Recent Advances in Safety, Optimization, Learning, and Control

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University of California

CoE Review @ UC Santa Cruz - May 30, 2024

UF UNIVERSITY of FLORIDA Duke UNIVERSITY TEXAS The University of Texas at Austin UC SANTA CRUZ

Outline of Recent Results

1. **Safety**
   - Safety Certificates
     ACC23a, CDC23a, CDC23b, TAC (accepted) w/ Warren Dixon
   - Reinforcement Learning
     RLC 2024 w/ Zachary Bell poster here!
2. **Optimization**
   - Dynamical systems approach
     ACC23c, Optimization journal (almost ready)
     Automatica 2023, ACC23d w/ Matt Hale
   - Optimization with Computational Constraints
     CPSWeek-IoT 24 Workshop
3. **Motion Planning for Hybrid Systems**
   - RRT for feasibility and optimality
     CDC22, CCTA22b, CDC23c, ADHS24 poster here!
4. **Learning-based Hybrid Control**
   - Learning Lyapunov functions for hybrid systems
     HSCC 2024 Carlos will present it next

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     CPSWeek-IoT 24 Workshop
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New PhD student and postdoc arriving in Fall

Visiting Z. Bell at AFRL/RW in two weeks

CDC 2024 Pre-Conference Workshop Proposal on Hybrid Estimation

CDC 2024 Tutorial Session Proposal on Hybrid Control

A Data-Driven Approach for Certifying Asymptotic Stability and Cost Evaluation for Hybrid Systems

Carlos A. Montenegro G., Santiago J. Leudo, and Ricardo G. Sanfelice
University of California, Santa Cruz, USA

CoE Review

May 30, 2024

Hybrid Systems Laboratory

Motivation

Control Theory + Learning

Quadruped Robot. Mutiple time domains.

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- Existing numerical methods only apply to limited classes of systems (to certify formal guarantees)
- Challenges: Hybrid systems pose additional challenges due to interaction of discrete and continuous dynamics

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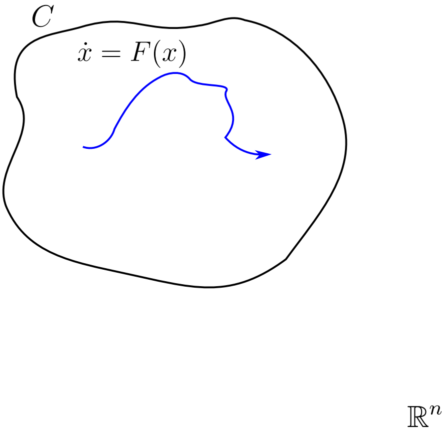
Thus, we propose a **learning-based approach** to **certify stability** for systems with such **complex dynamics**.

discrete and continuous dynamics

Modeling Hybrid Dynamics



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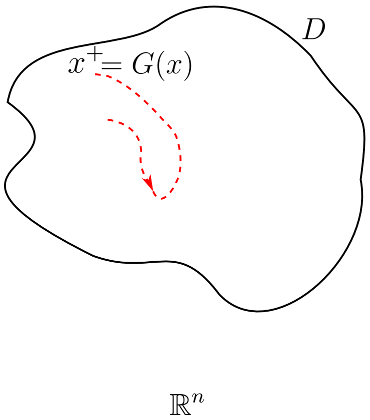

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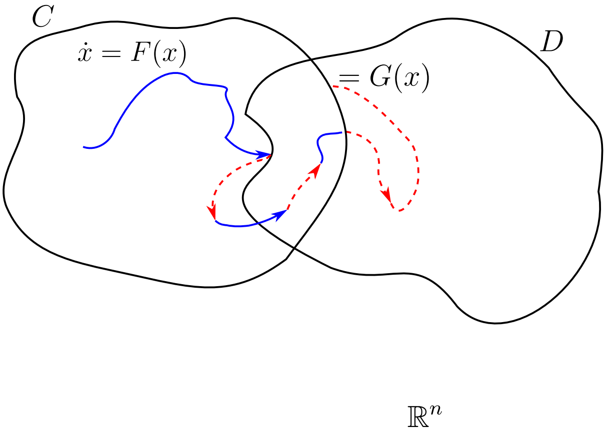

Hybrid Systems

A hybrid system $\mathcal{H}$ with state x as in [Goebel, et.al., PUP 2012]:

$$\mathcal{H} \begin{cases} \dot{x} &= F(x) \quad x \in C \\ x^+ &= G(x) \quad x \in D \end{cases}$$

- C is the *flow set*
- F is the *flow map*
- D is the *jump set*
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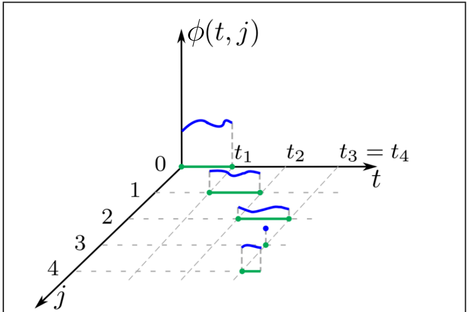

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Connections to Other Frameworks

Switched Systems

$$\dot{x} = f_{\sigma(t)}(x)$$
$$\sigma(t) \in \{1, 2, \dots\}$$

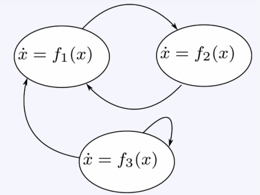
Impulsive Systems

$$\dot{x} = f(x(t))$$
$$x(t^+) = g(x(t)) \quad t \in \{t_1, t_2, \dots\}$$

Differential-Algebraic Equations

$$\dot{x} = f(x, w)$$
$$0 = \eta(x, w)$$

Hybrid Automata



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To the best of our knowledge, there does not exist previous work to synthesize **Lyapunov functions** based on **learning methods** for **hybrid systems** modeled in such framework.

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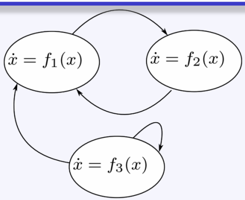

Data-Driven Stability Certificates

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- Extending Lyapunov conditions from samples
- Application to an oscillator with impacts

Stability for Hybrid Systems

Pre-asymptotic stability (pAS)

Given a hybrid system $\mathcal{H} = (C, F, D, G)$, a nonempty set $\mathcal{A} \subset \mathbb{R}^n$ is said to be

- **stable** for $\mathcal{H}$ if, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\phi(0,0)|_{\mathcal{A}} \leq \delta \quad \Longrightarrow \quad |\phi(t,j)|_{\mathcal{A}} \leq \epsilon \quad \forall (t,j) \in \operatorname{dom} \phi$$

  for each solution ϕ to $\mathcal{H}$;

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- **pre-attractive (pA)** for $\mathcal{H}$ if there exists $\ell > 0$ such that every solution ϕ to $\mathcal{H}$ with

$$|\phi(0,0)|_{\mathcal{A}} \leq \ell$$

  is such that $(t,j) \mapsto |\phi(t,j)|_{\mathcal{A}}$ is bounded and if ϕ is complete $\lim_{\substack{(t,j)\in \operatorname{dom}\phi \\ t+j\to\infty}} |\phi(t,j)|_{\mathcal{A}} = 0$;

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- **pre-asymptotically stable (pAS)** for $\mathcal{H}$ if it is stable and pre-attractive for $\mathcal{H}$.

