Overview of Advances in Hybrid Systems and Control

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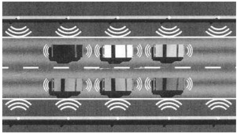
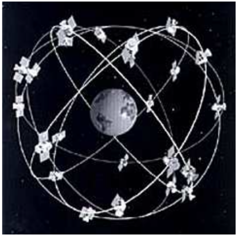
CoE Review @ AFOSR - April 7, 2025

Impulsive Coupling Systems

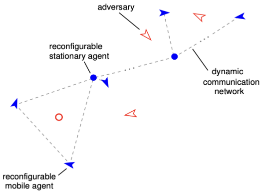
Smart national air space traffic management



Smart Highway Systems



Decentralized satellite constellation control



Reconfigurable Multi-agent Systems

Formation Control w/ Intermittency

Impulsive Coupling Systems

Communication Events

Intermittent events at time instances $t_1 \leq t_2 \leq \cdots \leq t_j \leq \ldots$ can be generated by an autonomous hybrid inclusion

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Idea: To capture all possible event sequences $\{t_\ell^i\}_{\ell=1}^\infty$ while removing dependency on time and stochastic dynamics, define

- τ_i is a timer that, when expires, generates the communication events from the neighbors of the i-th agent

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with dynamics

$$\begin{cases} \dot{\tau}_i = -1 & \tau_i \in [0, T_2^i] \\ \tau_i^+ \in [T_1^i, T_2^i] & \tau_i = 0 \end{cases}$$

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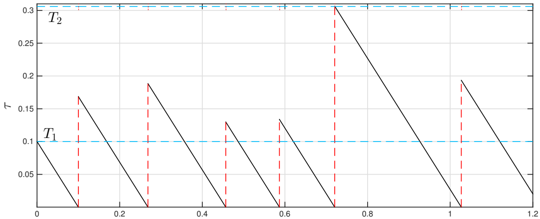

Synchronization Over a Network

Let the dynamics of the i-th node of the network be

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Design of a **feedback controller** for z_i that runs at *each agent* using local information transmitted at communication event times $\{t_\ell^i\}_{\ell=1}^{\infty}$ satisfying $T_1^i \leq t_{\ell+1}^i - t_\ell^i \leq T_2^i$ where

- T_1^i defines the fastest communication rate (> 0)
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Hybrid State-Feedback Control: Assign $u_i = \eta_i$ with dynamics

$$\dot{\eta}_i = f_c(\eta_i, z_i) \qquad \text{when } t \notin \{t_\ell^i\}_{\ell=1}^{\infty}$$

$$\eta_i^+ = \sum_{k=1}^{N} g_{ik} G_{ik}(z_i, z_k, \eta_i, \eta_k) \qquad \text{when } t \in \{t_\ell^i\}_{\ell=1}^{\infty} 1$$

where gains f_c and G_{ik} are to be designed and g_{ik} are the elements of the adjacency matrix.

Synchronization of a Network of Oscillators

Four 2-D systems

- **Comm. Parameters:**

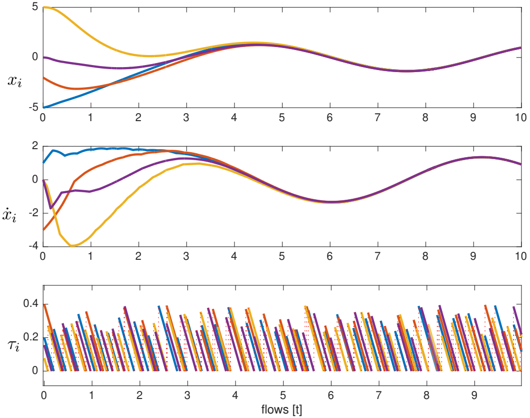
  $T_1 = 0.2, \quad T_2 = 0.6$

- i-th **Controller Gain:**

  $K = -[0.4; 1.7]$ and
  $E = -1.2$

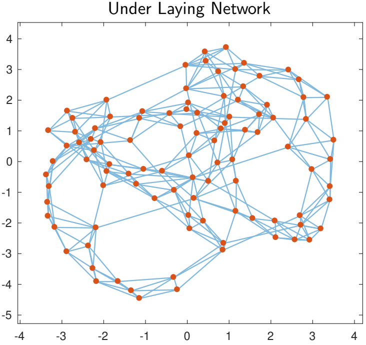
- i-th **Network Graph:**

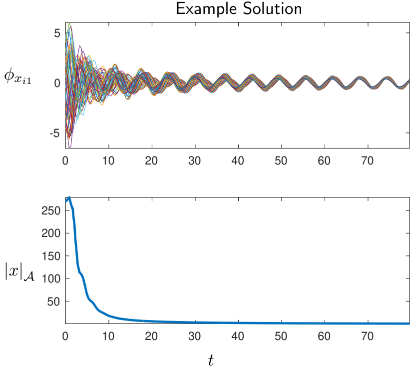
$$G = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$



Synchronization in a Small-World Network

Small-world networks:





$$N = 100$$

Nominal Robustness For Networked Systems

Assured robust autonomy:

$\rho_i > 0$

- Parameter uncertainty
  - Skewed clocks

$$\dot{\tau}_i = -1 \qquad \to \qquad \dot{\tau}_i \in -1 + \rho_i \mathbb{B}$$

  - Rate uncertainty

$$[T_1^i, T_2^i] \qquad \to \qquad [T_1^i - \rho_i, T_2^i + \rho_i]$$

- Unmodeled dynamics
  - Additive dynamics

$$\dot{z}_i = \tilde{f}_i(z_i, u_i) \qquad \to \qquad \dot{z}_i \in \tilde{f}_i(z_i, u_i) + \rho_i \mathbb{B}$$

