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Tracking and Regulation Control of a Mobile Robot System With Kinematic Disturbances: A Variable Structure-Like Approach

This paper presents the design of a variable structure-like tracking controller for a mobile robot system. The controller provides robustness with regard to bounded disturbances in the kinematic model. Through the use of a dynamic oscillator and a Lyapunov-based stability analysis, we demonstrate that the position and orientation tracking errors exponentially converge to a neighborhood about zero that can be made arbitrarily small (i.e., the controller ensures that the tracking error is globally uniformly ultimately bounded (GUUB)). In addition, we illustrate how the proposed tracking controller can also be utilized to achieve GUUB regulation to an arbitrary desired setpoint. An extension is also provided that illustrates how a smooth, time-varying control law can be utilized to achieve setpoint regulation despite parametric uncertainty in the kinematic model. Simulation results are presented to demonstrate the performance of the proposed controllers.
[S0022-0434(00)00504-9]

1 Introduction

Over the past twenty years the control of wheeled mobile robots (WMRs) has been heavily studied (see [1,2], and the references therein for an in-depth review of the previous work) due to the challenging theoretical nature of the problem (i.e., WMRs are nonlinear, underactuated systems subject to nonholonomic constraints imposed by a pure rolling and nonslipping assumption) and the wide range of applications that are well suited to their use (*e.g.*, munitions handling, exploration, security, and monitoring, etc.). Several researchers have examined the regulation [3,4] and the tracking control [5] problems for WMRs that are subject to disturbances in the kinematic model that violate the pure rolling and nonslipping assumption. Specifically, a quasi-sliding mode controller was presented by Canudas de Wit et al. in [3] that achieved exponential regulation of the position/orientation of a WMR subject to either a constant matched disturbance or a state vanishing disturbance that violates the nonholonomic constraint. In [4], Corradini et al. proposed a discrete-time, quasi-sliding mode controller that regulated the position of a WMR subject to a similar disturbance to a neighborhood about the origin. Note that the quasi-sliding mode regulation controllers presented in [3,4] are not differentiable, and unfortunately, the standard backstepping procedure, often used for incorporating the mechanical dynamics, requires that the kinematic controller be differentiable (see the discussion in [1]). In [6], d'Andrea-Novel et al. proposed a singular perturbation formulation that led to robustness results for feedback linearizing control laws with sufficiently small slipping and skidding effects. In [5], Leroquais et al. used the results in [6] to design a linear, differentiable time-varying feedback law that achieved local uniformly asymptotically stable tracking of a time-varying reference trajectory; however, due to restrictions on the reference trajectory, the tracking controller cannot be applied to the regulation problem. It should be noted that the controller proposed in [5] included the dynamic model of the WMR. In addition to problems concerning disturbances in the kinematic model, researchers have also investigated the effects of parametric uncertainty in the kinematic model. For example, [7–9] examined regulating a WMR with uncertain parameters multiplied by the control inputs of the kinematic model. Specifically, in [7], Hespanha et al. utilized a supervisory control strategy to switch between a suitably defined family of candidate control laws to solve the so-called ''parking problem'' for a WMR. In [8], Jiang proposed a switching controller to exponentially regulate a WMR, and in [9], Jiang extended the results to the general chained form.

In this paper, we present the design of a variable structure-like tracking controller for a mobile robot system that is subject to kinematic disturbances. The kinematic disturbances considered in this paper represent a broader class of disturbances than previously examined (i.e., the kinematic model presented in [3,4] are special cases of the kinematic model examined in this paper). Through the use of a dynamic oscillator and a Lyapunov-based stability analysis, we show that the position and orientation tracking errors exponentially converge to a neighborhood about zero that can be made arbitrarily small (i.e., the controller ensures that the tracking error is globally uniformly ultimately bounded (GUUB)). In addition, since we only require that the reference trajectory be bounded, the proposed tracking controller can also be utilized to achieve GUUB regulation; hence, a unified control framework for both the tracking and the regulation problem is proposed. Finally, we also illustrate how the differentiable, time-varying controller presented in [2] can be utilized to solve the regulation problem of a WMR with the same uncertain kinematic model as examined in [7–9].

The paper is organized as follows. In Section 2, we develop the kinematic model of a WMR that is subject to kinematic disturbances and then transform the model into a form which facilitates the subsequent control development. In Section 3, we present the control law and the corresponding closed-loop error system. In Section 4, we provide a Lyapunov based stability analysis that illustrates global uniformly ultimately bounded tracking. In Section 5, we present a setpoint extension for the proposed controller. In Section 6, we present an extension regarding the setpoint regulating problem for a WMR with uncertain parameters in the kinematic model. In Section 7, we provide simulation results to illustrate the performance of the proposed controllers, and in Section 8, we present some concluding remarks.

Contributed by the Dynamic Systems and Control Division for publication in the JOURNAL OF DYNAMIC SYSTEMS, MEASUREMENT, AND CONTROL. Manuscript received by the Dynamic Systems and Control Division February 4, 2000. Associate Technical Editors: E. Misawa and V. Utkin.

2 Problem Formulation

2.1 WMR Kinematic Model. The kinematic model for the so-called kinematic wheel is given as follows

$$\dot{q}=S(q)v+[\rho_1(t) \quad \rho_2(t) \quad \rho_3(t)]^T \tag{1}$$

where $q(t),\ \dot{q}(t)\in\Re^3$ are defined as

$$q=[x_c \quad y_c \quad \theta]^T \quad \dot{q}=[\dot{x}_c \quad \dot{y}_c \quad \dot{\theta}]^T \tag{2}$$

$x_c(t)$ and $y_c(t)$ denote the Cartesian position of the center of mass (COM) of the WMR along the X and Y-coordinate axis of the Cartesian plane (see Fig. 1), $\theta(t)\in\Re^1$ represents the orientation of the WMR (see Fig. 1), $\dot{x}_c(t),\ \dot{y}_c(t)$ denote the Cartesian components of the linear velocity of the COM, $\dot{\theta}(t)\in\Re^1$ denotes the angular velocity of the COM, the matrix $S(q)\in\Re^{3\times 2}$ is defined as follows

$$S(q)=\begin{bmatrix}\cos\theta & 0\\ \sin\theta & 0\\ 0 & 1\end{bmatrix}, \tag{3}$$

the velocity vector $v(t)\in\Re^2$ is defined as follows

$$v=[v_1 \quad v_2]^T=[v_l \quad \dot{\theta}]^T \tag{4}$$

with $v_l(t)\in\Re^1$ denoting the linear velocity of the COM, and $\rho_1(t),\rho_2(t),\rho_3(t)\in\Re^1$ represent unknown disturbances that are assumed to be upper bounded as shown below

$$|\rho_1(t)|\le\zeta_1, \quad |\rho_2(t)|\le\zeta_2, \quad |\rho_3(t)|\le\zeta_3 \tag{5}$$

where $\zeta_1,\zeta_2,\zeta_3\in\Re^1$ are positive bounding constants. Note that if $\rho_1(t),\rho_2(t),\rho_3(t)=0$, the standard kinematic model for the pure rolling and nonslipping kinematic wheel is recovered.