Stability for Hybrid Systems

Theorem. Sufficient Lyapunov conditions for pre-asymptotic stability

Consider

- A set $\mathcal{U} \subset \mathbb{R}^n$ and a compact set $\mathcal{A} \subset \mathbb{R}^n$,
- a function $V : \operatorname{dom} V \to \mathbb{R}$ defining a Lyapunov function candidate on $\mathcal{U}$ with respect to $\mathcal{A}$ for a system $\mathcal{H}$.

If $\mathcal{H}$ satisfies the hybrid basic conditions, $V \in \mathcal{PD}(\mathcal{A})$[1], and

$$\langle \nabla V(x), F(x) \rangle < 0 \quad \forall x \in (C \cap \mathcal{U}) \setminus \mathcal{A}$$
$$V(G(x)) - V(x) < 0 \quad \forall x \in (D \cap \mathcal{U}) \setminus \mathcal{A}$$

then $\mathcal{A}$ is pAS for $\mathcal{H}$.

[1] We say that a function $g : \operatorname{dom} g \to \mathbb{R}_{\geq 0}$ is positive definite with respect to a set K, also written as $g \in \mathcal{PD}(K)$, if $g(x) = 0$ for any $x \in \operatorname{dom} g \cap K$ and $g(x) > 0$ for any $x \in \operatorname{dom} g \setminus K$.

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Hybrid basic conditions:

- C and D are closed sets of $\mathbb{R}^n$
- F is a single-valued continuous map defined on C
- G is a single-valued continuous map defined on D

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Learning-based Lyapunov Functions

Modeling Lyapunov functions using function approximators has been studied:

- Through **feature maps**[2] $x \mapsto \eta(x) := [\eta_1(x), \dots, \eta_\ell(x)]^\top \in \mathbb{R}^\ell$,

$$\widehat{V}_\theta(x) := \sum_{j=1}^{\ell} \theta_j \eta_j(x) = \langle \theta, \eta(x) \rangle$$

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- Stacking generalized linear models (GLMs) yields a **neural network**[3], described by the following recursive equations

$$x^0 = x, \quad x^{k+1} = \varphi(W^k x^k + b^k),\ k \in \{0, \ldots, \ell - 1\}, \quad \widehat{V}_\theta(x) = W^\ell x^\ell + b^\ell$$

  where $\theta = (W^\ell, b^\ell)$ and $z \mapsto \varphi(z)$ denotes the **activation function**.

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Learning-based Lyapunov Functions

M

We define the notion of **complexity of a function**. E.g., every finite dimensional reproducing kernel Hilbert space (RKHS) $\mathcal{H}_K$ can be described as

$$f(x) = \langle \kappa(\cdot, x), f(\cdot) \rangle_{\mathcal{H}_K} \quad \forall x \in \mathcal{X}, \forall f \in \mathcal{H}_K$$

where, for all $x, x' \in \mathcal{X}$,

$$\kappa(x, x') = \langle \eta(x), \eta(x') \rangle$$

Then, it follows

$$\|f\|^2_{\mathcal{H}_K} := \langle f, f \rangle_{\mathcal{H}_K}$$

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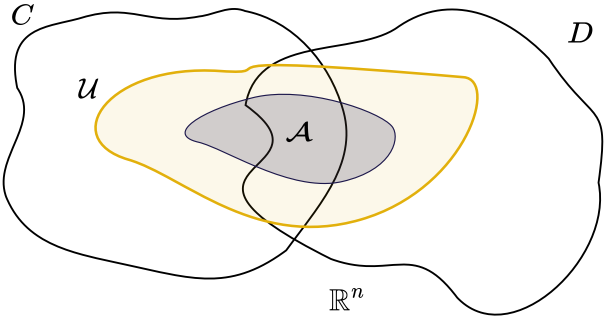
Coverings via ε-nets



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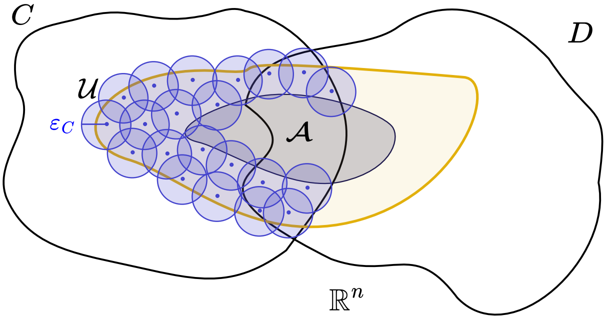
$$\mathcal{F}_C := \Big\{ \text{centers of } \bigcirc \Big\}$$



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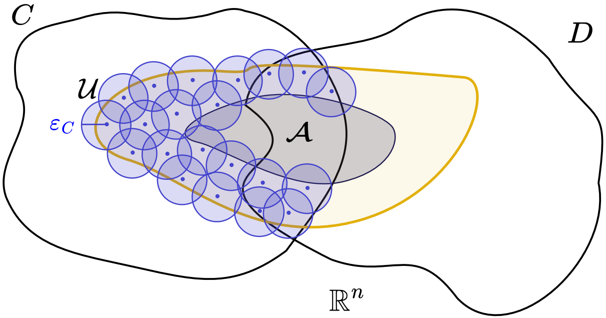

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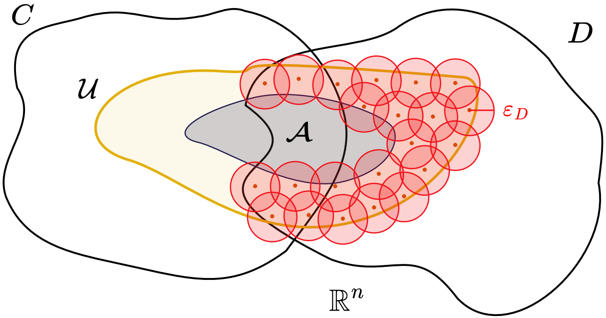

