Updates on Research and Collaborations

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Spring 2024 Review
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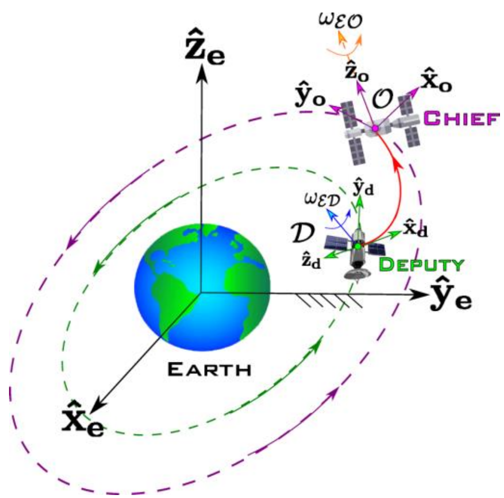
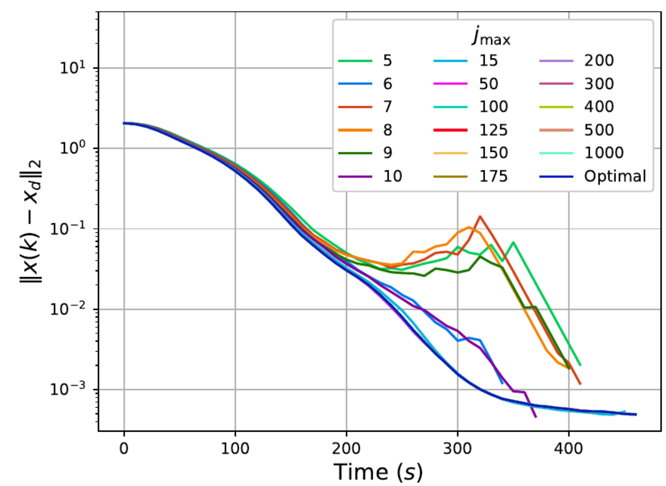
Our work is running on a space-grade processor

- Joint work with Sean Phillips and Alex Soderlund (both at RV) led to time-constrained MPC for satellites
- Gabriel Behrendt (PhD student) created a time-constrained MPC algorithm that has been implemented in the SPACER Lab at RV

- Underlying idea: space-grade processors are slow, so we often cannot solve MPC sub-problems exactly
- Main question: with limited time, do we achieve stability/high performance?

Unibap SpaceCloud iX10-101





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Collaborations with Air Force Colleagues

- For Summer 2024:
  - William Warke is at RW with Kevin Brink
    - The paper “Pose Graph Optimization over Planar Unit Dual Quaternions: Improved Accuracy with Provably Convergent Riemannian Optimization” is under review
    - Working on a joint journal paper extension
  - Adam Pooley is at RW with Adrienne Dorr
  - Alexander Benvenuti is at RW with Mitzi Dennis
    - The joint paper “Differentially Private Reward Functions for Multi-Agent Markov Decision Processes” was just accepted to CCTA 2024
    - A journal extension will be submitted soon
  - Gabriel Behrendt is at RW with Zach Bell
    - The paper “Distributed Asynchronous Discrete-Time Feedback Optimization” is under review
    - Multiple time-varying non-convex optimization papers are in preparation

Efficient Verification of High Order Control Barrier Functions

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Safety-critical systems are widely used

- Safety-critical systems…
  - Have state-based constraints that designate a safe region
  - Must prioritize safety over optimality
  - Require a guarantee that safety can be maintained ahead of time
- Example systems:
  - Autonomous vehicles
  - Medical robots
  - Industrial robots