Remark 1. Note that the kinematic model for a WMR subject to the so-called matched disturbance is given as follows [3]

$$\dot{q}=S(q)v+\rho_M(t)[\cos\theta \quad \sin\theta, \quad 0]^T \tag{6}$$

where $\rho_M(t)\in\Re^1$ denotes a bounded disturbance. In addition, the kinematic model for a WMR subject to the so-called unmatched disturbance is given as follows [3]

$$\dot{q}=S(q)v+\rho_U(t)[\sin\theta \quad -\cos\theta \quad 0]^T \tag{7}$$

where $\rho_U(t)\in\Re^1$ denotes a bounded disturbance. Note that in order to obtain the exponential regulation result presented in [3], the unmatched disturbance $\rho_U(t)$ must be upper bounded by a function of the states, whereas the GUUB result obtained in [4] required the disturbance be upper bounded by a constant. From a control point of view, it is easy to see from (1), (6), and (7) that the matched disturbance and unmatched disturbance problems are both special cases of the model used in (1).

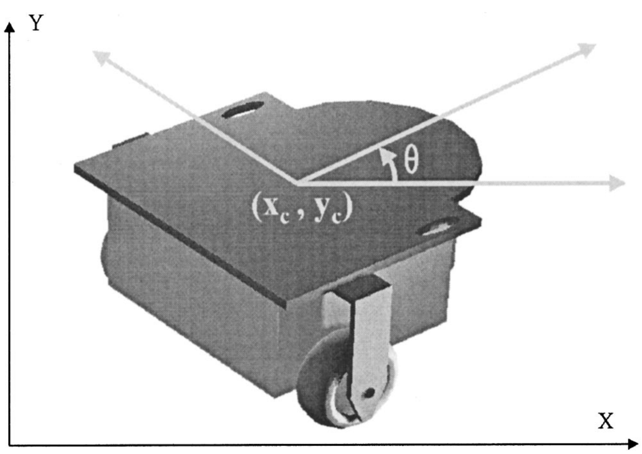

Fig. 1 Two-dimensional kinematic wheel

2.2 Model Transformation. As defined in previous work (e.g., see [10] and [11]), the time-varying reference trajectory for the WMR is generated via a reference robot which moves according to the following dynamic trajectory

$$\dot{q}_r=S(q_r)v_r \tag{8}$$

where $S(\cdot)$ was defined in (3), $q_r(t)=[x_{rc}(t)\ y_{rc}(t)\ \theta_r(t)]^T\in\Re^3$ denotes the desired time-varying position/orientation trajectory, and $v_r(t)=[v_{r1}(t)\ v_{r2}(t)]^T\in\Re^2$ denotes the reference time-varying linear/angular velocity. With regard to (8), it is assumed that the signal $v_r(t)$ is constructed to produce the desired motion and that $v_r(t),\ \dot{v}_r(t),\ q_r(t)$, and $\dot{q}_r(t)$ are bounded for all time.

To facilitate the subsequent control synthesis and the corresponding stability proof, we define the following global invertible transformation

$$\begin{bmatrix}w\\ z_1\\ z_2\end{bmatrix}=\begin{bmatrix}-\tilde{\theta}\cos\theta+2\sin\theta & -\tilde{\theta}\sin\theta-2\cos\theta & 0\\ 0 & 0 & 1\\ \cos\theta & \sin\theta & 0\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{\theta}\end{bmatrix} \tag{9}$$

where $w(t)\in\Re^1$ and $z(t)=[z_1(t)\ z_2(t)]^T\in\Re^2$ are auxiliary tracking error variables and $\tilde{x}(t),\tilde{y}(t),\tilde{\theta}(t)\in\Re^1$ denote the difference between the actual Cartesian position and orientation of the COM and the reference position and orientation of the COM as follows

$$\tilde{x}=x_c-x_{rc} \quad \tilde{y}=y_c-y_{rc} \quad \tilde{\theta}=\theta-\theta_r. \tag{10}$$

2.3 Open-Loop Tracking Error Development. After taking the time derivative of (9), and using (1)–(4), (8)–(10) we can rewrite the open-loop tracking error dynamics in a more convenient form as follows

$$\begin{aligned}\dot{w}&=u^TJ^Tz+f+\chi_1\\ \dot{z}&=u+\chi_2\end{aligned} \tag{11}$$

where $J\in\Re^{2\times 2}$ is a skew-symmetric matrix defined as

$$J=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \tag{12}$$

$f(z,v_r,t)\in\Re^1$ is an auxiliary signal defined as

$$f=2(v_{r2}z_2-v_{r1}\sin z_1) \tag{13}$$

the auxiliary kinematic control input, denoted by $u(t)=[u_1(t)\ u_2(t)]^T\in\Re^2$, is defined in terms of the WMR position/orientation, the WMR linear velocities, and the reference trajectory as follows

$$\begin{aligned}u&=T^{-1}v-\begin{bmatrix}v_{r2}\\ v_{r1}\cos\tilde{\theta}\end{bmatrix}\\ v&=Tu+\begin{bmatrix}v_{r1}\cos\tilde{\theta}+v_{r2}(\tilde{x}\sin\theta-\tilde{y}\cos\theta)\\ v_{r2}\end{bmatrix}\end{aligned} \tag{14}$$

the auxiliary matrix $T\in\Re^{2\times 2}$ is defined as follows

$$T=\begin{bmatrix}(\tilde{x}\sin\theta-\tilde{y}\cos\theta) & 1\\ 1 & 0\end{bmatrix} \tag{15}$$

and $\chi_1(t)\in\Re^1$ and $\chi_2(t)=[\chi_{21}\ \chi_{22}]\in\Re^2$ are auxiliary signals defined as follows

$$\begin{aligned}\chi_1=&\,2(\rho_1\sin\theta-\rho_2\cos\theta)+\rho_3(z_2+z_1(\tilde{x}\sin\theta-\tilde{y}\cos\theta))\\ &-z_1(\rho_1\cos\theta+\rho_2\sin\theta)\end{aligned} \tag{16}$$

$$\chi_2=[\rho_3 \quad \rho_1\cos\theta+\rho_2\sin\theta-\rho_3(\tilde{x}\sin\theta-\tilde{y}\cos\theta)]^T. \quad (17)$$