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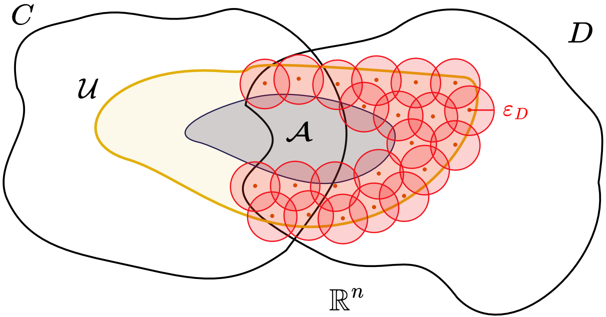

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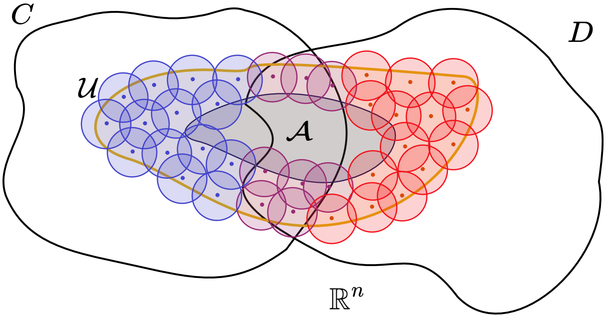

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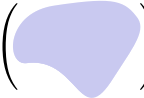
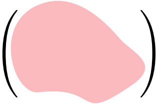
Learning-based Lyapunov Functions

Optimization Problem for Lyapunov Functions

Robust Program (RP)

$$\underset{\theta}{\text{minimize}} \quad \|\widehat{V}_\theta\|^2_{\mathcal{H}_K}$$

subject to $\dot{\widehat{V}}_\theta\Big($  $\Big) < 0$ along flows

$\Delta\widehat{V}_\theta\Big($  $\Big) < 0$ along jumps

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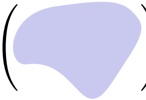
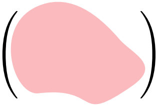
subject to $\dot{\widehat{V}}_\theta\Big($  $\Big) < 0$ along flows

$\Delta\widehat{V}_\theta\Big($  $\Big) < 0$ along jumps

Scenario Program (SP)

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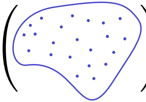
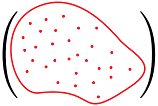
subject to $\dot{\widehat{V}}_\theta\Big($  $\Big) < 0$

$\Delta\widehat{V}_\theta\Big($  $\Big) < 0$

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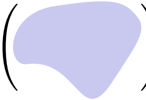
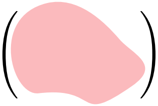
subject to $\dot{\widehat{V}}_\theta\Bigg($  $\Bigg) < 0$ along flows

$\Delta\widehat{V}_\theta\Bigg($  $\Bigg) < 0$ along jumps

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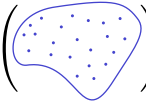
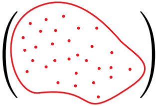
subject to $\dot{\widehat{V}}_\theta\Bigg($  $\Bigg) < 0$

$\Delta\widehat{V}_\theta\Bigg($  $\Bigg) < 0$

- Does solving the SP guarantee Lyapunov constraints satisfaction for points that were not sampled?

Learning-based Lyapunov Functions

Optimization Problem for Lyapunov Functions

Robust Program (RP)

$$\underset{\theta\in\mathbb{R}^r}{\text{minimize}} \quad \|\theta\|_2$$

$$\text{s.t.} \quad \dot{\widehat{V}}_\theta(x) < 0 \quad \forall x \in (C \cap \mathcal{U}) \setminus \mathcal{A},$$

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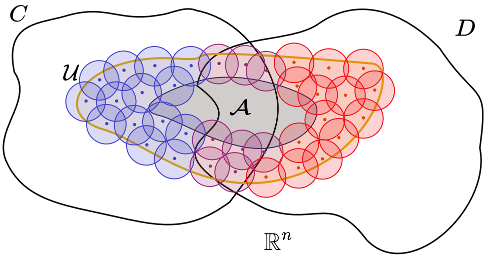

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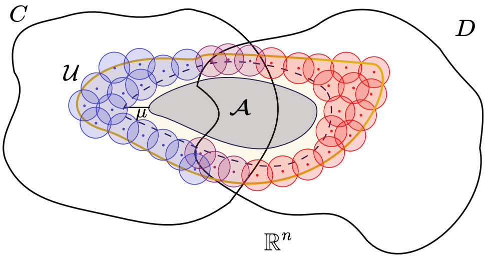
Scenario Program (SP)

$$\begin{aligned}\underset{\theta\in\mathbb{R}^r}{\text{minimize}} \quad & \|\theta\|_2 \\ \text{s.t.} \quad & \dot{\hat{V}}_\theta(x') < -\tau_C \quad \forall x' \in \mathcal{F}_C \setminus (\mathcal{A} + \mu\mathbb{B}), \\ & \Delta\widehat{V}_\theta(x') < -\tau_D \quad \forall x' \in \mathcal{F}_D \setminus (\mathcal{A} + \mu\mathbb{B})\end{aligned}$$



Learning-based Lyapunov Functions

Optimization Problem for Lyapunov Functions

Robust Program (RP)

$$\underset{\theta\in\mathbb{R}^r}{\text{minimize}} \quad \|\theta\|_2$$

$$\text{s.t.} \quad \dot{\widehat{V}}_\theta(x) < 0 \quad \forall x \in (C \cap \mathcal{U}) \setminus \mathcal{A},$$
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Scenario Program (SP)

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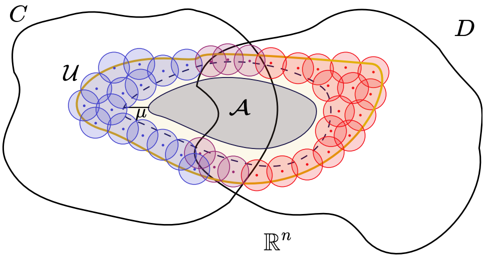

Slack variables.

Extending Lyapunov Conditions from Samples

- **Goal**: Generalize from sample data to compact set $\mathcal{U}$.