  - Event conditions

$$\tau_i = 0 \qquad \to \qquad \tau_i \in \rho_i \mathbb{B}$$

- Disturbances
  - Actuator noise (ISS)
  - Measurement noise (ISS)
- Information dropouts

Modeling Hybrid Systems: Closed Loop

Hybrid closed-loop systems are given by *hybrid inclusions*

$$\mathcal{H} \quad \begin{cases} \dot{x} &= F(x) \qquad x \in C \\ x^+ &= G(x) \qquad x \in D \end{cases}$$

where x is the *state*

- C is the *flow set*
- F is the *flow map*
- D is the *jump set*
- G is the *jump map*

Solutions are functions parameterized by hybrid time (t, j):

- Flows parameterized by $t \in \mathbb{R}_{\geq 0} := [0, +\infty)$
- Jumps parameterized by $j \in \mathbb{N}_{\geq 0} := \{0, 1, 2, \ldots\}$

Then, solutions to $\mathcal{H}$ are given by hybrid arcs x defined on

$$([0, t_1] \times \{0\}) \cup ([t_1, t_2] \times \{1\}) \cup \ldots ([t_j, t_{j+1}] \times \{j\}) \cup \ldots$$

The hybrid system $\mathcal{H}$ satisfies the hybrid basic conditions if C, D are closed and F, G are "continuous"

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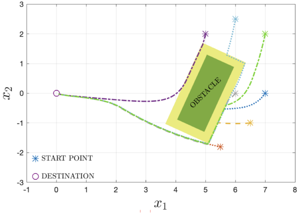
From Robust Asymptotic Stability to ....

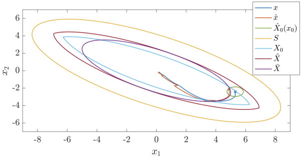
- **Forward invariance and safety properties:**
  - Tangent cone conditions and barrier functions
  - Optimal invariance-based control
  - **Applications:** *vehicle control, security*

$$\kappa^*(x) \triangleq \arg\min_{u\in\mathbb{R}^m} Q(x,u)$$
$$s.t.\ \Gamma(x,u) \leq -\gamma(x),$$
$$\psi(x,u) \leq 0,$$



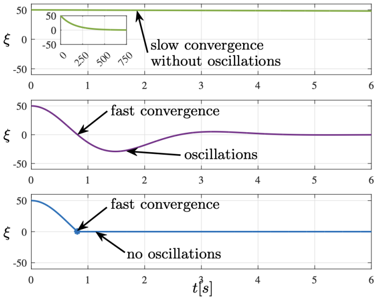
$$\dot{\hat{x}} = \begin{cases} A\hat{x} + Bu + L(Cx - C\hat{x}) & \text{if} \quad t \notin \mathcal{T}_a, \\ A\hat{x} + Bu + \tilde{L}(\tilde{C}x - \tilde{C}\hat{x}) & \text{if} \quad t \in \mathcal{T}_a, \end{cases}$$

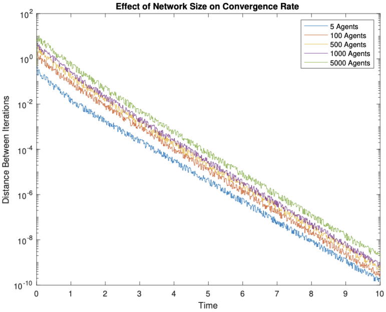
$$\dot{e} = \begin{cases} (A - LC)e & \text{if} \quad t \notin \mathcal{T}_a, \\ (A - \tilde{L}\tilde{C})e & \text{if} \quad t \in \mathcal{T}_a \end{cases}$$



From Robust Asymptotic Stability to ....

- **Optimality guarantees:**
  - Hybrid heavy ball and Nesterov methods
  - Constrained optimization
  - **Applications:** *distributed optimization with intermittency, multi-agent systems*





From Robust Asymptotic Stability to ....

- **Advanced feedback control:**
  - Model predictive control
  - Learning-based control
  - **Applications:** *all of the above*

Dynamic optimization:

Given an initial condition $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} \underset{(x,u)\in\widehat{\mathcal{S}}_{\mathcal{H}}(x_0)}{\text{minimize}} \quad & \mathcal{J}(x,u) \\ \text{subject to} \quad & (T,J) \in \mathcal{T} \\ & x(T,J) \in X, \end{aligned}$$

Mayer form:

$$\begin{aligned} \underset{x\in\widehat{\mathcal{S}}_{\mathcal{H}}}{\text{minimize}} \quad & \mathcal{J}(x(0,0),(T,J),x(T,J)) \\ \text{subject to} \quad & (x(0,0),(T,J),x(T,J)) \in \Omega, \end{aligned}$$

Algorithm 1 Hybrid MPC Implementation

```
i = 0, (T_0, J_0) = (0, 0), x_0 = x(0, 0).
while true do
    Solve Problem 3.2 to obtain an optimal pair (x_i^*, u_i^*).
    while (t − T_i) < τ/2 and j − J_i < 1 do
        Apply u_i^* to H to generate the state trajectory x.
    end while
    i = i + 1, (T_i, J_i) = (t, j), x_0 = x(T_i, J_i).
end while
```

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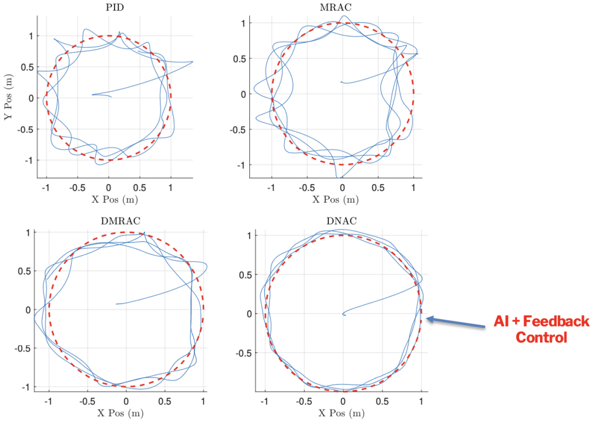

