) needs to be satisfied according to the initial condition, and the region of attraction can be made arbitrarily large to include any initial condition by increasing the gains α_1, α_2, and k accordingly; therefore, the result is semiglobal.

The parameter estimate $\hat{\theta} \in \mathcal{L}_\infty$ due to the projection operation, which implies $\tilde{\theta}(t) = \theta(t) - \hat{\theta}(t)$ is bounded, because the parameter $\theta \in \mathcal{L}_\infty$ by Assumption 4. Since $e_1, e_2, r \in \mathcal{L}_\infty$, and because $q_d, \dot{q}_d, \ddot{q}_d \in \mathcal{L}_\infty$ by Assumption 5, using (7)–(9) implies that $q, \dot{q}, \ddot{q} \in \mathcal{L}_\infty$. Furthermore, the regression matrix $Y_d \in \mathcal{L}_\infty$ by Assumption 5, because Y is locally bounded due to Properties 2 and 3. Therefore, by Corollary 1 in the Appendix, $\dot{\hat{\theta}} \in \mathcal{L}_\infty$. The expression in (12) indicates that $\mu \in \mathcal{L}_\infty$, because among the remaining terms in (12), $M(q)r$ and $Y_d\tilde{\theta}$ comprise bounded terms because M is locally bounded, and $S \in \mathcal{L}_\infty$ because its definition comprises terms that are locally bounded functions of the bounded errors and states due to Assumption 2. From the expression in (11), since $\hat{\theta}, Y_d, \mu \in \mathcal{L}_\infty, \tau \in \mathcal{L}_\infty$. Moreover, differentiating the right-hand side in (11) yields terms that are bounded, which implies $\dot{\tau} \in \mathcal{L}_\infty$; therefore, τ is continuous. Hence, all the closed-loop signals are bounded.

V. Simulation Example

To demonstrate the efficacy of the developed method, a simulation example of a horizontal two-link manipulator system is provided, and the results are compared with an e-modification (e-mod)-based controller [18]. The dynamics of the manipulator system can be represented in the form of (2), with $M(q,t) = \begin{bmatrix} p_1(t) + 2p_3(t)c_2 & p_2(t) + p_3(t)c_2 \\ p_2(t) + p_3(t)c_2 & p_2(t) \end{bmatrix}$, $V_m(q,\dot{q},t) = \begin{bmatrix} -p_3(t)s_2\dot{q}_2 & -p_3(t)s_2(\dot{q}_1 + \dot{q}_2) \\ p_3(t)s_2\dot{q}_1 & 0 \end{bmatrix}$, $F(\dot{q},t) = \begin{bmatrix} F_{d1}(t)\dot{q}_1 \\ F_{d2}(t)\dot{q}_2 \end{bmatrix}$, and $\tau_d(t) = [\,\tau_{d1}(t)\;\tau_{d2}(t)\,]^T$, where $c_2 \triangleq \cos(q_2)$, $s_2 \triangleq \sin(q_2)$, and $p_1, p_2, p_3, F_{d1}, F_{d2}, \tau_{d1}, \tau_{d2} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, and the gravity term $G(q,t)$ is ignored for a horizontal manipulator. The augmented time-varying parameter vector for the manipulator system is given by $\theta(t) = [\,p_1(t)\; p_2(t)\; p_3(t)\; F_{d1}(t)\; F_{d2}(t)\; \tau_{d1}(t)\; \tau_{d2}(t)\,]^T$. The control objective is to track a given reference trajectory $q_d(t) = [\,\cos(0.5t)\;2\cos(t)\,]^T$. The time-varying parameters used in the simulation are $p_1(t) = 3.473 + 0.5\sin(3t)$, $p_2(t) = 0.196 + 0.2\exp(-\sin(t))$, $p_3(t) = 0.242 + 0.1\cos(10t)$, $F_{d1}(t) = 5.3 + 2\exp(-0.1t)$, $F_{d2}(t) = 1.1 + \cos(5t)$, and the disturbance terms $\tau_{d1}(t) = 0.5\cos(0.5t)$ and $\tau_{d2}(t) = \sin(t)$. The initial conditions used in the simulation are $q(0) = [\,-1\;1\,]^T$, $\dot{q}(0) = [\,0\;0\,]^T$, and $\hat{\theta}(0) = [\,0\;0\;0\;0\;0\;0\;0\,]^T$.[10]