In order to facilitate the subsequent stability analysis, we utilize the fact that

$$\tilde{x}\sin\theta-\tilde{y}\cos\theta=\frac{1}{2}(w+z_1z_2) \quad (18)$$

to rewrite (16) and (17) in terms of the auxiliary variables defined in (9) as follows

$$\chi_1=2(\rho_1\sin\theta-\rho_2\cos\theta)+\rho_3\left(z_2+\frac{z_1}{2}(w+z_1z_2)\right)-z_1(\rho_1\cos\theta+\rho_2\sin\theta) \quad (19)$$

$$\chi_2=\left[\rho_3 \quad \rho_1\cos\theta+\rho_2\sin\theta-\frac{\rho_3}{2}(w+z_1z_2)\right]^T. \quad (20)$$

3 Smooth Variable Structure-Like Control

Our control objective is to design a controller for the transformed kinematic model given in (11) that forces the actual position/orientation of the WMR to track the reference time-varying position/orientation generated in (8). To facilitate the subsequent control development, we define an auxiliary error signal $\tilde{z}(t)\in\Re^2$ as the difference between the subsequently designed auxiliary signal $z_d(t)\in\Re^2$ and the transformed variable $z(t)$, defined in (9), as follows

$$\tilde{z}=z_d-z. \quad (21)$$

3.1 Control Formulation. Based on the open-loop tracking error dynamics given in (11) and the subsequent stability analysis, we design the auxiliary kinematic control signal $u(t)$ as follows

$$u=u_a-k_2z \quad (22)$$

where the auxiliary control signal $u_a(t)\in\Re^2$ is defined as

$$u_a=\left(\frac{k_1w+f}{\delta_d^2}\right)Jz_d+\Omega_1z_d, \quad (23)$$

the auxiliary signal $z_d(t)$ is defined by the following oscillator-like relationship

$$\dot{z}_d=\frac{\dot{\delta}_d}{\delta_d}z_d+\left(\frac{k_1w+f}{\delta_d^2}+w\Omega_1\right)Jz_d \quad z_d^T(0)z_d(0)=\delta_d^2(0), \quad (24)$$

the auxiliary terms $\Omega_1(t)\in\Re^1$ and $\delta_d(t)\in\Re^1$ are defined as

$$\Omega_1=k_2+\frac{\dot{\delta}_d}{\delta_d}+w\left(\frac{k_1w+f}{\delta_d^2}\right) \quad (25)$$

and

$$\delta_d=\alpha_0\exp(-\alpha_1t)+\varepsilon_1 \quad (26)$$

respectively, $f(z,v_r,t)$ was defined in (13), $k_1(t)$, $k_2(t)\in\Re^1$ are positive, time-varying functions selected as follows

$$k_1=k_s+\frac{\kappa_1^2}{\kappa_1|w|+\varepsilon_{c1}} \quad k_2=k_s+\frac{\kappa_2^2}{\kappa_2\|\tilde{z}\|+\varepsilon_{c2}}, \quad (27)$$

$\kappa_1(w,\tilde{z},t)$, $\kappa_2(w,\tilde{z},t)\in\Re^1$ are positive bounding functions, and k_s, α_0, α_1, ε_1, ε_{c1}, $\varepsilon_{c2}\in\Re^1$ are positive, constant control gains.

Remark 2. In order to facilitate the subsequent stability analysis, we note that the auxiliary signals defined in (19) and (20) can be upper bounded as follows

$$|\chi_1|\leq\kappa_1 \quad \|\chi_2\|\leq\kappa_2 \quad (28)$$

where the positive bounding functions $\kappa_1(w,\tilde{z},t)$, $\kappa_2(w,\tilde{z},t)\in\Re^1$ *are defined as follows*

$$\kappa_1\geq2\zeta_4+\left(\zeta_3+\zeta_4+\frac{\zeta_3}{2}|w|\right)(\|z_d\|+\|\tilde{z}\|)+\frac{\zeta_3}{2}(\|z_d\|+\|\tilde{z}\|)^3 \quad (29)$$

$$\kappa_2\geq\sqrt{\zeta_3^2+\left(\zeta_4+\frac{\zeta_3}{2}(\|z_d\|+\|\tilde{z}\|)^2+\frac{\zeta_3}{2}|w|\right)^2} \quad (30)$$

and $\zeta_4\in\Re^1$ *is a positive bounding constant selected as follows*

$$\zeta_4\geq\rho_1+\rho_2. \quad (31)$$

Remark 3. Motivation for the structure of (24) is obtained by taking the time derivative of $z_d^Tz_d$ *as follows*

$$\frac{d}{dt}(z_d^Tz_d)=2z_d^T\dot{z}_d=2z_d^T\left(\frac{\dot{\delta}_d}{\delta_d}z_d+\left(\frac{k_1w+f}{\delta_d^2}+w\Omega_1\right)Jz_d\right) \quad (32)$$

where (24) *has been utilized. After noting that the matrix J of (12) is skew symmetric, we can rewrite (32) as follows*

$$\frac{d}{dt}(z_d^Tz_d)=2\frac{\dot{\delta}_d}{\delta_d}z_d^Tz_d. \quad (33)$$

As result of the selection of the initial conditions given in (24), it is easy to verify that

$$z_d^Tz_d=\|z_d\|^2=\delta_d^2 \quad (34)$$

is a unique solution to the differential equation given in (33). The relationship given by (34) will be used during the subsequent error system development and stability analysis.

Remark 4. Note that the exponential term in (26) is not necessary for the subsequent stability analysis. That is, if α_0 *is selected as* $\alpha_0=0$ *then the subsequent stability proof is still valid. The motivation for selecting* $\alpha_0\neq0$ *is to provide the designer with increased flexibility with regard to ensuring that the control effort is maintained at a reasonable magnitude. Specifically, for the case of large initial tracking error, the magnitude of the control could possibly be reduced through the selection of* α_0.

3.2 Closed-Loop Error System Development. To determine the closed-loop tracking error dynamics for $w(t)$, we substitute (22) for $u(t)$ in the open-loop tracking error system given by (11) add and subtract $u_a^TJz_d$ to the resulting expression, and then utilize (21) to rewrite the dynamics for $w(t)$, as follows

$$\dot{w}=u_a^TJ\tilde{z}-u_a^TJz_d+f+\chi_1 \quad (35)$$

where the fact that $J^T=-J$ was utilized. Finally, by substituting (23) for only the second occurrence of $u_a(t)$ in (35) and then utilizing the equality given by (34) and the fact that $J^TJ=I_2$ (note that I_2 denotes the standard 2×2 identity matrix), we can obtain the final expression for the closed-loop tracking error system for $w(t)$ as follows

$$\dot{w}=u_a^TJ\tilde{z}-k_1w+\chi_1. \quad (36)$$

To determine the closed-loop error system for $\tilde{z}(t)$, we take the time derivative of (21), substitute (24) for $\dot{z}_d(t)$, and then substitute (11) for $\dot{z}(t)$ to obtain the following expression

$$\dot{\tilde{z}}=\frac{\dot{\delta}_d}{\delta_d}z_d+\left(\frac{k_1w+f}{\delta_d^2}+w\Omega_1\right)Jz_d-u-\chi_2. \quad (37)$$