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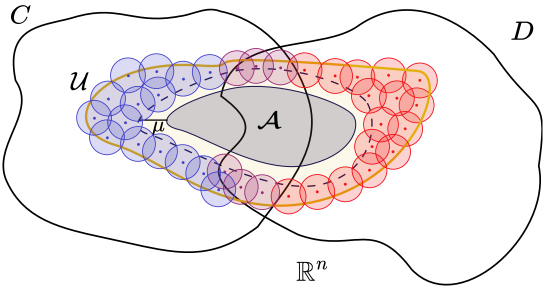

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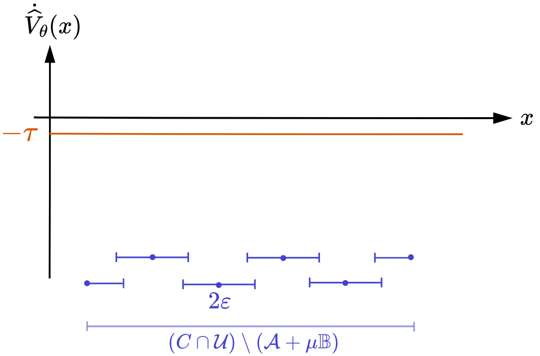

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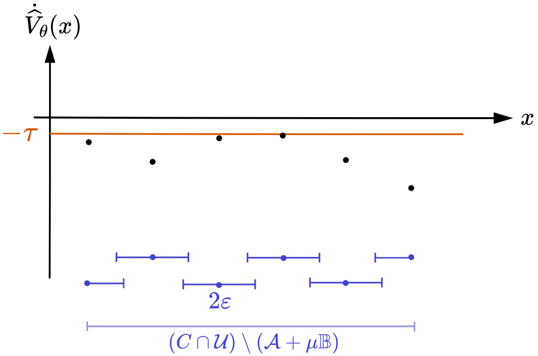

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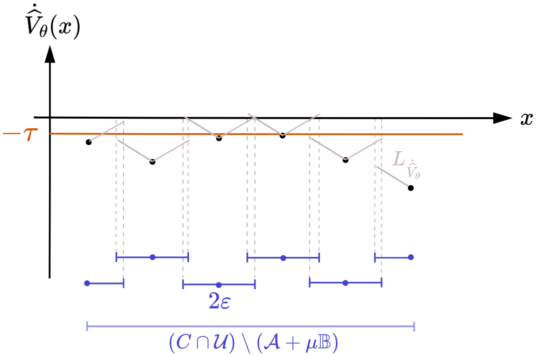

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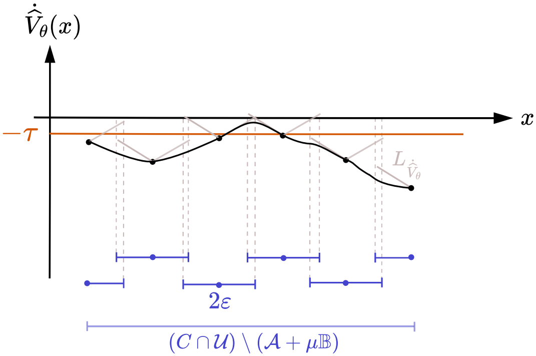

Extending Lyapunov Conditions from Samples

Lipschitz Continuity

Consider

- the function $\widehat{V}_\theta$ defined as a **neural network** with d layers and network parameter θ,
- a hybrid system $\mathcal{H} = (C, F, D, G)$, and
- a compact set $\mathcal{U} \subset \mathbb{R}^n$ (sample set).

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Lemma. Lipschitz continuity of the Lyapunov function $\widehat{V}_\theta$

If the activation function φ is L_φ-Lipschitz continuous, then $\widehat{V}_\theta$ is $L_{\widehat{V}_\theta}-$Lipschitz continuous.

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Lemma. Lipschitz continuity of the gradient of the Lyapunov function $\widehat{V}_\theta$

If the activation function φ is $\mathcal{C}^2$, then $\nabla\widehat{V}_\theta$ is $L_{\nabla\widehat{V}_\theta}-$Lipschitz continuous.

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Proposition. Lipschitz continuity of $\dot{\widehat{V}}_\theta$

If

- the flow map F is L_F-Lipschitz,
- there exists $\eta_F > 0$ such that $\|F(x)\| \leq \eta_F$ for all $x \in C \cap \mathcal{U}$, and
- the activation function φ is L_φ-Lipschitz and its gradient $\nabla\varphi$ is $L_{\nabla\varphi}-$Lipschitz,

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then, the function $\dot{\widehat{V}}_\theta(x) := \langle \nabla\widehat{V}_\theta(x), F(x) \rangle$ is $L_{\dot{\widehat{V}}_\theta}-$Lipschitz with $L_{\dot{\widehat{V}}_\theta} := L_{\nabla\widehat{V}_\theta}\eta_F + L_{\widehat{V}_\theta}L_F$.

Extending Lyapunov Conditions from Samples

Main Result

Proposition. Generalized Lyapunov Conditions

Given

- compact sets $\mathcal{U} \subset \mathbb{R}^n$ (sample set) and $\mathcal{A} \subset \mathbb{R}^n$ (set to render stable),

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if, for some $\tau_C > L_{\dot{\widehat{V}}_\theta} \varepsilon$, $\tau_D > L_{\widehat{V}_\theta}(1 + L_G)\varepsilon$, $\mu > \varepsilon$, we have

$$\dot{\widehat{V}}_\theta(x') \leq -\tau_C \quad \forall x' \in \mathcal{F}_C \setminus (\mathcal{A} + \mu\mathbb{B}),$$
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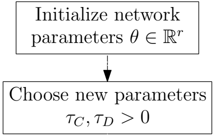
then,

$$\dot{\widehat{V}}_\theta(x) < 0 \quad \forall x \in (C \cap \mathcal{U}) \setminus (\mathcal{A} + \mu\mathbb{B}),$$
$$\Delta\widehat{V}_\theta(x) < 0 \quad \forall x \in (D \cap \mathcal{U}) \setminus (\mathcal{A} + \mu\mathbb{B}).$$

Extending Lyapunov Conditions from Samples

Bootstrap Evaluation

Iterative search for a learning-based Lyapunov function.