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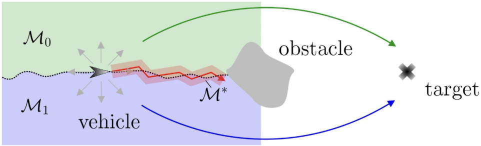

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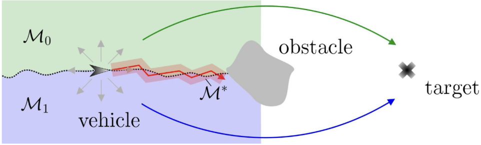

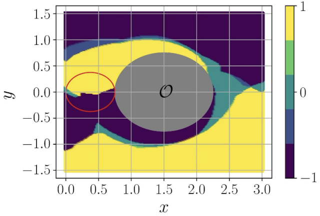

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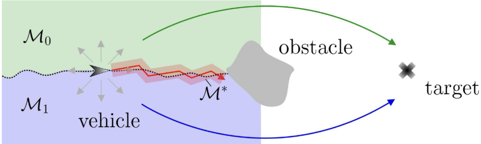

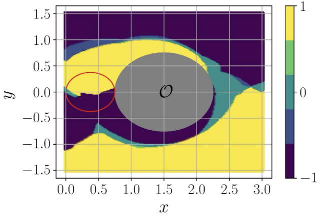

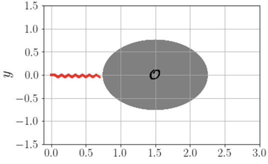

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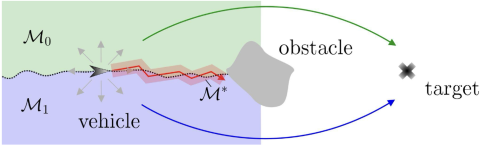

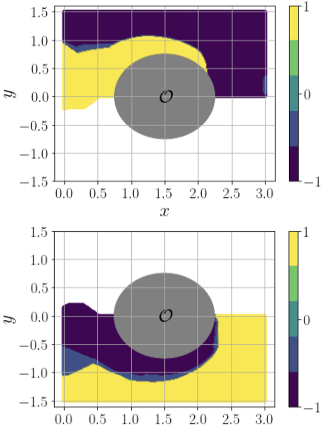

Modeling Hybrid Systems: Hybrid Plant

Hybrid closed-loop systems are given by *hybrid inclusions* with inputs

$$\mathcal{H} \quad \begin{cases} \dot{x} & \in & F(x,u) & \qquad (x,u) \in C \\ x^+ & \in & G(x,u) & \qquad (x,u) \in D \end{cases}$$

where x is the *state* and u the *input*,

- C is the *flow set*
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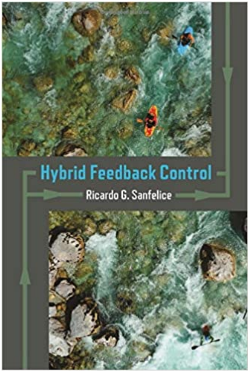
$$([0,t_1] \times \{0\}) \cup ([t_1,t_2] \times \{1\}) \cup \ldots ([t_j,t_{j+1}] \times \{j\}) \cup \ldots$$

The state x can have logic, memory, and timer components.

Modeling Hybrid Systems: Hybrid Plant



Hybrid Feedback Control
Princeton University Press
2021

Research Areas during AFOSR CoE

1. **Intermittent Information**
   - Estimation
   - Synchronization
2. **Safety**
   - Safety Certificates
   - Reinforcement Learning
3. **Optimization**
   - Hybrid Model Predictive Control
   - Distributed Optimization
   - Optimization with Computational Constraints
4. **Security**
   - Attack Detection
   - Hybrid Control for Attack Recovery

During this project, we published 91 conference papers, 44 journal papers, 4 book chapters, and 1 textbook related to these topics.

HyRRT/HySST: Sampling-based Motion Planning Algorithms for Hybrid Dynamical Systems

Nan Wang, Ricardo Sanfelice
Hybrid Systems Laboratory, UC Santa Cruz

HYBRID SYSTEMS LABORATORY

UNIVERSITY OF CALIFORNIA SANTA CRUZ

AACE Program Final Review
April 7, 2025

Motivation

Hybrid system exhibits combined continuous and discrete behaviors within respective constraint sets.

$$\mathscr{H}[3]: \begin{cases} \dot{x} = f(x,u) & (x,u) \in C \\ x^{+} = g(x,u) & (x,u) \in D \end{cases}$$