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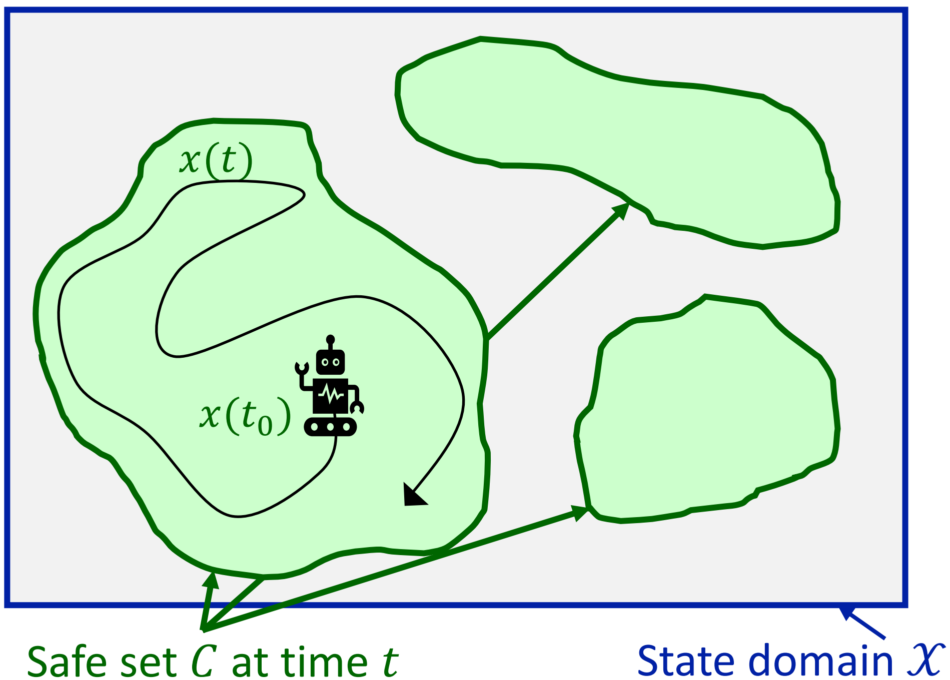
Forward invariance encodes safety

- Safety ↔ Bad things never happen
- Forward invariance is a method used for safety-critical systems
  - Identify a safe region $C \subseteq \mathcal{X}$
  - Goal is to make C forward invariant through control-action



Definition:

A set $C \subset \mathbb{R}^n$ is forward invariant if every solution with $x(0) \in C$ implies $x(t) \in C$ for all $t \geq 0$.

Control barrier functions are commonly applied

- Control Barrier Functions (CBFs) are used in nonlinear, control-affine systems

$$\dot{x} = f(x) + g(x)u$$

  - The state, $x \in \mathcal{X} \subset \mathbb{R}^n$
  - The input, $u \in \mathcal{U} \subset \mathbb{R}^q$

Polynomial functions

- The CBF function $b: C \to \mathbb{R}$ defines the safe area C

$$C = \{x \in \mathcal{X} \mid b(x) \geq 0\}$$

  - Assume that (1.) b is polynomial, (2.) C is compact, (3.) $C \subseteq \mathcal{X}$, for this talk

Problem 0, Safety verification:

Develop conditions that guarantee that safe control action is always available, with respect to the CBF and other system-specific requirements.

CBFs apply forward invariance to a system

Definition (Ames et al., 2019):

A function $b: C \to \mathbb{R}$ is a CBF if there exists a class $\mathcal{K}_\infty$ function α such that

$$\sup_{u \in U}\left[L_f b(x) + L_g b(x) u + \alpha\big(b(x)\big)\right] \geq 0$$

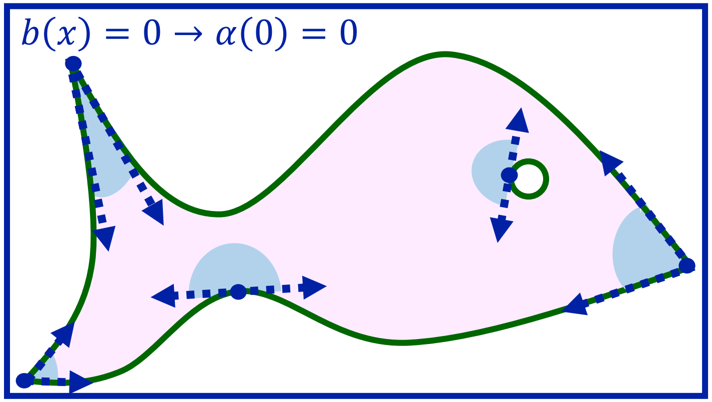
for all $x \in C$.