SP:

$$
\begin{aligned}
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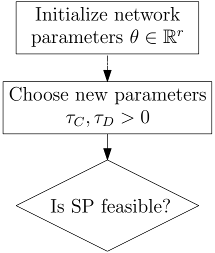

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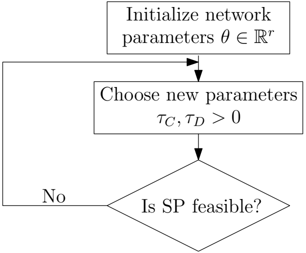

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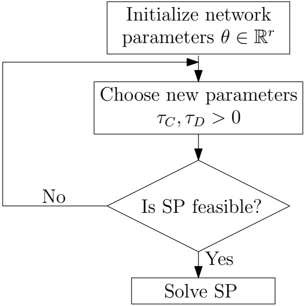

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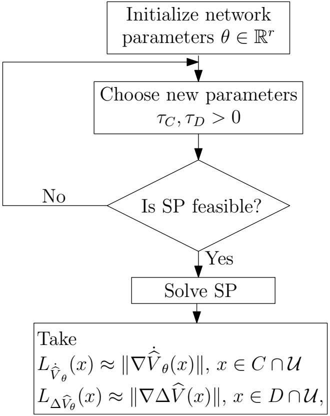

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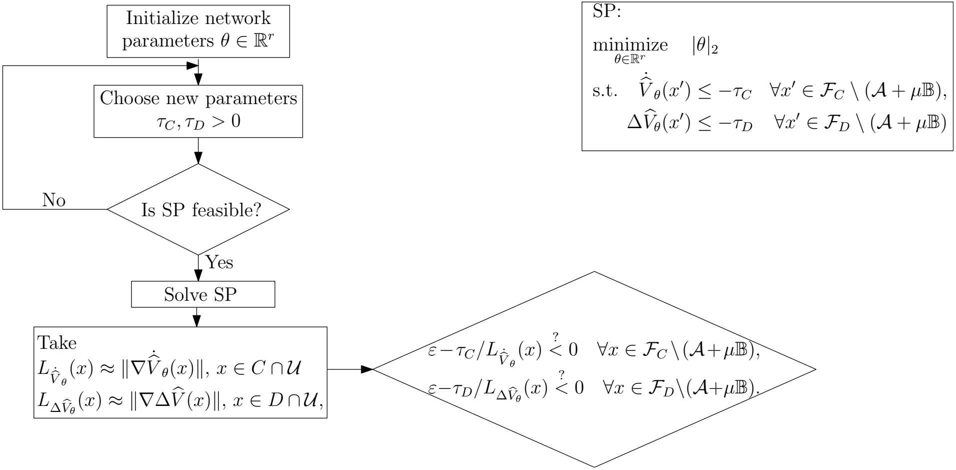

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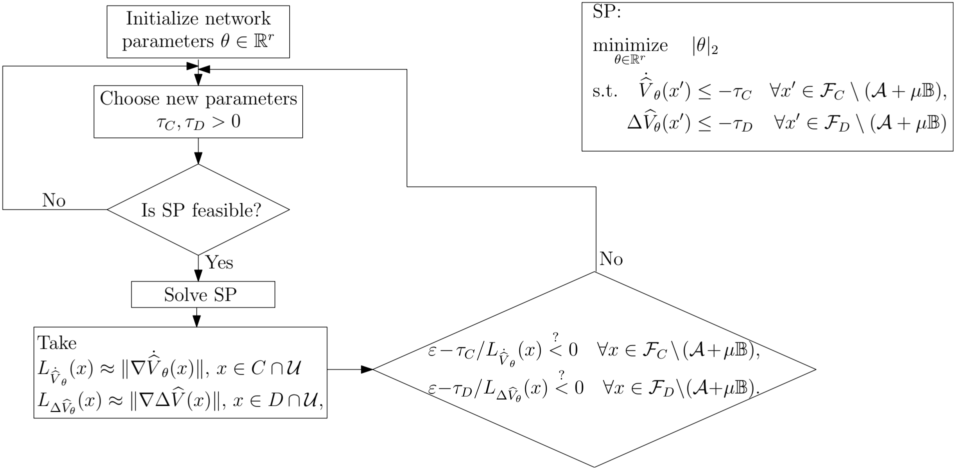

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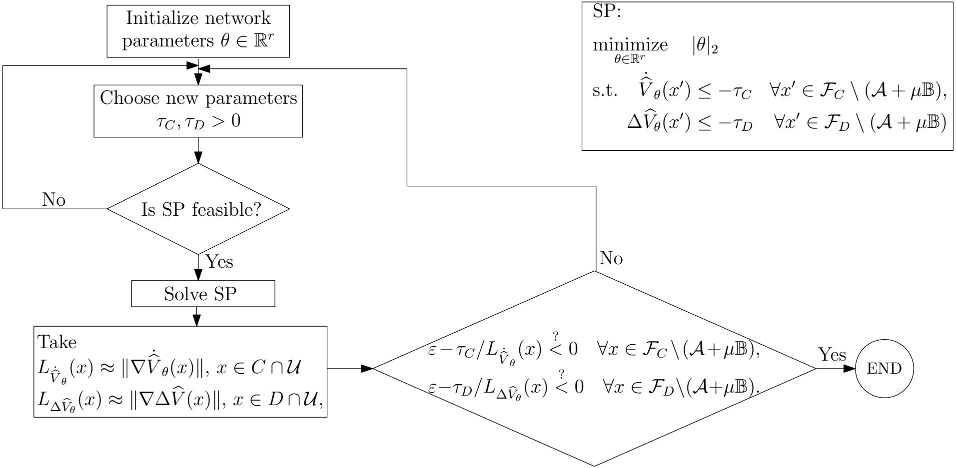

Learning-based Sufficient Conditions for Stability

Main Result

Theorem. Practical pre-Asymptotic Stability

Given

- compact sets $\mathcal{U} \subset \mathbb{R}^n$ (sample set) and $\mathcal{A} \subset \mathbb{R}^n$ (set to render stable),
- a hybrid system $\mathcal{H} = (C, F, D, G)$, with F locally L_F-Lipschitz on $C \cap \mathcal{U}$ and G locally L_G-Lipschitz on $D \cap \mathcal{U}$,
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- a $L_{\widehat{V}_\theta}-$ Lipschitz function $\widehat{V}_\theta$ over $(C \cup D) \cap \mathcal{U}$, and $L_{\dot{\widehat{V}}_\theta}-$Lipschitz time derivative on $C \cap \mathcal{U}$, such that $\alpha_1(|x|_\mathcal{A}) \leq \widehat{V}_\theta(x) \leq \alpha_2(|x|_\mathcal{A})$ on $(C \cup D) \cap \mathcal{U}$ for some $\alpha_1, \alpha_2 \in \mathcal{K}$.

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If for $\mu > \varepsilon$ and some $\tau_C > L_{\dot{\widehat{V}}_\theta} \varepsilon$, $\tau_D > L_{\widehat{V}_\theta}(1 + L_G)\varepsilon$, we have

$$\dot{\widehat{V}}_\theta(x') \leq -\tau_C \quad \forall x' \in \mathcal{F}_C \setminus (\mathcal{A} + \mu\mathbb{B}),$$
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Then, $\mathcal{A}$ is **practically pre-asymptotically stable (PpAS) for $\mathcal{H}$ with respect to** ε.