- C is the *flow set*
- f is the *flow map*
- D is the *jump set*
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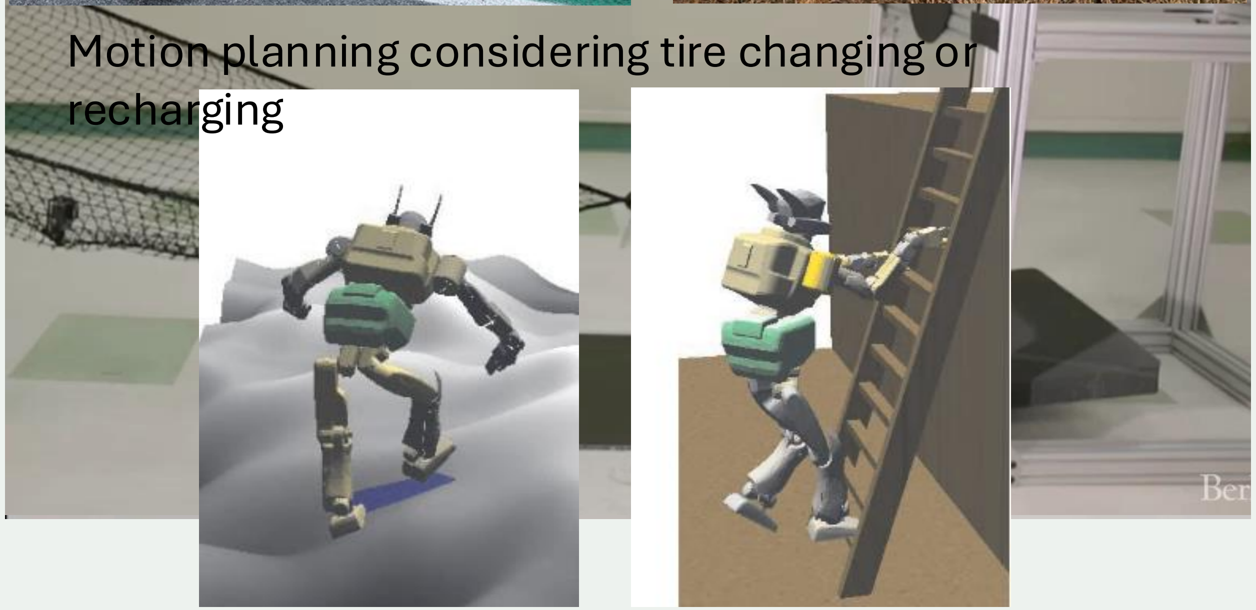
Motion planning considering tire changing or recharging



Jumping insect robot [2]

Collision-resilient aerial vehicle [1]

Robotic systems with multimodal structure

[3] Sanfelice, Ricardo G. *Hybrid feedback control*. Princeton University Press, 2021.

[1] J. Zha and M. W. Mueller, "Exploiting collisions for sampling-based multicopter motion planning," *2021 IEEE International Conference on Robotics and Automation (ICRA)*, Xi'an, China, 2021, pp. 7943-7949, doi: 10.1109/ICRA48506.2021.9561166.

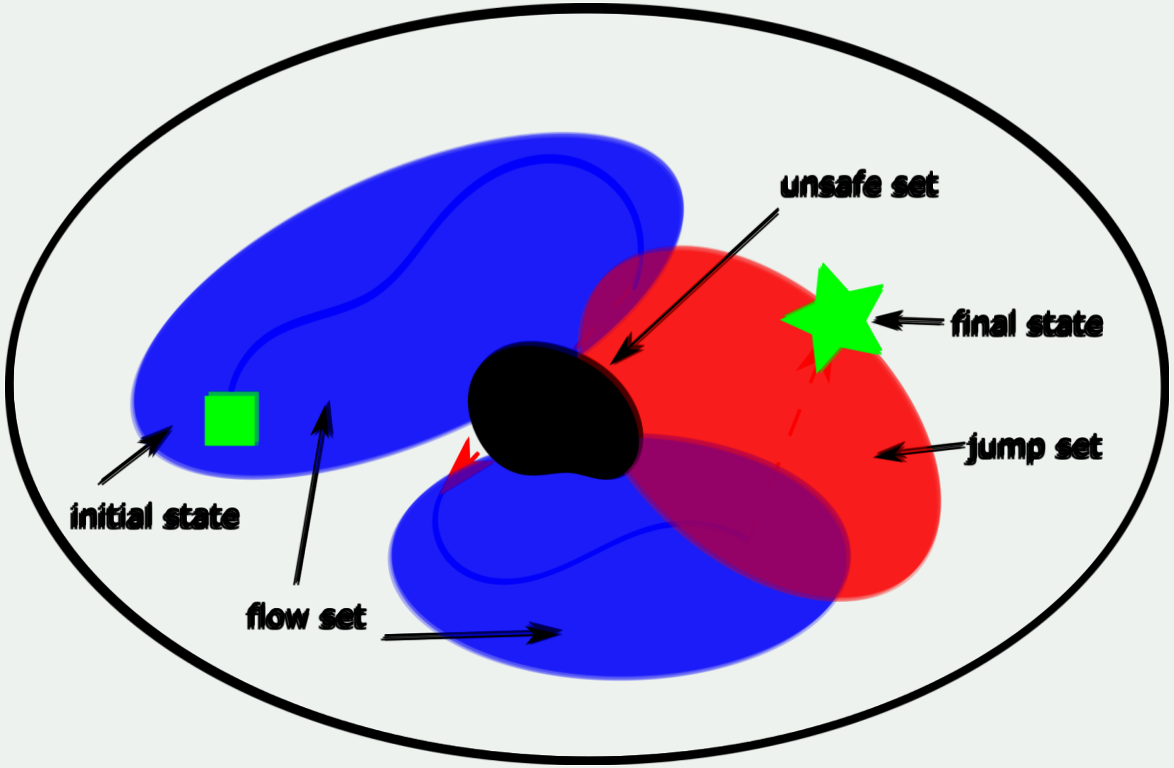
[2] Duncan W. Haldane et al. Robotic vertical jumping agility via series-elastic power modulation Sci. Robot.1,eaag 2048(2016) DOI:10.1126/scirobotics.aag2048