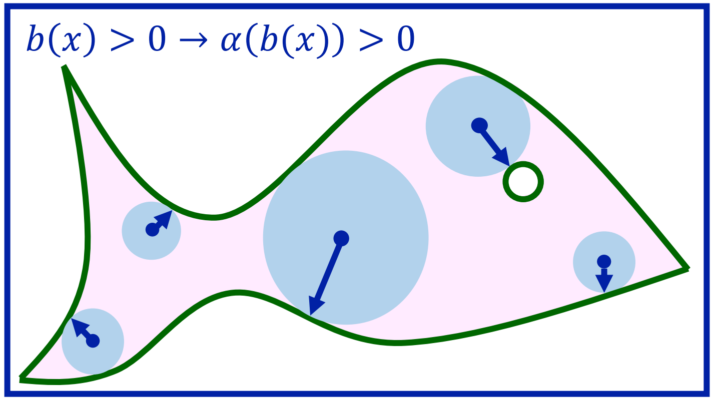
- The admissible input set is

$$\Psi_1(x) = \{u \in U \mid L_f b(x) + L_g b(x) u + \alpha\big(b(x)\big) \geq 0\}$$

Theorem (Ames et al., 2019):

Given a CBF b, any Lipschitz continuous controller $u: [0, \infty) \to U$ such that $u(t) \in \Psi_1(x(t))$ for all $t \geq 0$ renders C forward invariant.





Aaron D Ames, Samuel Coogan, Magnus Egerstedt, Gennaro Notomista, Koushil Sreenath, and Paulo Tabuada, Control barrier functions: Theory and applications, 2019 18th European control conference (ECC), IEEE, 2019, pp. 3420–3431.

- High Order Control Barrier Functions (HOCBFs) are an extension of CBFs
  - The input does not appear on the first derivative

$$L_f b(x) + L_g b(x) u(t) + \alpha\big(b(x)\big) \geq 0$$

$L_g b(x) \to 0$

Control action is rendered ineffective

  - The relative degree r is the number of times b must be differentiated until the input shows up
- The high-degree functions:

$$\psi_0(x) = b(x)$$
$$\psi_1(x) = \dot{\psi}_0(x) + \alpha_1(\psi_0(x))$$
$$\vdots$$
$$\psi_{r-1}(x) = \dot{\psi}_{r-2}(x) + \alpha_{r-1}(\psi_{r-2}(x))$$
$$\psi_r(x,u) = \underbrace{\dot{\psi}_{r-1}(x,u)} + \alpha_r(\psi_{r-1}(x))$$

The input shows up here!

Wei Xiao and Calin Belta. Control barrier functions for systems with high relative degree. In *2019 IEEE 58th conference on decision and control (CDC)*, pages 474-479. IEEE, 2019.

HOCBFs are a natural extension of CBFs

Definition:

A function $b: C_1 \rightarrow \mathbb{R}$ is an HOCBF with relative degree r if there exist r class $\mathcal{K}_\infty$ functions α_i such that

$$\sup_{u \in U}[\psi_r(x,u)] \geq 0 \qquad for\ all \quad x \in C_r$$

- The admissible input set is

$$\Psi_r(x) = \{u \in U \mid \psi_r(x,u) \geq 0\}$$

Theorem:

Given a relative degree r HOCBF, any Lipschitz continuous controller $u: [0,\infty) \rightarrow U$ such that $u(t) \in \Psi_r(x(t))$ for all $t \geq 0$ renders C_1 forward invariant.

Wei Xiao and Calin Belta. Control barrier functions for systems with high relative degree. In *2019 IEEE 58th conference on decision and control (CDC)*, pages 474-479. IEEE, 2019.

The forward invariance connection

Theorem:
Given a relative degree r HOCBF, any Lipschitz continuous controller $u: [0, \infty) \to U$ such that $u(t) \in \Psi_r(x(t))$ for all $t \geq 0$ renders C_1 forward invariant.

$C = \{ x \in \mathcal{X} \mid b(x) \geq 0 \}$

$$
\begin{aligned}
C_1 &= \{x \in \mathcal{X} \mid \psi_0(x) \geq 0\} \\
C_2 &= \{ x \in \mathcal{X} \mid \psi_1(x) \geq 0 \} \\
&\vdots \\
C_{r-1} &= \{ x \in \mathcal{X} \mid \psi_{r-2}(x) \geq 0 \} \\
C_r &= \{x \in \mathcal{X} \mid \psi_{r-1}(x) \geq 0\}
\end{aligned}
$$

Information about one companion set provides information about all companion sets.

Theorem 1 (Pond & Hale, In Preparation):
Assume: All C_i are compact and nonempty

$C_i = \{ x \in \mathcal{X} \mid \psi_{i-1}(x) \geq 0 \}$ is forward invariant $\iff$ $C_{i+1} = \{ x \in \mathcal{X} \mid \psi_i(x) \geq 0 \}$ is forward invariant

Aaron D Ames, Samuel Coogan, Magnus Egerstedt, Gennaro Notomista, Koushil Sreenath, and Paulo Tabuada, Control barrier functions: Theory and applications, 2019 18th European control conference (ECC), IEEE, 2019, pp. 3420–3431.