After substituting (22) for $u(t)$, and then substituting (23) for $u_a(t)$ in the resulting expression, we can rewrite (37) as follows

$$\dot{\tilde{z}}=\frac{\dot{\delta}_d}{\delta_d}z_d+w\Omega_1Jz_d-\Omega_1z_d+k_2z-\chi_2. \quad (38)$$

After substituting (25) for only the second occurrence of $\Omega_1(t)$ in (38) and using the fact that $JJ=-I_2$, we can cancel common terms and rearrange the resulting expression to obtain

$$\dot{\tilde{z}} = -k_2\tilde{z} + wJ\left[\left(\frac{k_1 w + f}{\delta_d^2}\right)Jz_d + \Omega_1 z_d\right] - \chi_2 \tag{39}$$

where (21) has been utilized. Finally, since the bracketed term in (39) is equal to $u_a(t)$ defined in (23), we can obtain the final expression for the closed-loop tracking error system for $\tilde{z}(t)$ as follows

$$\dot{\tilde{z}} = -k_2\tilde{z} + wJu_a - \chi_2. \tag{40}$$

4 Stability Analysis

Theorem 1. The kinematic control law given in (22)–(27) ensures the position and orientation tracking errors defined in (10) are GUUB in the sense that

$$|\tilde{x}(t)|,|\tilde{y}(t)|,|\tilde{\theta}(t)| \leq \sqrt{\beta_0 \exp(-\gamma_0 t) + \beta_1(\varepsilon_{c1} + \varepsilon_{c2})} + \beta_2 \exp(-\gamma_1 t) + \beta_3 \varepsilon_1 \tag{41}$$

where ε_1 was defined in (26), ε_{c1}, ε_{c2} were defined in (27) and β_0, β_1, β_2, β_3, γ_0, and $\gamma_1 \in \Re^1$ are some positive constants. With regard to (41) we note that ε_1, ε_{c1}, ε_{c2} can be made arbitrarily small.

Proof: To prove *Theorem 1*, we define a non-negative, scalar function denoted by $V(t) \in \Re^1$ as follows

$$V = \frac{1}{2}w^2 + \frac{1}{2}\tilde{z}^T\tilde{z}. \tag{42}$$

After taking the time derivative of (42), making the appropriate substitutions from (36) and (40), and then cancelling common terms, we obtain the following expression

$$\dot{V} = -k_1 w^2 - k_2\tilde{z}^T\tilde{z} + w\chi_1 - \tilde{z}^T\chi_2 \tag{43}$$

where we utilized the fact that $J^T = -J$. After substituting (27) for $k_1(t)$ and $k_2(t)$, we can upperbound $\dot{V}(t)$ of (43) as follows

$$\dot{V} \leq -k_s w^2 - k_s\|\tilde{z}\|^2 + \left[\kappa_1|w| - \frac{\kappa_1^2 w^2}{\kappa_1|w| + \varepsilon_{c1}}\right] + \left[\kappa_2\|\tilde{z}\| - \frac{\kappa_2^2\|\tilde{z}\|^2}{\kappa_2\|\tilde{z}\| + \varepsilon_{c2}}\right] \tag{44}$$

where (28) was utilized. We can now utilize (42) and the facts that

$$\left[\kappa_1|w| - \frac{2\kappa_1^2 w^2}{\kappa_1|w| + \varepsilon_{c1}}\right] \leq \varepsilon_{c1} \quad \left[\kappa_2\|\tilde{z}\| - \frac{2\kappa_2^2\|\tilde{z}\|^2}{\kappa_2\|\tilde{z}\| + \varepsilon_{c2}}\right] \leq \varepsilon_{c2} \tag{45}$$

to upper bound $\dot{V}(t)$ of (44) as follows

$$\dot{V} \leq -2k_s V + \varepsilon_{c1} + \varepsilon_{c2}. \tag{46}$$

After solving the differential inequality given in (46), we obtain the following expression

$$V \leq \exp(-2k_s t)V(0) + \frac{\varepsilon_{c1} + \varepsilon_{c2}}{2k_s}(1 - \exp(-2k_s t)). \tag{47}$$

Finally, we can utilize (42) to rewrite the inequality given by (47) as

$$\|\Psi(t)\| \leq \sqrt{\exp(-2k_s t)\|\Psi(0)\|^2 + \frac{\varepsilon_{c1} + \varepsilon_{c2}}{k_s}(1 - \exp(-2k_s t))} \tag{48}$$

where the vector $\Psi(t) \in \Re^3$ is defined as

$$\Psi = [w \quad \tilde{z}^T]^T. \tag{49}$$

Based on (48) and (49), it is clear that $w(t)$, $\tilde{z}(t) \in \mathcal{L}_\infty$. After utilizing (21), (34), and the fact that $\tilde{z}(t)$, $\delta_d(t) \in \mathcal{L}_\infty$, we can conclude that $z(t)$, $z_d(t) \in \mathcal{L}_\infty$. From (5) and the fact that $w(t)$, $z(t)$, $\tilde{z}(t)$, $z_d(t) \in \mathcal{L}_\infty$, it is clear from (19), (20), (28)–(30) that $\chi_1(t)$, $\chi_2(t)$, $\kappa_1(w, \tilde{z}, t)$, $\kappa_2(w, \tilde{z}, t) \in \mathcal{L}_\infty$. Based on these facts, we can now use the assumption that $v_{r1}(t), v_{r2}(t) \in \mathcal{L}_\infty$, (13), (22)–(26), and (34) to show that $f(z, v_r, t)$, $u_a(t)$, $\dot{z}_d(t)$, $\Omega_1(t)$, $u(t) \in \mathcal{L}_\infty$. Now, in order to illustrate that the Cartesian position and orientation signals defined in (1) are bounded, we utilize the inverse of the transformation given in (9) as follows

$$\begin{bmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{\theta}\end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sin\theta & 0 & \frac{1}{2}(\tilde{\theta}\sin\theta + 2\cos\theta)\\ -\frac{1}{2}\cos\theta & 0 & -\frac{1}{2}(\tilde{\theta}\cos\theta - 2\sin\theta)\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}w\\ z_1\\ z_2\end{bmatrix}. \tag{50}$$

Since $z(t) \in \mathcal{L}_\infty$, it is clear from (10) and (50) that $\tilde{\theta}(t)$, $\theta(t) \in \mathcal{L}_\infty$. Furthermore, from (10), (50), and the fact that $w(t)$, $z(t)$, $\theta(t) \in \mathcal{L}_\infty$, we can conclude that $\tilde{x}(t)$, $\tilde{y}(t)$, $x_c(t)$, $y_c(t) \in \mathcal{L}_\infty$. We can utilize (14) the assumption that $v_{r1}(t)$, $v_{r2}(t) \in \mathcal{L}_\infty$, and the fact that $\theta(t)$, $u(t)$, $\tilde{x}(t)$, $\tilde{y}(t) \in \mathcal{L}_\infty$, to show that $v(t) \in \mathcal{L}_\infty$; therefore, it follows from (1)–(5) that $\dot{\theta}(t)$, $\dot{x}_c(t)$, $\dot{y}_c(t) \in \mathcal{L}_\infty$. Standard signal chasing arguments can now be employed to conclude that all of the remaining signals in the control and the system remain bounded during closed-loop operation.