Problem Formulation: Feasible Motion Planning Problem

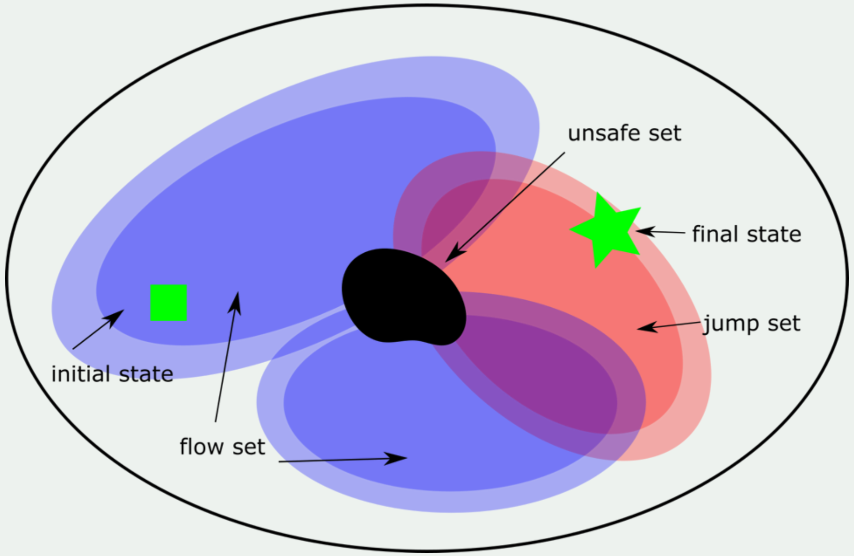
Problem 1: (Feasible motion planning problem for hybrid systems) Given a hybrid system $\mathcal{H}$ as in (1) with state $x \in \mathbb{R}^n$ and input $\mathrm{u} \in \mathbb{R}^m$, the initial state set $X_0 \subset \mathbb{R}^n$, the final state set $X_f \subset \mathbb{R}^n$, the unsafe set $X_u \subset \mathbb{R}^n \times \mathbb{R}^m$, find a pair (ϕ, u) such that for some $(T, J) \in \mathrm{dom}(\phi, u)$, the following hold:

- $\phi(0,0) \in X_0$;
- $\phi(T,J) \in X_f$;
- (ϕ, u) is a solution pair to $\mathcal{H}$;
- For any $(t,j) \in \mathrm{dom}(\phi, u)$ such that $t + j \leq T + J$, $\big(\phi(t,j), u(t,j)\big) \notin X_u$.

$$\mathcal{H}: \begin{cases} \dot{x} = f(x,u) & (x,u) \in C \\ x^+ = g(x,u) & (x,u) \in D \end{cases}$$



HyRRT [1]

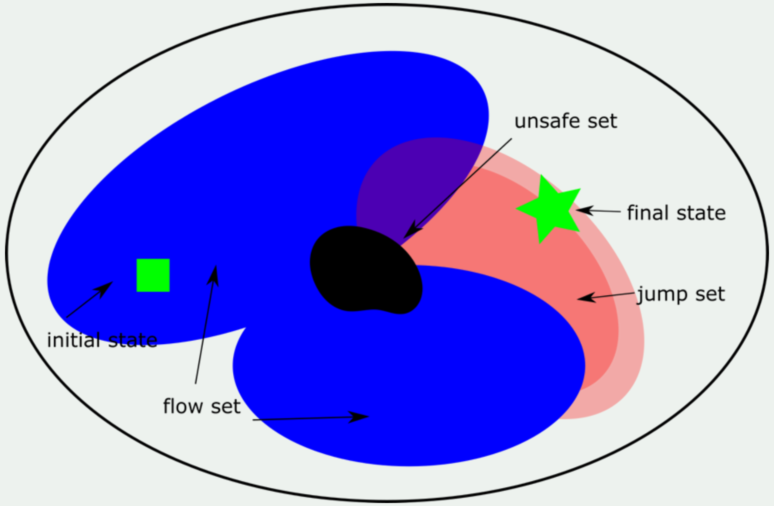


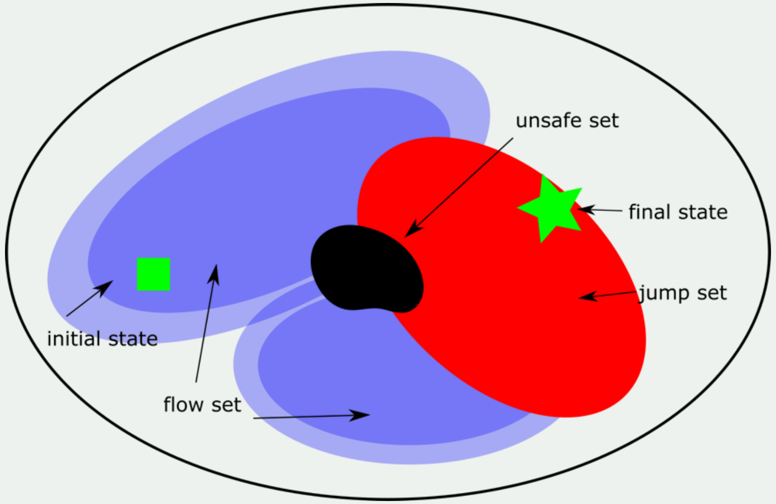
Step 1. Inflation of hybrid systems

$$\mathcal{H}: \begin{cases} \dot{x} = f(x,u) & (x,u) \in C \\ x^+ = g(x,u) & (x,u) \in D \end{cases}$$