Potential conflicts can arise in real-time

CBFs/HOCBFs are typically implemented in real-time with sequential quadratic programs (SQPs)

SQP

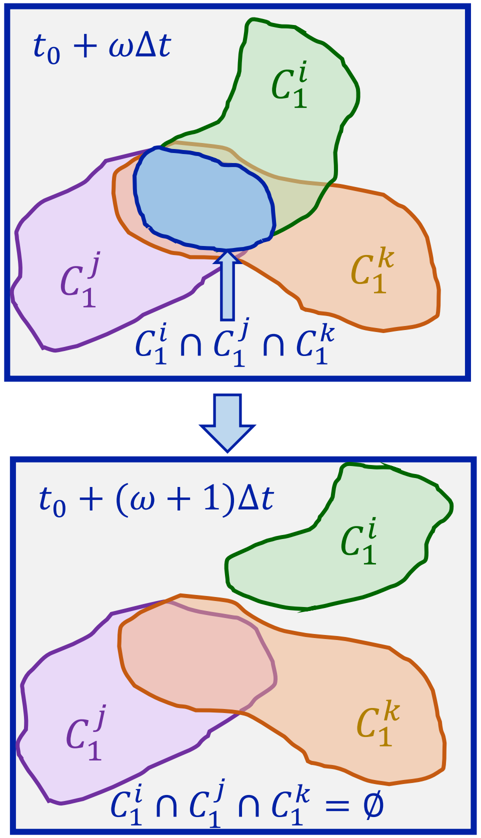
$$\min_{u(t)\in\mathcal{U}} ||u(t)||^2$$

$$s.t\ \psi^1_{r_1}(x(t),u(t)) \geq 0$$

$$\vdots$$

$$\psi^J_{r_J}(x(t),u(t)) \geq 0 \quad \Longrightarrow \text{HOCBFs}$$

$$L_f V(x(t)) + L_g V(x(t))u(t) + \gamma\left(V(x(t))\right) \leq 0 \quad \Longrightarrow \text{CLF}$$



Possible conflicts:

1. **Validation** of the HOCBF definition – Does b always satisfy its definition?
2. **Actuation limits** – Will the input needed for safety ever be outside the system's input bounds?
3. **Safe stabilization** – Can the CLF and CBF constraint be satisfied by the same input?
4. **Multiple HOCBFs** – Will there be a time when the intersection of the safe sets is empty?

Problem 1, Single HOCBF Verification:

For a single HOCBF b, determine a sufficient condition to guarantee the continued feasibility of the SQP.

Definition:

A system has a guarantee of continued feasibility if the existence of a continuous input that satisfies:

1. The actuation limits $u \in \mathcal{U} = \{Au(x) \leq c\}$
2. The HOCBF definition validation
3. The CLF definition validation

has been established.

SQP

$$\min_{u(t)\in\mathcal{U}} ||u(t)||^2$$

$$s.t\ \ \psi_r(x(t), u(t)) \geq 0$$

$$L_f V(x(t)) + L_g V(x(t)) u(t) + \gamma\left(V(x(t))\right) \leq 0$$

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Algebraic geometry has a natural connection

- Nonnegativity certificates
  - **Question:** Is the polynomial $g \in \mathbb{R}[x]$ nonnegative over the semialgebraic set
    $$K_S = \{x \in \mathbb{R}^n \mid v_1(x) \geq 0, \dots, v_m(x) \geq 0\}?$$
  - **Answer:** If $g \in Q_S$, then yes!

Q_S is an algebraic object – a specifically structured polynomial set that requires nonnegativity for membership

Definition (Powers, 2021):
The quadratic module generated by the set $S = \{v_1, \dots, v_m\}$ is
$$Q_S = \{q \in \mathbb{R}[x] \mid q(x) = s_0(x) + \sum_{i=1}^{m} s_i(x) v_i(x)\}$$

Sums-of-squares polynomial:
$s_i(x) = \sum p_i^2(x)$

- Recall, we need to know if $\psi_r(x, u) \geq 0$ over the set
  $$C_r = \{x \in \mathcal{X} \mid \psi_{r-1}(x) \geq 0\}$$
- Nonnegativity certificates can be solved using semidefinite programs (SDPs)

The verification for a single HOCBF

Theorem 2 (Pond & Hale, In Preparation):