In order to prove that $z(t)$ defined in (9) is GUUB, we can now apply the triangle inequality to (21) to obtain the following bound for $z(t)$

$$\|z\| \leq \|\tilde{z}\| + \|z_d\| \leq \sqrt{\exp(-2k_s t)\|\Psi(0)\|^2 + \frac{\varepsilon_{c1} + \varepsilon_{c2}}{k_s}(1 - \exp(-2k_s t))} + \alpha_0 \exp(-\alpha_1 t) + \varepsilon_1 \tag{51}$$

where (26), (34), (48) and (49) have been utilized. Based on (48)–(51), the result given in (41) can now be directly obtained. □

Remark 5. It is clear that if the control terms ε_{c1} and ε_{c2} are set to zero in (44), then the stability analysis would be the same as for a variable structure controller. We refer to the control scheme as ''variable structure-like'' due to the close resemblance of the stability analysis to the classic variable structure stability analysis.

5 Setpoint Extension

Unlike some of the previously proposed tracking controllers (see [12,10,11], etc.) we have not imposed any restrictions on the desired trajectory (other than the assumption that $v_r(t)$, $\dot{v}_r(t)$, $q_r(t)$, and $\dot{q}_r(t) \in \mathcal{L}_\infty$); hence, the position and orientation tracking problem reduces to the position and orientation regulation problem in a similar manner as illustrated in [13]. That is, if the control objective is targeted at the regulation problem, the desired position and orientation vector, denoted by $q_r = [x_{rc}\; y_{rc}\; \theta_r]^T \in \Re^3$ and originally defined in (8), becomes an arbitrary desired constant vector. Based on the fact that $q_r(t)$ is now defined as a constant vector, it is straightforward that $v_r(t)$ given in (8), and consequently $f(z, v_r, t)$ defined in (13) equal zero. We also note that the auxiliary variable $u(t)$ originally defined in (14), is now defined as follows

$$u = T^{-1}v \quad v = Tu \tag{52}$$

where the matrix T was defined in (15). Based on the above simplifications, it is easy to show that the result given by *Theorem 1* is valid for the regulation problem as well. Furthermore, we note that the proposed kinematic control law given in (22), (23), (24), (25), (26), and (27) can be slightly modified as illustrated in Remark 5.5 of [14] to reject parametric uncertainty and bounded disturbances in the dynamic model.

6 Uncertain Kinematic Model Extension

Several researchers have examined the setpoint regulation problem for a WMR with a kinematic model containing parametric uncertainties (see [7–9]). Specifically, the kinematics of the WMR are given as follows

$$\dot{q}=S(q)Av \tag{53}$$

where $q(t)$ was defined in (2) $S(\cdot)$ was defined in (3), $v(t)=[v_1 \quad v_2]$ was defined (4), and $A\in\Re^{2\times 2}$ is defined as follows

$$A=\begin{bmatrix} p_1^* & 0\\ 0 & p_2^*\end{bmatrix} \tag{54}$$

where p_1^*, $p_2^*\in\Re^1$ are uncertain positive parameters that represent the radius of the wheels and the distance between them.

6.1 Error System Development. In order to facilitate the subsequent control design, we utilize the following global invertible transformation [2]

$$\begin{bmatrix} e_1\\ e_2\\ e_3\end{bmatrix}=\begin{bmatrix} \sin\theta & -\cos\theta & 0\\ 0 & 0 & 1\\ \cos\theta & \sin\theta & 0\end{bmatrix}\begin{bmatrix} \tilde{x}\\ \tilde{y}\\ \tilde{\theta}\end{bmatrix} \tag{55}$$

where $e_1(t)$, $e_2(t)$, $e_3(t)\in\Re^1$ are auxiliary tracking error variables, and x_{rc}, y_{rc}, θ_r, originally defined in (8), are now defined as arbitrary desired constants. After taking the time derivative of (55) and using (2)–(4), (53)–(55), we can rewrite the open-loop error dynamics in a more convenient form as follows

$$\begin{bmatrix} \dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix}=\begin{bmatrix} p_2^* v_2 e_3\\ p_2^* v_2\\ p_1^* v_1 - p_2^* v_2 e_1\end{bmatrix}. \tag{56}$$

Based on the open-loop error dynamics given in (56) and the subsequent stability analysis, we utilize the following smooth, time-varying controller [2]

$$\begin{bmatrix} v_1\\ v_2\end{bmatrix}=\begin{bmatrix} -k_2 e_3 - e_1 v_2\\ -k_1 e_2 + e_1^2\sin(t)\end{bmatrix}. \tag{57}$$

After substituting (57) into (56) and performing some algebraic manipulation, we obtain the following closed-loop error system

$$\begin{bmatrix} \frac{1}{p_2^*}\left(1+\frac{p_2^*}{p_1^*}\right)\dot{e}_1\\ \frac{1}{p_2^*}\dot{e}_2\\ \frac{1}{p_1^*}\dot{e}_3\end{bmatrix}=\begin{bmatrix} \left(1+\frac{p_2^*}{p_1^*}\right)v_2 e_3\\ -k_1 e_2 + e_1^2\sin(t)\\ -k_2 e_3 - e_1 v_2\left(1+\frac{p_2^*}{p_1^*}\right)\end{bmatrix}. \tag{58}$$

Remark 6. Note that the closed-loop dynamics for $e_2(t)$ given in (58) represent a stable linear system subjected to an additive disturbance given by the product $e_1^2(t)\sin(t)$. If the additive disturbance is bounded (i.e., if $e_1(t)\in\mathcal{L}_\infty$), then it is clear that $e_2(t)\in\mathcal{L}_\infty$. Furthermore, if the additive disturbance asymptotically vanishes (i.e., if $\lim_{t\to\infty} e_1(t)=0$) then it is clear that $\lim_{t\to\infty} e_2(t)=0$.