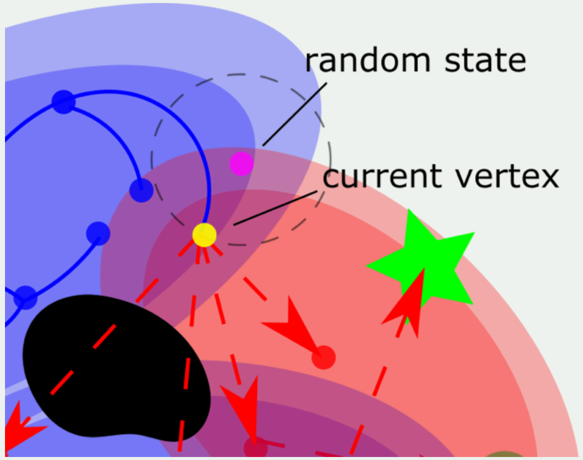
Create clearance!

$$\mathcal{H}_\delta: \begin{cases} \dot{x} = f_\delta(x,u) & (x,u) \in C_\delta \\ x^+ = g_\delta(x,u) & (x,u) \in D_\delta \end{cases}$$





Step 2: Randomly select from flow or jump scheme



Step 3: Find the closest vertex in the search tree to the random state

[1] Wang, Nan, and Ricardo G. Sanfelice. "Motion planning for hybrid dynamical systems: Framework, algorithm template, and a sampling-based approach." The International Journal of Robotics Research (2025)

HyRRT

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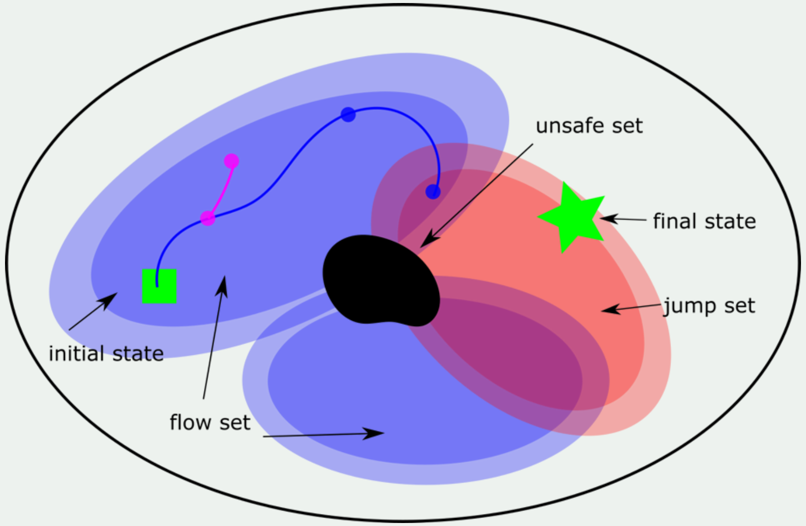
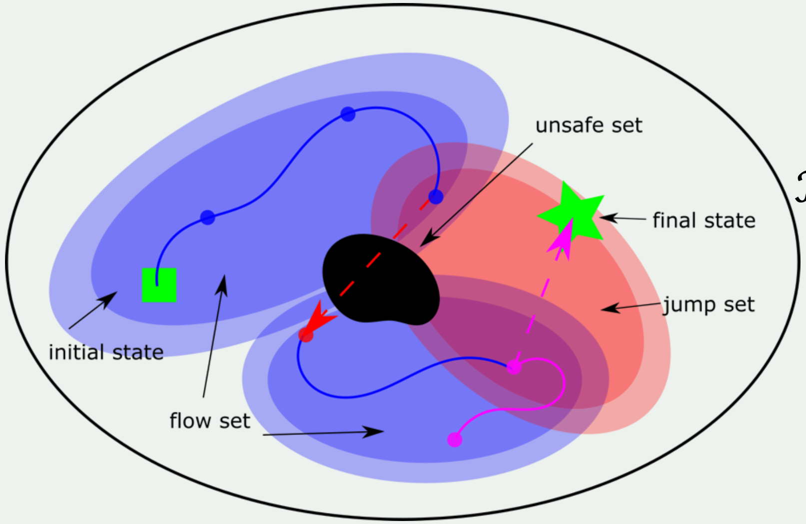
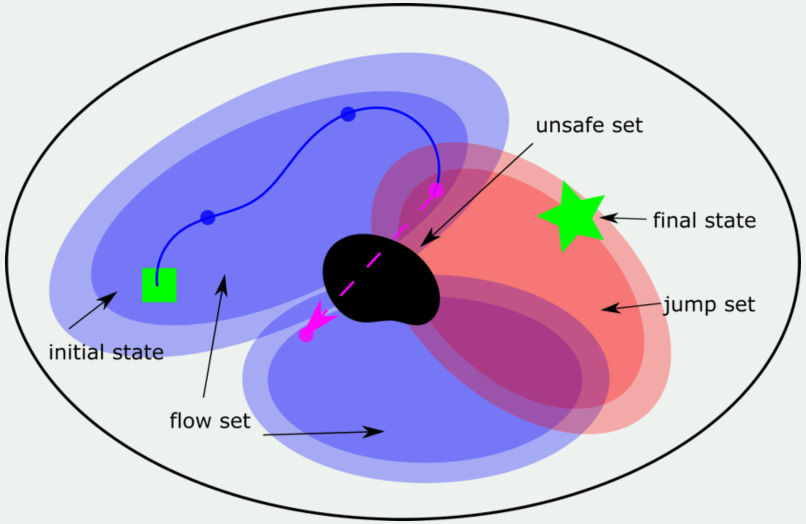