The control gains for each method are obtained using a Monte-Carlo method; an appropriate range is qualitatively determined for each gain, and 10 000 iterations are subsequently run with a uniform random gain sampling within those ranges in an attempt to minimize

$$J = \int_0^{10} \left(a\,\|e_1(t)\|^2 + b\,\|\tau(t)\|^2\right) dt \tag{39}$$

with $a = 1$ and $b = 0.01$. The gains[11] that minimized (39) for the developed method are $K = 18.1502$, $\alpha_1 = 0.8982$, $\alpha_2 = 1.0552$, $\beta =$

[8]The set of times $T \triangleq \{t \in [t_0, \infty) : r(t)^T\beta\mathrm{SGN}(e_2(t)) - r(t)^T\beta\mathrm{SGN}(e_2(t)) \neq \{0\}\} \subset \mathbb{R}_{\geq 0}$ is equal to the set of times $\{t : e_2(t) = 0 \wedge r(t) \neq 0\}$. Using $r = \dot{e}_2 + \alpha_2 e_2$, this set can also be represented by $\{t : e_2(t) = 0 \wedge \dot{e}_2(t) \neq 0\}$. Since e_2 is continuously differentiable because the right-hand side of (30) is continuous, [27, Lemma 2] can be used to show that the set of time instances $\{t : e_2(t) = 0 \wedge \dot{e}_2(t) \neq 0\}$ is isolated, and hence, measure zero; hence, T is measure zero. Therefore, $\dot{\tilde{V}}_L = \{\dot{V}_L\}$ a.e. in time, and an upper bound on $\dot{V}_L$ can be obtained a.e. in time, using the right-hand side of (36).

[9]Though the sets $\mathcal{S}$ and $\mathcal{D}$ are defined to include $y(t)$ instead of $\psi(t)$, one can easily construct the bounded sets $\mathcal{S}_\psi$ and $\mathcal{D}_\psi$ such that $y(t_0) \in \mathcal{S}$ and $y(t) \in \mathcal{D}$ imply $\psi(t_0) \in \mathcal{S}_\psi$ and $\psi(t) \in \mathcal{D}_\psi$, respectively, to conclude the uniform boundedness of all Filippov trajectories $\psi(t)$ initializing in the set $\mathcal{S}_\psi$.

[10]Note that in practice, the best guess estimates of the uncertain parameters or their approximate mean should be used for improved performance. The estimates were initialized to zero to illustrate adaptation with no prior knowledge.

[11]This set of gains might not satisfy the gain conditions in (27) and (28), however, those conditions are not necessary, rather only sufficient. The gains were selected from the Monte–Carlo simulation to provide the best performance and an equal comparison with the e-mod method.

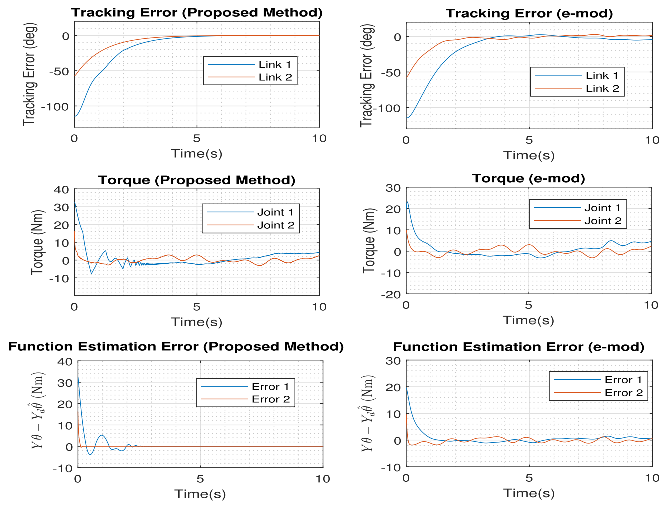

Fig. 1. Plots of tracking error (degrees), torque input (Nm), and function estimation error $(Y\theta - Y_d\hat{\theta})$ versus time (s) with the proposed method and e-mod.

TABLE I
CONTROLLER PERFORMANCE COMPARISON

Method	$\|e_{\mathrm{rms}}\|$	$\|e_{\mathrm{rms,ss}}\|$	$e_{\max,\mathrm{ss}}$	$\|\tilde{Y}_{\mathrm{rms}}\|$	$\|\tau_{\mathrm{rms}}\|$
(11)	29.1340	0.5515	1.4666	1.3654	3.2726
e-mod	52.4838	3.5624	5.2500	6.1292	7.4722

36.2946, and $\Gamma = I_2$. For the projection algorithm, $\zeta_0 = 5000$ and the corresponding function $f(\hat{\theta}) = \|\hat{\theta}\|^2 - \zeta_0^2$. For the e-mod update law, i.e., $\dot{\hat{\theta}} = \Gamma_e Y_d^T r - \sigma\|e_1\|\hat{\theta}$ and the corresponding controller $\tau = Y_d\hat{\theta} - k_e r$, the gains are $\Gamma_e = 12.5$, $k_e = 9.7877$, and $\sigma = 9.7319$.