6.2 Stability Analysis. *Theorem 2. The smooth, time-varying kinematic control law given in (57) ensures global asymptotic position and orientation regulation in the sense that*

$$\lim_{t\to\infty}\tilde{x}(t),\tilde{y}(t),\tilde{\theta}(t)=0. \tag{59}$$

Proof: To prove *Theorem 2*, we define a non-negative, scalar function denoted by $V_2(t)\in\Re^1$ as follows

$$V=\frac{1}{2}\frac{1}{p_2^*}\left(1+\frac{p_2^*}{p_1^*}\right)e_1^2+\frac{1}{2}\frac{1}{p_1^*}e_3^2. \tag{60}$$

After taking the time derivative of (60), making the appropriate substitutions from (58), and then canceling common terms, we obtain the following expression

$$\dot{V}=-k_2 e_3^2. \tag{61}$$

Based on (61) and (60), it is clear that $e_1(t)$, $e_3(t)\in\mathcal{L}_\infty$ and that $e_3(t)\in\mathcal{L}_2$. Since $e_1(t)\in\mathcal{L}_\infty$, it is clear from Remark 6 that $e_2(t)\in\mathcal{L}_\infty$. Based on the fact that $e_1(t)$, $e_2(t)$, $e_3(t)\in\mathcal{L}_\infty$, we can utilize (57) and (58) to prove that $v_1(t)$, $v_2(t)$, $\dot{e}_1(t)$, $\dot{e}_2(t)$, $\dot{e}_3(t)\in\mathcal{L}_\infty$. Since $\dot{e}_1(t)$, $\dot{e}_2(t)$, $\dot{e}_3(t)\in\mathcal{L}_\infty$, we can conclude that $e_1(t)$, $e_2(t)$, $e_3(t)$ are uniformly continuous. After taking the time derivative of (57) and utilizing the aforementioned facts, we can show that $\dot{v}_1(t)$, $\dot{v}_2(t)\in\mathcal{L}_\infty$, and hence, $v_1(t)$, $v_2(t)$ are uniformly continuous.

Based on the facts that $e_3(t)\in\mathcal{L}_2$ and that $e_3(t)$ is uniformly continuous, we can now utilize Barbalat's lemma [15] to prove that $\lim_{t\to\infty} e_3(t)=0$. After taking the time derivative of the product $e_1(t)e_3(t)$ and substituting (58) in the resulting expression, we can conclude that

$$\frac{d}{dt}(e_1 e_3)=-\left[p_1^*\left(1+\frac{p_2^*}{p_1^*}\right)e_1^2 v_2\right]+e_3(\dot{e}_1-k_2 p_1^* e_1). \tag{62}$$

Since the bracketed term in (62) is uniformly continuous (*i.e.*, $e_1(t), v_2(t)$ are uniformly continuous) and $\lim_{t\to\infty} e_3(t)=0$, we can utilize an extension of Barbalat's lemma [2] to conclude that

$$\lim_{t\to\infty}\frac{d}{dt}(e_1 e_3)=0 \qquad \lim_{t\to\infty} p_1^*\left(1+\frac{p_2^*}{p_1^*}\right)e_1^2 v_2=0. \tag{63}$$

From the second limit in (63), it is clear that $\lim_{t\to\infty} e_1(t)v_2(t)=0$. From the facts that $\lim_{t\to\infty} e_1(t)v_2(t)=0$ and $\lim_{t\to\infty} e_3(t)=0$, we can utilize (57) to conclude that $\lim_{t\to\infty} v_1(t)=0$, and hence, from (58), we can prove that $\lim_{t\to\infty}\dot{e}_1(t)=0$ and $\lim_{t\to\infty}\dot{e}_3(t)=0$.

To facilitate further analysis, we take the time derivative of the product $e_1(t)v_2(t)$ and utilize (58) to obtain the following expression

$$\frac{d}{dt}(e_1 v_2)=[e_1^3\cos(t)]+\dot{e}_1(v_2+2e_1\sin(t))-p_2^* k_1 e_1 v_2. \tag{64}$$

Since the bracketed term in (64) is uniformly continuous, $\lim_{t\to\infty}\dot{e}_1(t)=0$, and $\lim_{t\to\infty} e_1(t)v_2(t)=0$, we can utilize an extension of Barbalat's lemma [2] to conclude that

$$\lim_{t\to\infty}\frac{d}{dt}(e_1 v_2)=0 \qquad \lim_{t\to\infty} e_1^3\cos(t)=0. \tag{65}$$

From the second limit in (65), it is clear that $\lim_{t\to\infty} e_1(t)=0$. Since we have shown that $\lim_{t\to\infty} e_1(t), e_2(t), e_3(t)=0$, we can now utilize the inverse of the transformation defined in (55), given as follows

$$\begin{bmatrix} \tilde{x}\\ \tilde{y}\\ \tilde{\theta}\end{bmatrix}=\begin{bmatrix} \sin\theta & 0 & \cos\theta\\ -\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix} e_1\\ e_2\\ e_3\end{bmatrix} \tag{66}$$

to obtain the result given in (59). □

7 Simulation Results

The control law given in (22)–(27) was simulated based on the kinematic model given in (1) where $\rho_1(t)$, $\rho_2(t)$, $\rho_3(t)$ were selected in a similar manner as in [4] as follows

$$\rho_1=[0.01H(t-2)-0.01H(t-4)]\sin\theta \tag{67}$$

$$\rho_2 = -[0.01H(t-2) - 0.01H(t-4)]\cos\theta \tag{68}$$

$$\rho_3 = 0.01H(t-2) - 0.01H(t-4) \tag{69}$$

where $H(\cdot)$ denotes the standard Heaviside step function. The desired reference linear and angular velocity were selected as

$$v_{r1} = 1\,(\text{m/s}) \quad v_{r2} = \frac{-\sin(x_r)\cos\theta_r}{1+\tan^2\theta_r}\,(\text{rad/s}), \tag{70}$$

respectively, where

$$x_{cr}(0) = 0(m), \quad y_{cr}(0) = 0(m), \quad \theta_r(0) = 0.78539\,(\text{rad}) \tag{71}$$

(see Fig. 2 for the resulting reference time-varying Cartesian position and orientation).

The Cartesian position/orientation and the auxiliary signal $z_d(t)$ were initialized as follows

$$x_c(0) = -1(m), \quad y_c(0) = -1(m),$$
$$\theta_r(0) = 0\,(\text{rad}), \quad z_d(0) = [2 \quad 0]^T. \tag{72}$$

The control gains that resulted in the best performance are given below

$$k_s = 10, \quad \alpha_0 = 2, \quad \alpha_1 = 10,$$
$$\varepsilon_1 = 0.001, \quad \varepsilon_{c1} = 0.002, \quad \varepsilon_{c1} = 0.002 \tag{73}$$

where the bounding terms ζ_1, ζ_2, ζ_3, and ζ_4 given in (5) and (31) were selected as follows

$$\zeta_1 = 0.5, \quad \zeta_2 = 0.05, \quad \zeta_3 = 0.1 \text{ and } \zeta_4 = 1.0. \tag{74}$$