The continued feasibility for a system with

1. A HOCBF
2. A CLF
3. Actuation constraints

is guaranteed if there is a solution to the SDP:

Decision Variables
$u \in \mathbb{R}^q[x]$
$s_0, s_1, s_2, s_3 \in \Sigma[x]$
$s_4, s_5 \in \Sigma^q[x]$

$$\begin{aligned} \min \quad & 0 \\ s.t. \quad & \psi_r(x, u(x)) = s_0(x) + s_1(x)\psi_{r-1}(x) \\ & -L_f V(x) - L_g V(x) u(x) - \gamma(V(x)) = s_2(x) + s_3(x)\psi_0(x) \\ & -Au(x) + c = s_4(x) + s_5(x)\psi_0(x) \end{aligned}$$

Nonnegativity certificates:

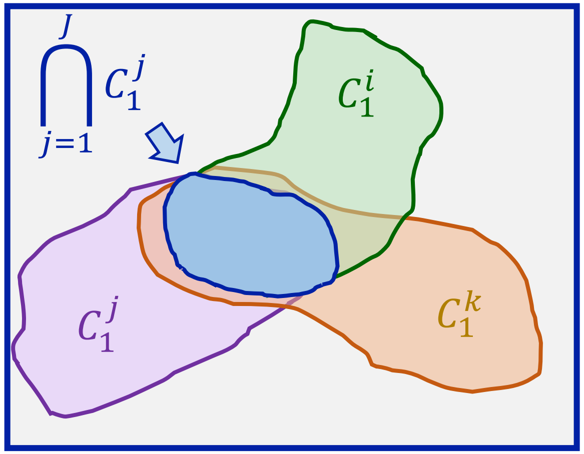
$\psi_r(x, u(x)) \geq 0$ for all $x \in C_r$ ⇒ HOCBF definition satisfied and C_1 forward invariant

$L_f V(x) + L_g V(x) u(x) + \gamma(V(x)) \leq 0$ for all $x \in C_1$ ⇒ Safe stabilization is possible

$Au(x) \leq c$ for all $x \in C_1$ ⇒ Actuation limits satisfied

SQP
$\min_{u(t)\in\mathcal{U}} \lvert\lvert u(t)\rvert\rvert^2$ $s.t\ \ \psi^1_{r_1}\big(x(t), u(t)\big) \geq 0$ $\vdots$ $\psi^J_{r_J}\big(x(t), u(t)\big) \geq 0$ $L_f V\big(x(t)\big) + L_g V\big(x(t)\big) u(t) + \gamma\Big(V\big(x(t)\big)\Big) \leq 0$

Problem 2, Multiple HOCBF Verification: Given a collection of HOCBFs $\{b_1, \dots, b_J\}$, determine a sufficient condition to guarantee the continued feasibility of the SQP.



- Now, we must establish
  - HOCBF definition validation for each $j \in [J]$
  - CLF definition validation
  - Actuation constraint compliance

over the forward invariant intersection set

$$\bigcap_{j=1}^{J} C_1^j = \{x \in \mathcal{X} \mid \psi_0^1(x) \geq 0, \dots, \psi_0^J(x) \geq 0\}$$

The verification for multiple HOCBFs

Theorem 3 (Pond & Hale, In Preparation):

The continued feasibility for a system with

1. J HOCBFs
2. A CLF
3. Actuation constraints

is guranteed if there is a solution to the SDP:

Decision Variables
$u \in \mathbb{R}^q[x]$
$s_i^1, \ldots, s_i^J, s_i^{J+1} \in \Sigma[x]$
$s_i^{J+2} \in \Sigma^q[x]$

$$\begin{aligned} \min \quad & 0 \\ s.t. \quad & \psi_{r_1}^1(x, u(x)) = s_0^1(x) + s_1^1(x)\psi_{r_1-1}^1(x) + \cdots + s_J^1(x)\psi_0^J(x) \\ & \vdots \\ & \psi_{r_J}^J(x, u(x)) = s_0^J(x) + s_1^J(x)\psi_0^1(x) + \cdots + s_J^J(x)\psi_{r_J-1}^J(x) \\ & -L_fV(x) - L_gV(x)u(x) - \gamma(V(x)) = s_0^{J+1}(x) + s_1^{J+1}(x)\psi_0^1(x) + \cdots + s_J^{J+1}(x)\psi_0^J(x) \\ & -Au(x) + c = s_0^{J+2}(x) + s_1^{J+2}(x)\psi_0^1(x) + \cdots + s_J^{J+2}(x)\psi_0^J(x) \end{aligned}$$

$$\psi_{r_j}^j(x, u(x)) \geq 0 \text{ for all } x \in C_1^1 \cap \cdots \cap C_{r_j}^j \cap \cdots \cap C_1^J$$