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- compact sets $\mathcal{U} \subset \mathbb{R}^n$ (sample set) and $\mathcal{A} \subset \mathbb{R}^n$ (set to render stable),
- a hybrid system $\mathcal{H} = (C, F, D, G)$, with F locally L_F-Lipschitz on $C \cap \mathcal{U}$ and G locally L_G-Lipschitz on $D \cap \mathcal{U}$,
- $\varepsilon > 0$ defining $\mathcal{F}_C$ and $\mathcal{F}_D$ as $\varepsilon-$nets over $C \cap \mathcal{U}$ and over $D \cap \mathcal{U}$, respectively, and
- a $L_{\widehat{V}_\theta}-$ Lipschitz function $\widehat{V}_\theta$ over $(C \cup D) \cap \mathcal{U}$, and $L_{\dot{\hat{V}}_\theta}-$Lipschitz time derivative on $C \cap \mathcal{U}$, such that $\alpha_1(|x|_\mathcal{A}) \leq \widehat{V}_\theta(x) \leq \alpha_2(|x|_\mathcal{A})$ on $(C \cup D) \cap \mathcal{U}$ for some $\alpha_1, \alpha_2 \in \mathcal{K}$

We say that a set $\mathcal{A}$ is **PpAS** for $\mathcal{H}$ with respect to ε if there exists $\beta \in \mathcal{KL}$ such that each solution ϕ to $\mathcal{H}$ from $(\overline{C} \cup D) \cap \mathcal{U}$ that stays in $(\overline{C} \cup D \cup G(D)) \cap \mathcal{U}$, satisfies

$$|\phi(t,j)_\mathcal{A}| \leq \beta(|\phi(0,0)|_\mathcal{A}, t+j) + \mu \quad \forall (t,j) \in \operatorname{dom} \phi.$$

Learning-based Sufficient Conditions for Stability

Main Result

Proof Sketch. Practical pre-Asymptotic Stability

Given $\mu > \varepsilon > 0$, and since for some $\tau_C > L_{\dot{\hat{V}}_\theta}\varepsilon$ and $\tau_D > L_{\hat{V}_\theta}(1 + L_G)\varepsilon$, we have

$$\dot{\hat{V}}_\theta(x') \leq -\tau_C \quad \forall x' \in \mathcal{F}_C \setminus (\mathcal{A} + \mu\mathbb{B}),$$
$$\Delta\hat{V}_\theta(x') \leq -\tau_D \quad \forall x' \in \mathcal{F}_D \setminus (\mathcal{A} + \mu\mathbb{B}),$$

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then, from the Proposition on *Generalized Lyapunov Conditions* we have that

$$\dot{\widehat{V}}_\theta(x) < 0 \quad \forall x \in (C \cap \mathcal{U}) \setminus (\mathcal{A} + \mu\mathbb{B}),$$
$$\Delta\widehat{V}_\theta(x) < 0 \quad \forall x \in (D \cap \mathcal{U}) \setminus (\mathcal{A} + \mu\mathbb{B}).$$

Learning-based Sufficient Conditions for Stability

Main Result

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Since

$$\alpha_1(|x|_{\mathcal{A}}) \leq \widehat{V}_\theta(x) \leq \alpha_2(|x|_{\mathcal{A}}) \qquad \text{for all } x \in (C \cup D) \cap \mathcal{U},$$

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it can be shown that there exist $\alpha_C, \alpha_D \in \mathcal{K}$ such that

$$\dot{\widehat{V}}_\theta(x) \leq -\alpha_C(\widehat{V}_\theta(x)) \qquad \text{for all } x \in (C \cap \mathcal{U}) \setminus (\mathcal{A} + \mu\mathbb{B}),$$

and

$$\Delta\widehat{V}_\theta(x) \leq -\alpha_D(\widehat{V}_\theta(x)) \qquad \text{for all } x \in (D \cap \mathcal{U}) \setminus (\mathcal{A} + \mu\mathbb{B}).$$

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Define

$$x \mapsto \alpha(x) := \min\{\alpha_C(x), \alpha_D(x)\}$$

and, without loss of generality, assume it is locally Lipschitz.

Learning-based Sufficient Conditions for Stability

Main Result

Proof Sketch. Practical pre-Asymptotic Stability

Given a solution ϕ to $\mathcal{H}$ from $((C \cup D) \cap \mathcal{U}) \setminus (\mathcal{A} + \mu\mathbb{B})$, by **the comparison principle for hybrid systems** we have that

$$\widehat{V}_\theta(\phi(t,j)) \leq \tilde{\beta}\left(\widehat{V}_\theta(\phi(0,0)), t+j\right) \qquad \text{for all } (t,j) \in \operatorname{dom}\phi,$$

where $\tilde{\beta} \in \mathcal{KL}$.

Learning-based Sufficient Conditions for Stability

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where $\tilde{\beta} \in \mathcal{KL}$. This, together with

$$\alpha_1(|x|_{(\mathcal{A}+\mu\mathbb{B})}) < \alpha_1(|x|_{\mathcal{A}}) \leq \widehat{V}_\theta(x) \leq \alpha_2(|x|_{\mathcal{A}}) \qquad \text{for all } x \in ((C \cup D) \cap \mathcal{U}) \setminus (\mathcal{A} + \mu\mathbb{B}).$$

implies that

$$\alpha_1(|\phi(t,j)|_{(\mathcal{A}+\mu\mathbb{B})}) < \widehat{V}_\theta(\phi(t,j)) \leq \tilde{\beta}\left(\widehat{V}_\theta(\phi(0,0)), t+j\right) \leq \tilde{\beta}\left(\alpha_2\left(|\phi(0,0)|_{\mathcal{A}}\right), t+j\right).$$

Learning-based Sufficient Conditions for Stability

Main Result

Proof Sketch. Practical pre-Asymptotic Stability

Consequently,

$$|\phi(t,j)|_{(\mathcal{A}+\mu\mathbb{B})} \leq \alpha_1^{-1}\left(\tilde{\beta}\left(\alpha_2\left(|\phi(0,0)|_{\mathcal{A}}\right), t+j\right)\right)$$

where $(r, t+j) \mapsto \alpha_1^{-1}\left(\tilde{\beta}\left(\alpha_2\left(r\right), t+j\right)\right) \in \mathcal{KL}$.