The position/orientation tracking error of the COM of the WMR and the associated control inputs are shown in Fig. 3 and Fig. 4, respectively. Utilizing the same control gains and initial conditions, we also demonstrate the effectiveness of the proposed controller with regard to the regulation problem. That is, with the reference velocity signals in (70) set to zero and the desired position and orientation setpoint selected as zero, the proposed controller yields position/orientation regulation errors as shown in Fig. 5 with the associated control inputs given in Fig. 6. Note that by increasing the control terms ε_1, ε_{c1}, and ε_{c2}, the ''chattering'' effect observed in Fig. 4 and Fig. 6 can be eliminated; however, from (41) it is clear that steady-state position/orientation tracking error will be bounded by a larger neighborhood about the origin. To illustrate this fact, the control parameters ε_1, ε_{c1}, and ε_{c2}, were increased until the ''chattering'' effect was reduced. The resulting values of the control parameters are given below

$$\varepsilon_1 = 0.001, \quad \varepsilon_{c1} = 0.001, \quad \varepsilon_{c1} = 0.015. \tag{75}$$

The resulting position/orientation errors and the associated control torque input are given in Figs. 7 and 8 for the tracking controller and Figs. 9 and 10 for the regulation controller.

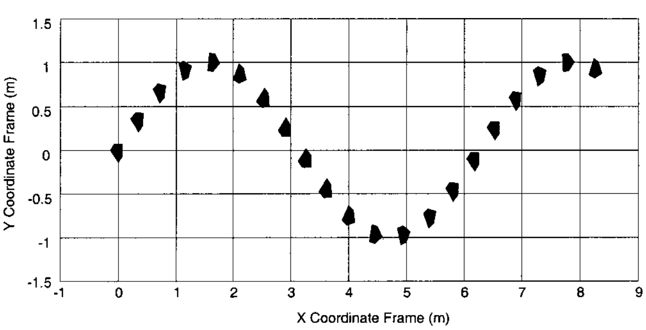

Fig. 2 Desired Cartesian trajectory

Simulation results for the kinematic model given in (53) and (54) are also presented to illustrate the effectiveness of the control law given in (57) where $\rho_1^*(t)$ and $\rho_2^*(t)$ were selected as follows