⇨ Each HOCBF definition is satisfied over the forward invariant operating region

- Dynamics

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

- Two HOCBFs, One CBF

$$b_1(x) = x_1^2 + x_2^2 - 1^2$$

$$b_2(x) = -x_1^2 - x_2^2 + 10^2$$

$$b_3(x) = -x_3^2 - x_4^2 + 15^2$$

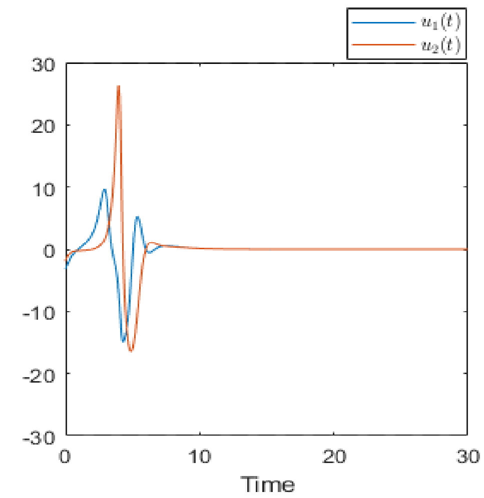
- One CLF

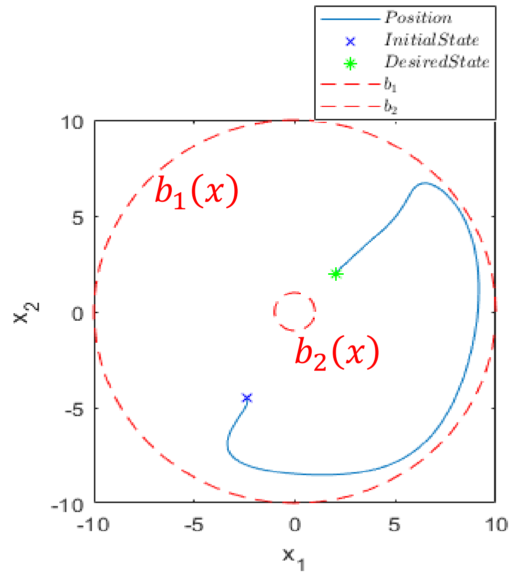
$$V(x) = \left(x - \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}\right)^T \left(x - \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}\right)$$

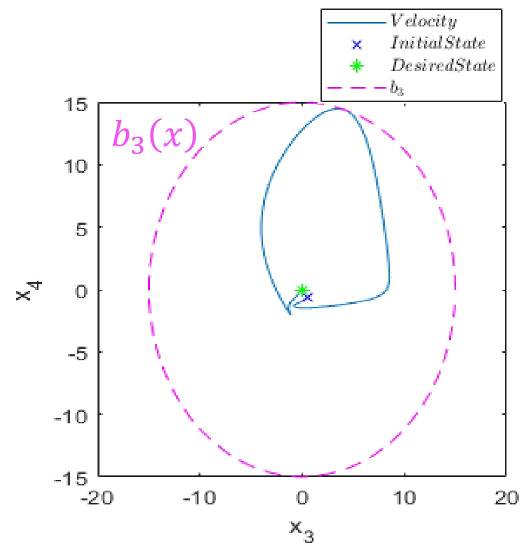
- Actuation Constraints

$$-30 \leq u \leq 30$$

Simulation of $u(x)$ and x resulting from verified SDP







Remaining Work

- Synthesizing HOCBFs
  - The class $\mathcal{K}_\infty$ functions lack intuition
  - Can be difficult to determine *any* feasible class $\mathcal{K}_\infty$ functions, let alone optimal class $\mathcal{K}_\infty$ functions
  - The class $\mathcal{K}_\infty$ functions determine restrictiveness for the intercession of safety
  - All issues with class $\mathcal{K}_\infty$ functions are exacerbated for HOCBFs

$$\psi_r(x) = L_f^r b(x) + L_g L_f^{r-1} b(x) u + \alpha_r\big(\psi_{r-1}(x)\big) + \sum_{i=1}^{r-1} L_f^i\big(\alpha_{i-1}\big(\psi_{r-i-1}(x)\big)\big)$$

  - Incorporate α_i functions as a decision variable

Thank you!

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