Learning-based Sufficient Conditions for Stability

Main Result

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Consequently,

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$$|x|_{\mathcal{A}} = |x|_{(\mathcal{A}+\mu\mathbb{B})} + \mu \qquad \text{for any } x \in ((C \cup D) \cap \mathcal{U}) \setminus (\mathcal{A}+\mu\mathbb{B}).$$

Learning-based Sufficient Conditions for Stability

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Then, the desired $\mathcal{KL}$ bound follows:

$$|\phi(t,j)|_{\mathcal{A}} = |\phi(t,j)|_{(\mathcal{A}+\mu\mathbb{B})} + \mu \leq \alpha^{-1}\left(\tilde{\beta}\left(\alpha_2\left(|\phi(0,0)|_{\mathcal{A}}\right), t+j\right)\right) + \mu$$

for every $(t,j) \in \operatorname{dom}\phi$. $\square$

Example: Oscillator with Impacts

$$\mathcal{H}\left\{\begin{array}{rcll}(\dot{x}_1, \dot{x}_2) & = & (x_2, -x_1 - \lambda_C x_2) & x_1 \geq 0 \\ (x_1^+, x_2^+) & = & (0, \lambda_D x_2) & x_1 = 0 \text{ and } x_2 \leq 0\end{array}\right.$$

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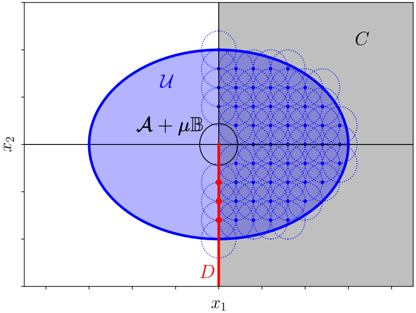

- **Goal:** Design learning-based Lyapunov function to certify stability of $\mathcal{A} := 0$.

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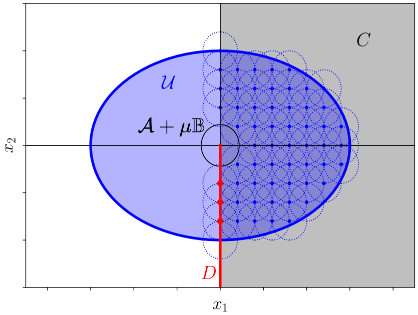

- **Goal:** Design learning-based Lyapunov function to certify stability of $\mathcal{A} := 0$.
- Sampling set

$$\mathcal{U} = \{x \in \mathbb{R}^2 \mid x_1^2/h_0^2 + x_2^2/v_0^2 \leq 1\}$$

where $h_0, v_0 > 0$.

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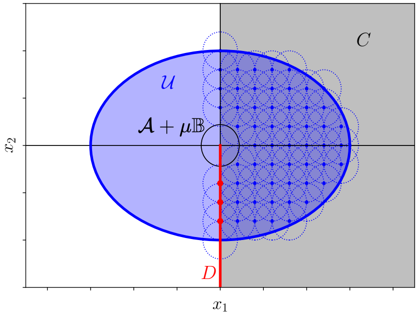

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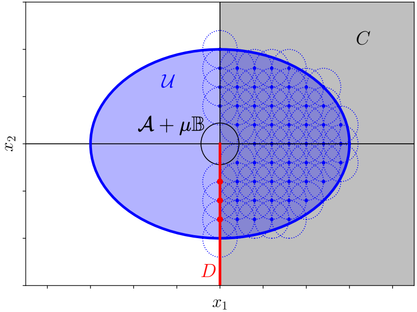

We enforce conditions at the centers of the balls, and generalize them to every point in $((C \cup D) \cap \mathcal{U}) \setminus (\mathcal{A} + \mu\mathbb{B})$.

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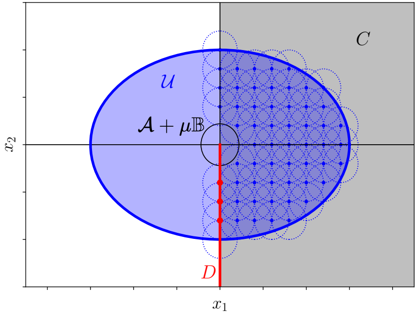

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- We guarantee practical asymptotic stability of $\mathcal{A}$ for $\mathcal{H}$ with respect to ε.

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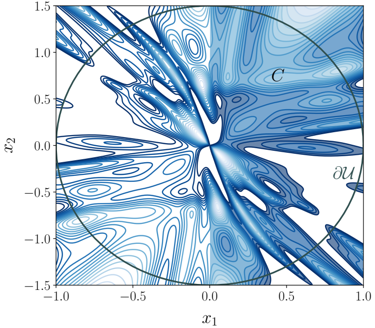
$\varepsilon_C = 0.01$



$\tau_C = 0.015,\ L_{\dot{\hat{V}}_\theta} = 4.482$

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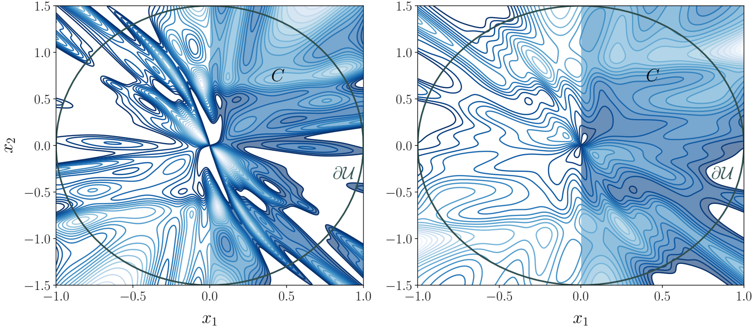

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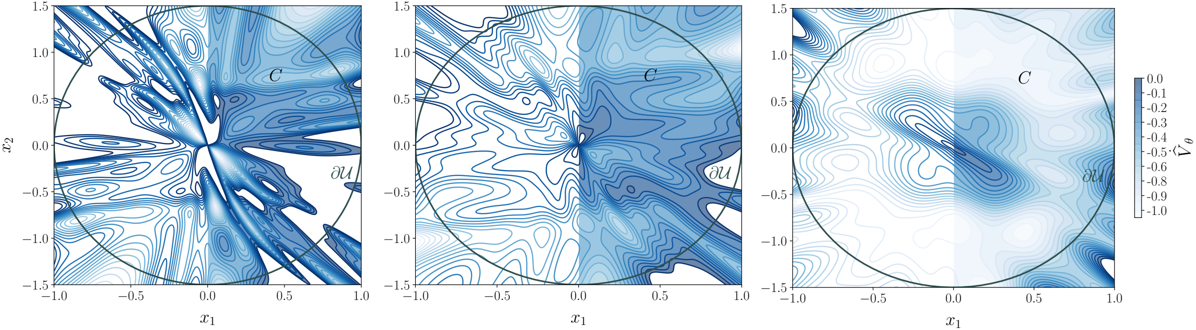

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$\tau_C = 0.028,\ L_{\dot{\hat{V}}_\theta} = 3.862$