Step 4: Random propagation via flow or jump

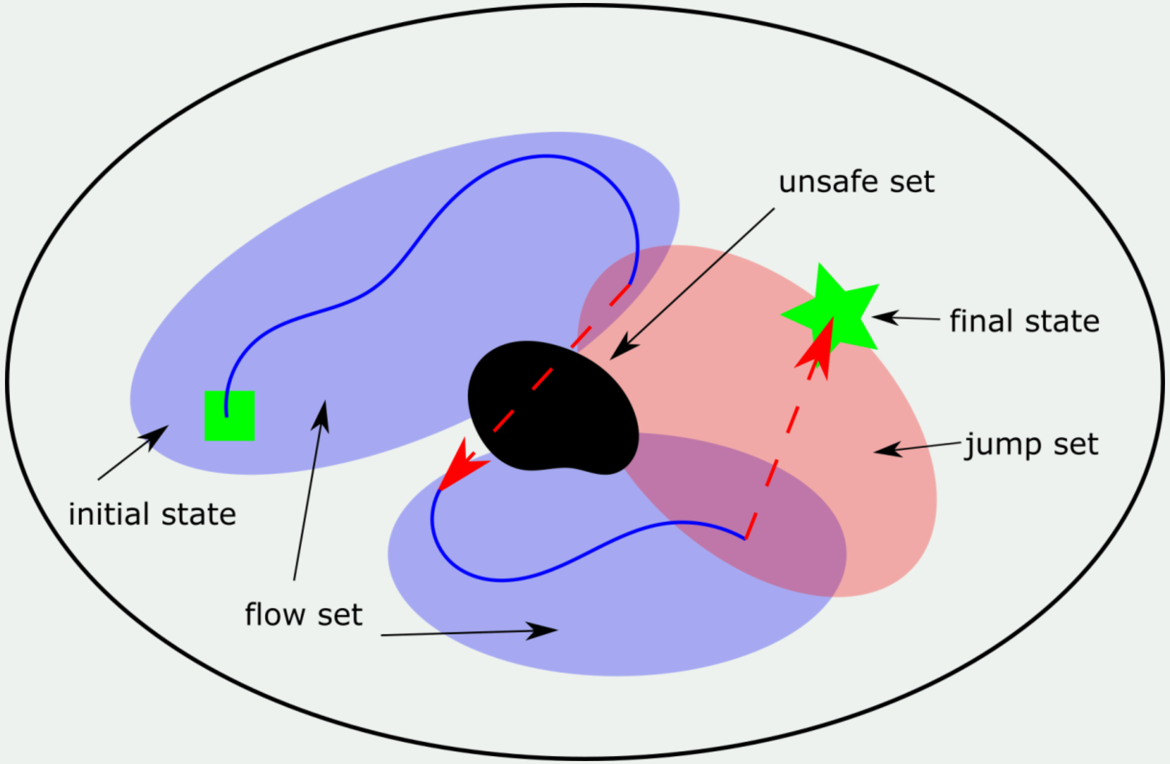
Theorem 1. (Probabilistic Completeness of HyRRT [1])
Suppose there exists a motion plan (ϕ, u) to $\mathcal{P} = (X_0, X_f, X_u, (C, f, D, g))$. When HyRRT is used to solve $\mathcal{P} = (X_0, X_f, X_u, (C_\delta, f_\delta, D_\delta, g_\delta))$, the probability that HyRRT fails to find a motion plan (ϕ', u') such that ϕ' is close to ϕ after k iterations is at most ae^{-bk}, where $a, b > 0$.

[1] Wang, Nan, and Ricardo G. Sanfelice. "Motion planning for hybrid dynamical systems: Framework, algorithm template, and a sampling-based approach." The International Journal of Robotics Research (2025)

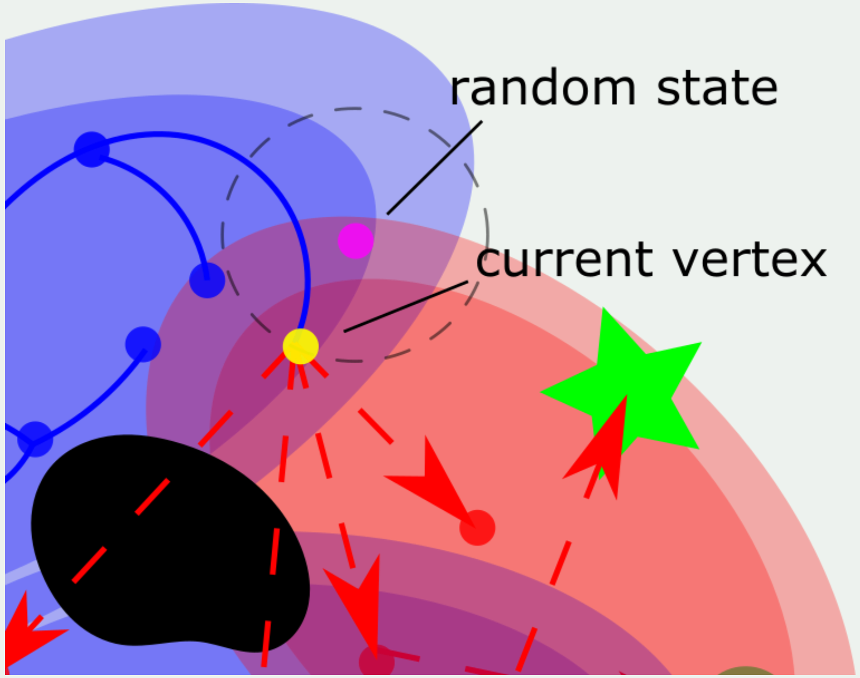
Problem Formulation: Optimal Motion Planning Problem

Problem 2: (Optimal motion planning problem for hybrid systems) Given ***Problem 1*** and a cost functional c, find a feasible motion plan (ϕ^*, u^*) to Problem 1 such that $(\phi^*, u^*) = \arg\min_{(\phi,u)} c(\phi)$.