Fig. 1 demonstrates the asymptotic convergence of the tracking error and the function estimation error $(Y\theta - Y_d\hat{\theta})$ [12] to zero with the developed method in the simulation, as opposed to the UUB tracking with the e-mod scheme. Table I provides a quantitative comparison of the controllers, where e_{rms} is the root-mean-square (rms) of e_1 (in deg) taken over the time interval [0, 10], $e_{\mathrm{rms,ss}}$ is the rms of e_1 over the time interval [5, 10] (i.e., after reaching the steady state), $e_{\max,\mathrm{ss}}$ is the maximum absolute value of the components of e_1 over the time interval [5, 10], $\tilde{Y}_{\mathrm{rms}}$ denotes the rms function estimation error (in Nm) over the interval [0, 10], and τ_{rms} denotes the rms simulated torque (in Nm) over the time interval [0, 10]. The developed method provides a significantly improved tracking and function estimation performance with less rms control effort, upon comparison with e-mod.

Fig. 2 demonstrates the tracking error performance in the presence of additive white Gaussian (AWG) noise with standard deviations of 2 deg and 2 deg/s in the q and $\dot{q}$ measurements, respectively. The rms steady state tracking error norms in the presence of measurement noise with the developed method and e-mod are 2.9427 and 4.5891, respectively.

[12]From an applied perspective, if the upper bound used for projection algorithm, i.e., ζ_0 is selected to be sufficiently high such that the parameter estimates never reach the boundary of the set $\mathcal{B}$, then $\mathrm{proj}(\Lambda_0(t)) = \Lambda_0(t)$, $\forall t \in [t_0, \infty)$, implying $\mu(t) = 0$, $\forall t \in [t_0, \infty)$. From (6), $Y\theta = \tau$, and $\tau - Y_d\hat{\theta} = \mu - ke_2$ using (11), therefore if $\mu(t) = 0\,\forall t \in [t_0, \infty)$, then the function approximation error $Y\theta - Y_d\hat{\theta} = \mu - ke_2 = -ke_2 \to 0$ as $t \to \infty$. In case the parameter estimates reach the boundary of $\mathcal{B}$, $Y\theta - Y_d\hat{\theta}$ may not converge to zero, yet it is guaranteed to be bounded using the stability analysis since μ is bounded.

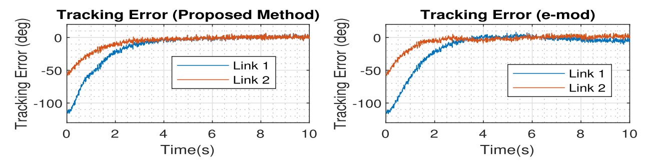

Fig. 2. Plots of tracking error (deg) versus time (s) in presence of AWG measurement noise with the proposed method and e-mod.

VI. CONCLUSION

A continuous adaptive control design was presented to achieve semiglobal asymptotic tracking for linearly parameterizable nonlinear systems with time-varying uncertain parameters. Through a unique analysis strategy, an adaptive feedforward term was developed along with specialized feedback terms to compensate for the time-varying uncertainty. Asymptotic tracking error convergence was guaranteed via a Lyapunov-based stability analysis for an EL system. Additionally, the time-varying uncertain function approximation error was shown to converge to zero. A simulation example of a two-link manipulator was provided to demonstrate the asymptotic tracking result, and a comparison with the e-mod scheme shows a better tracking performance with the proposed method. Future work may involve extension of the proposed approach to unstructured time-varying uncertainties using neural networks, compensation of time-varying uncertainty in the presence of sensor noise, and delays in input and state measurements.

APPENDIX

Lemma 1: Consider a positive-definite matrix $\Gamma \in \mathbb{R}^{(n+m)\times(n+m)}$ such that Γ has the block diagonal structure as $\Gamma \triangleq \begin{bmatrix} \Gamma_1 & 0_{m\times n} \\ 0_{n\times m} & \Gamma_2 \end{bmatrix}$, where $\Gamma_1 \in \mathbb{R}^{m\times m}$ and $\Gamma_2 \in \mathbb{R}^{n\times n}$. The matrix $Y\Gamma Y^T$ is positive-definite and therefore invertible. Furthermore, the inverse of this matrix satisfies the property $\|(Y\Gamma Y^T)^{-1}\|_2 \leq \frac{1}{\lambda_{\min}\{\Gamma_2\}}$, where $\|\cdot\|_2$ denotes the spectral norm and $\lambda_{\min}\{\cdot\}$ denotes the minimum eigenvalue of $\{\cdot\}$.