$\tau_C = 0.037,\ L_{\dot{\hat{V}}_\theta} = 1.110$

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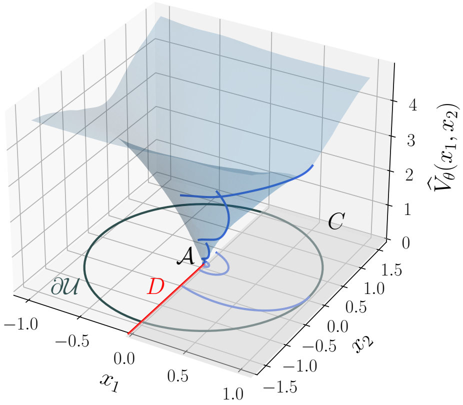

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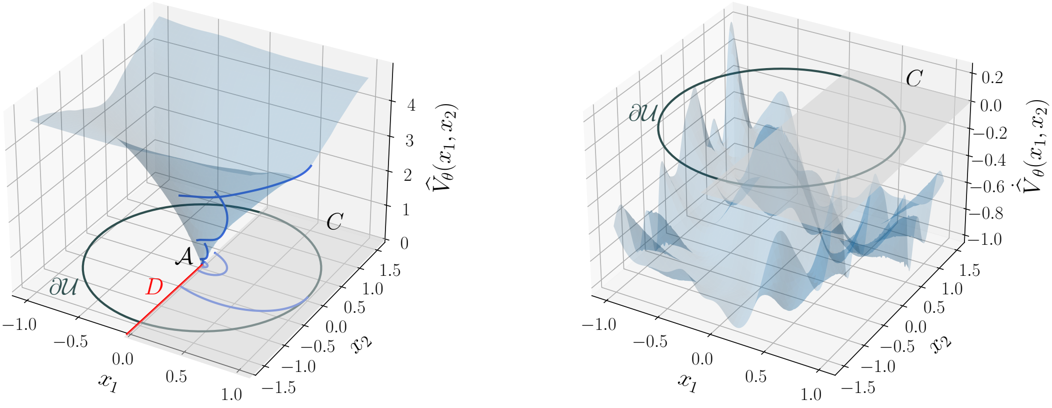

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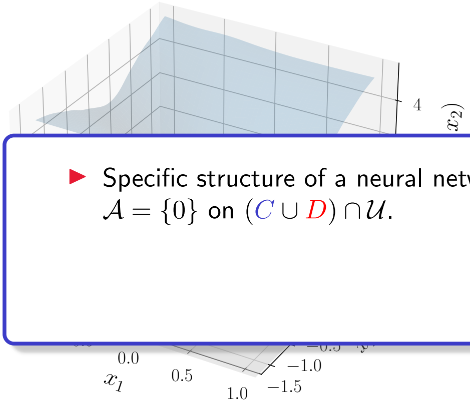

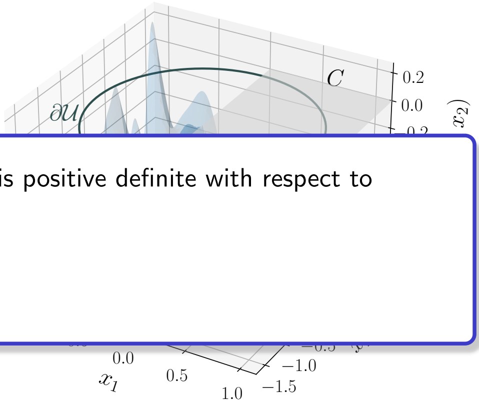

- Specific structure of a neural network that is positive definite with respect to $\mathcal{A} = \{0\}$ on $(C \cup D) \cap \mathcal{U}$.

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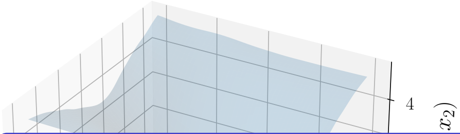
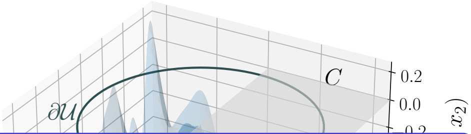
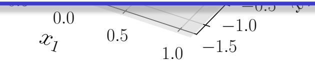
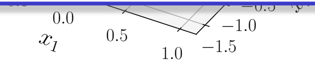




- Specific structure of a neural network that is positive definite with respect to $\mathcal{A} = \{0\}$ on $(C \cup D) \cap \mathcal{U}$.
- We solve the SP using JAX. Augmented Lagrangian to account for constraints.





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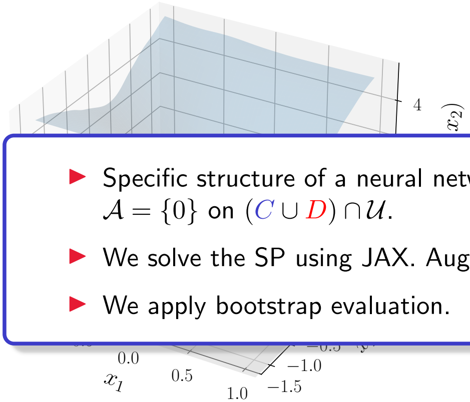

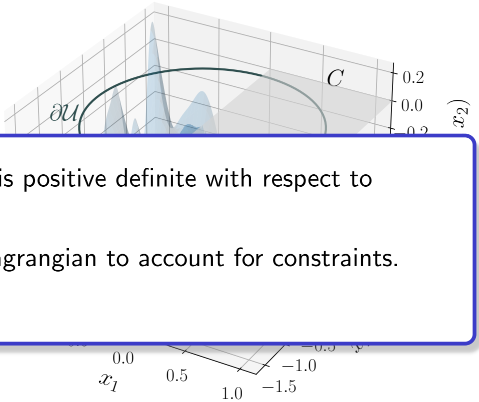

- Specific structure of a neural network that is positive definite with respect to $\mathcal{A} = \{0\}$ on $(C \cup D) \cap \mathcal{U}$.
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- We apply bootstrap evaluation.

Conclusions and Future Work

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- Application in oscillator with impacts.
- Future work: Evaluating different data-driven methods to learn the Lyapunov and value functions, and an extension to hybrid inclusions.

Acknowledgements

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- the National Science Foundation under Grant no. CNS-2039054 and Grant no. CNS-2111688,
- the Air Force Office of Scientific Research under Grant no. FA9550-19-1-0169, Grant no. FA9550-20-1-0238, Grant no. FA9550-23-1-0145, and Grant no. FA9550-23-1-0313,
- the Air Force Research Laboratory under Grant no. FA8651-22-1-0017 and Grant no. FA8651-23-1-0004,
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- and the Department of Defense Grant no. W911NF-23-1-0158.