$$\rho_1^* = 1.5, \quad \rho_2^* = 2.5 \tag{76}$$

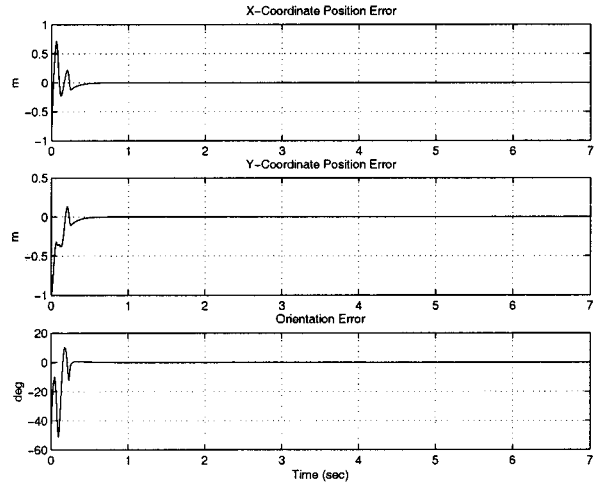

Fig. 3 Position/orientation tracking errors

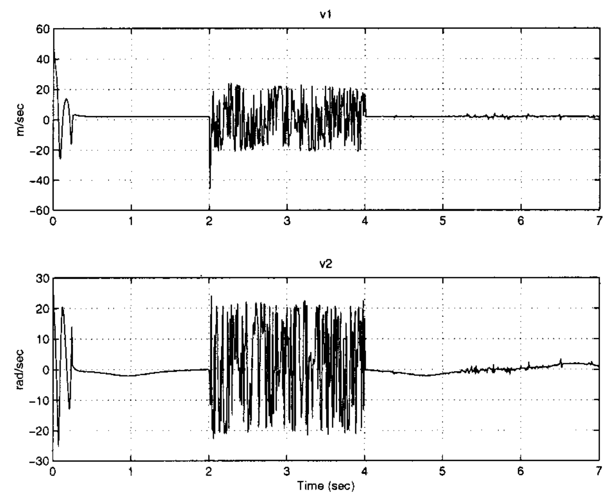

Fig. 4 Tracking control input

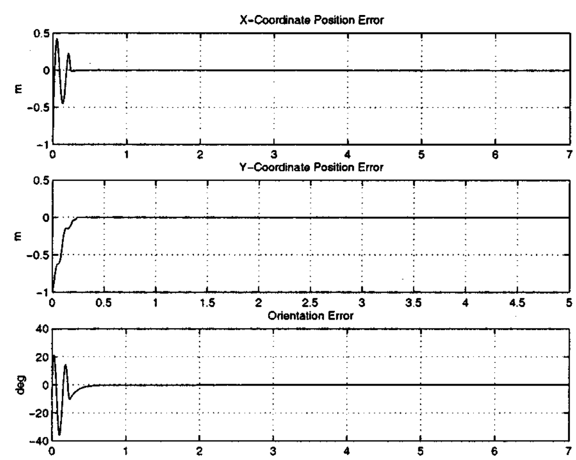

Fig. 5 Position/orientation regulation errors

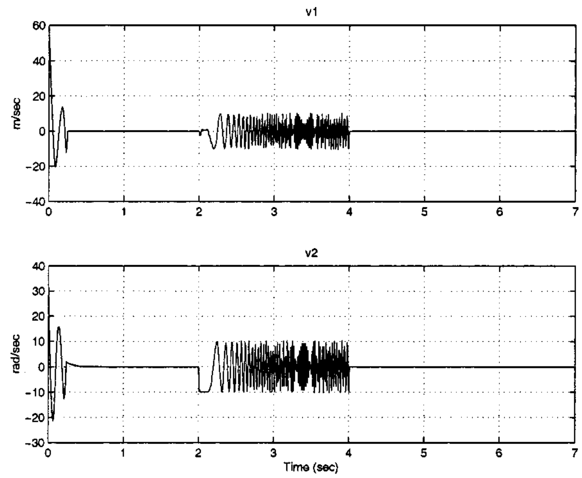

Fig. 6 Regulation control input

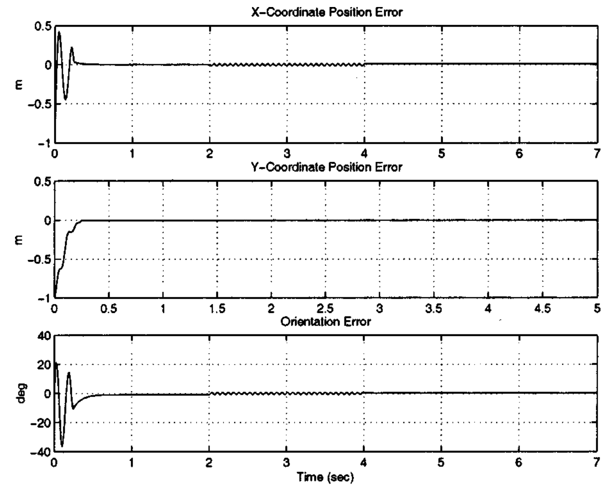

Fig. 9 Position/orientation regulation errors

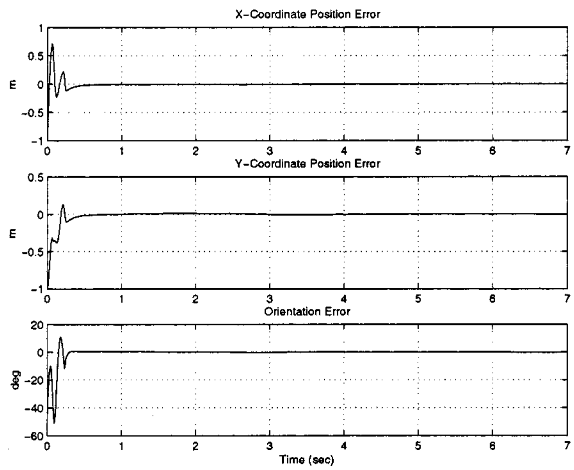

Fig. 7 Position/orientation tracking errors

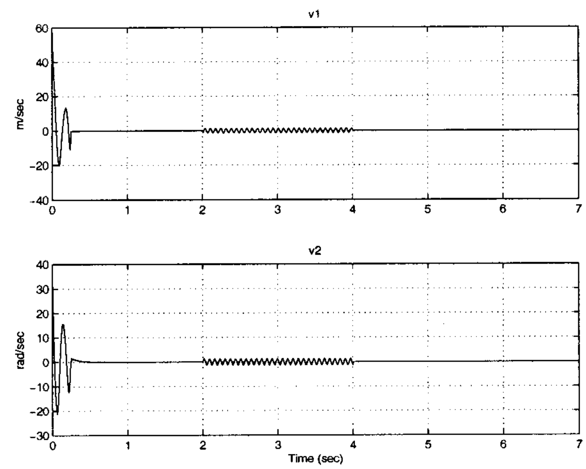

Fig. 10 Regulation control input

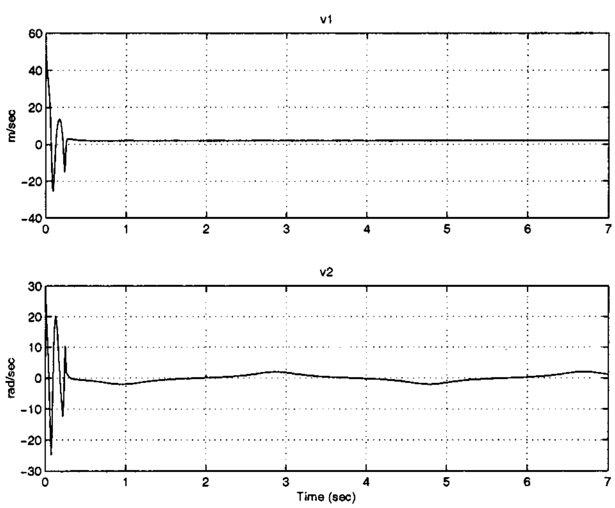

Fig. 8 Tracking control input

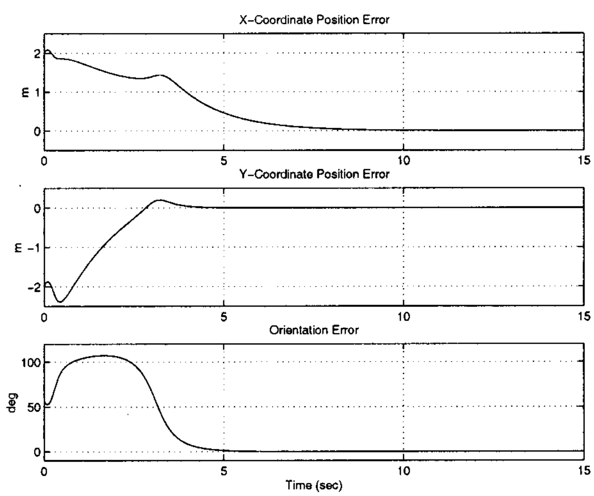

Fig. 11 Position/orientation regulation errors

and the Cartesian position/orientation was initialized as follows

$$x_{cr}(0)=2(m), \quad y_{cr}(0)=-2(m), \quad \theta_r(0)=1\ (\mathrm{rad}). \quad (77)$$

The control gains that resulted in the best performance are given below

$$k_1=0.75, \quad k_2=0.5. \quad (78)$$

The resulting position/orientation regulation error and the associated control inputs are shown in Figs. 11 and 12, respectively.

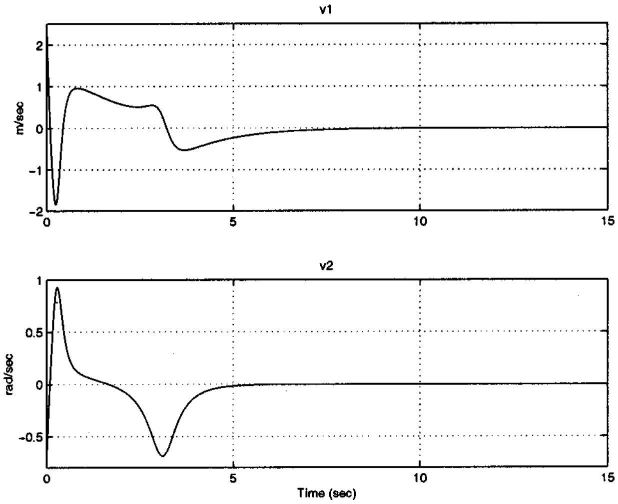

Fig. 12 Regulation control input

8 Conclusion

In this paper, we designed a variable structure-like tracking controller for a mobile robot system subject to bounded disturbances in the kinematic model. Through the use of a Lyapunov-based stability analysis, we have demonstrated that; (*i*): the position and orientation tracking errors exponentially converge to a neighborhood about zero that can be made arbitrarily small, and (*ii*) the controller provides robustness with regard to bounded disturbances in the kinematic model. In addition, we illustrated that the proposed controller can be utilized to regulate the position and orientation of the WMR to an arbitrary desired setpoint. Moreover, since the proposed tracking controller is smooth, we noted that it can be modified to include the dynamic model of the WMR to enhance the overall robustness. An additional extension was also provided to illustrate that the smooth, time-varying controller designed in [2] can be applied to solve the setpoint regulation problem of a WMR with parametric uncertainties in the kinematic model. It should also be noted that in addition to the WMR problem, the proposed controllers can be applied to other nonholonomic systems (see [16] for example). Finally, simulation results provide verification for the proposed controllers.

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