Proof: Substituting the definitions for Y and Γ in $Y\Gamma Y^T$ yields

$$\begin{aligned} Y\Gamma Y^T &= \begin{bmatrix} Y_p & I_n \end{bmatrix} \begin{bmatrix} \Gamma_1 & 0_{m\times n} \\ 0_{n\times m} & \Gamma_2 \end{bmatrix} \begin{bmatrix} Y_p^T \\ I_n \end{bmatrix} \\ &= Y_p\Gamma_1 Y_p^T + \Gamma_2. \end{aligned}$$

Since Γ is selected to be a positive-definite matrix, the block matrices Γ_1 and Γ_2 are both positive-definite, so $Y_p\Gamma_1 Y_p^T$ is positive semidefinite while the second term Γ_2 is positive-definite, hence the sum of these two terms, i.e., $Y\Gamma Y^T$ is positive-definite, and therefore, invertible. Furthermore, the spectral norm satisfies the property, $\|A\|_2 = \sqrt{\lambda_{\max}\{A^T A\}}$ for some $A \in \mathbb{R}^{p\times q}$, where $\lambda_{\max}\{\cdot\}$ denotes the maximum eigenvalue of $\{\cdot\}$. Utilizing this property with $\|(Y\Gamma Y^T)^{-1}\|_2$ yields

$$\begin{aligned} \left\|\left(Y\Gamma Y^T\right)^{-1}\right\|_2 &= \sqrt{\lambda_{\max}\left\{\left(\left(Y\Gamma Y^T\right)^{-1}\right)^T \left(Y\Gamma Y^T\right)^{-1}\right\}} \\ &= \lambda_{\max}\left\{\left(Y\Gamma Y^T\right)^{-1}\right\}. \\ &= \frac{1}{\lambda_{\min}\left\{Y\Gamma Y^T\right\}} \leq \frac{1}{\lambda_{\min}\left\{\Gamma_2\right\}}. \end{aligned} \tag{40}$$

Corollary 1: The norm of the time-derivative of the parameter estimate $\|\dot{\hat{\theta}}\|$ can be upper bounded by as $\|\dot{\hat{\theta}}\| \leq \gamma_4 + \gamma_5\|e_2\|$, where $\gamma_4, \gamma_5 \in \mathbb{R}_{>0}$ are known bounding constants.

Proof: Based on (14)

$$\begin{aligned}\left\|\dot{\hat{\theta}}\right\| &= \|\text{proj}(\Lambda_0)\| \leq \|\Lambda_0\| \\ &= \left\|\Gamma Y_d^T (Y_d \Gamma Y_d^T)^{-1} (\beta \text{sgn}(e_2) + k\alpha_2 e_2)\right\| \\ &\leq \left\|\Gamma Y_d^T (Y_d \Gamma Y_d^T)^{-1}\right\| (\beta + k\alpha_2 \|e_2\|) . \end{aligned} \tag{41}$$

Applying Holder's inequality to the right-hand side of (41) yields

$$\left\|\dot{\hat{\theta}}\right\| \leq \|\Gamma\|_2 \|Y_d\|_2 \left\|(Y_d \Gamma Y_d^T)^{-1}\right\|_2 (\beta + k\alpha_2 \|e_2\|) . \tag{42}$$

Using the result from Lemma 1 yields

$$\left\|\dot{\hat{\theta}}\right\| \leq \frac{\|\Gamma\|_2 \|Y_d\|_2}{\lambda_{\min}\{\Gamma_2\}} (\beta + k\alpha_2 \|e_2\|) .$$

Based on Assumption 5, the spectral norm of the desired regressor may be upper bounded by a constant $\overline{Y}_d \in \mathbb{R}_{>0}$, i.e., $\|Y_d\|_2 \leq \overline{Y}_d$, because Y_d is a continuously differentiable function. Therefore, selecting $\gamma_4 = \frac{\beta\|\Gamma\|_2 \overline{Y}_d}{\lambda_{\min}\{\Gamma_2\}}$ and $\gamma_5 = \frac{k\alpha_2\|\Gamma\|_2 \overline{Y}_d}{\lambda_{\min}\{\Gamma_2\}}$ yields

$$\begin{aligned}\left\|\dot{\hat{\theta}}\right\| &\leq \frac{\|\Gamma\|_2 \overline{Y}_d}{\lambda_{\min}\{\Gamma_2\}} (\beta + k\alpha_2 \|e_2\|) \\ &= \gamma_4 + \gamma_5 \|e_2\| .\end{aligned}$$

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