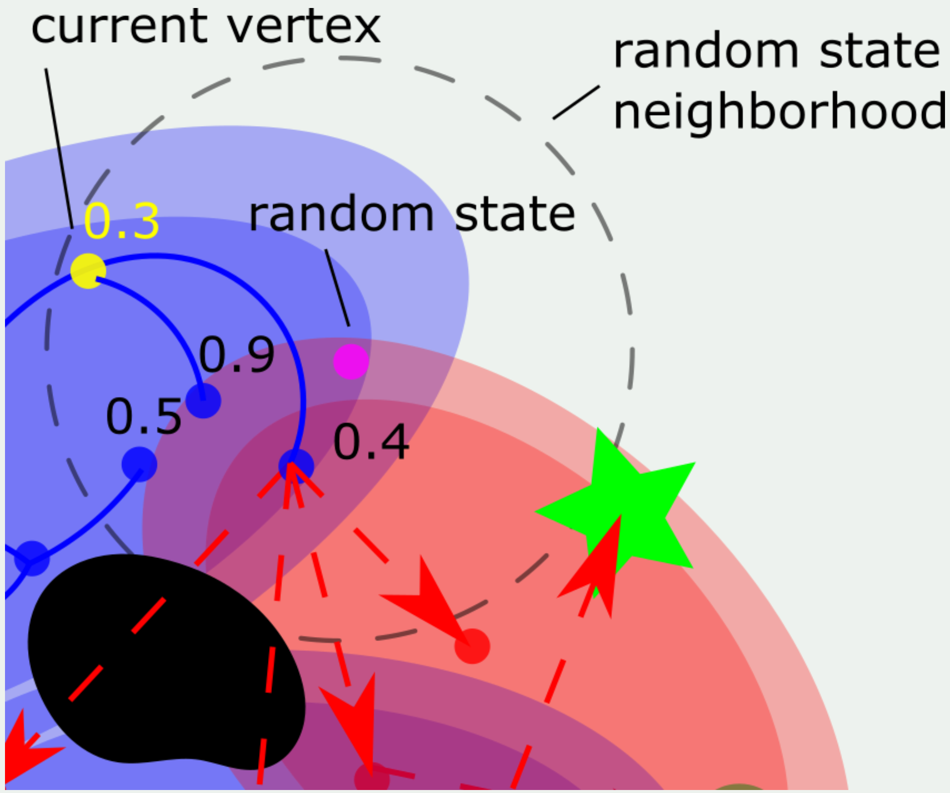
$$\mathcal{H}: \begin{cases} \dot{x} = f(x,u) & (x,u) \in C \\ x^+ = g(x,u) & (x,u) \in D \end{cases}$$



HySST [1]



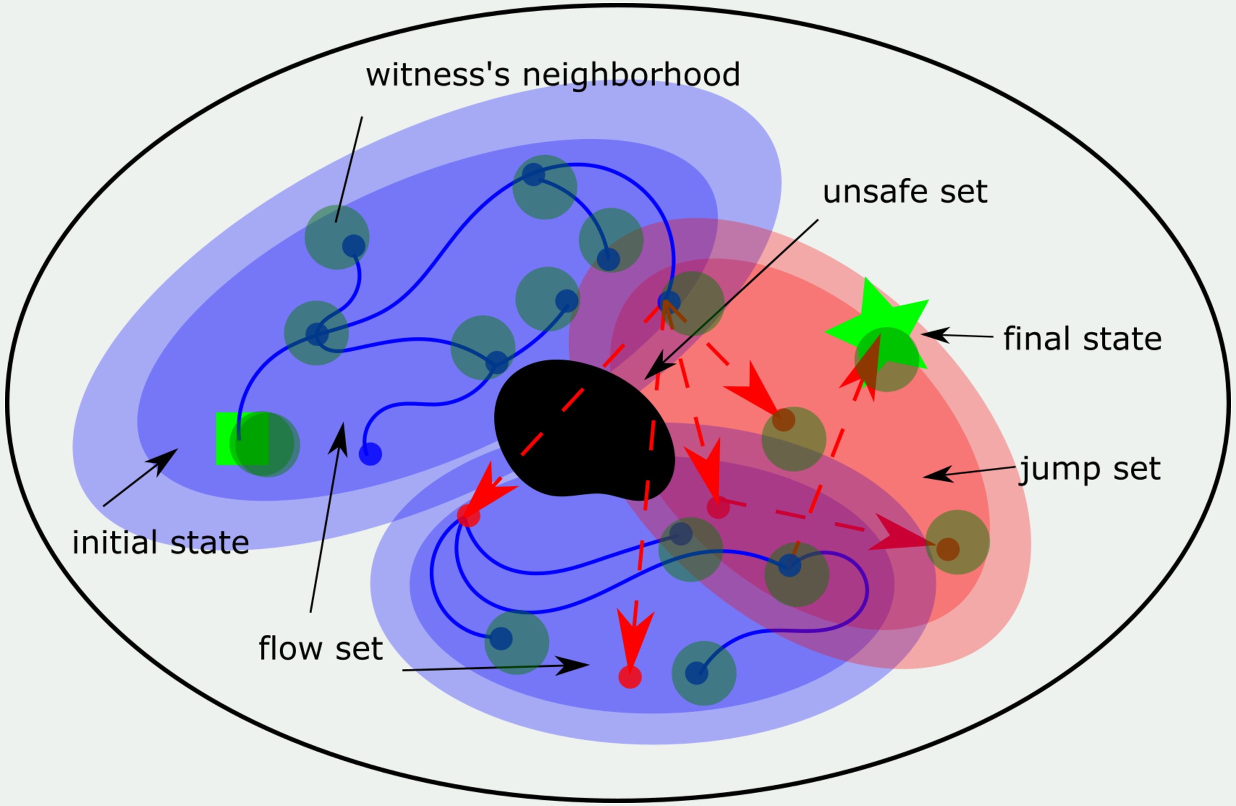
HyRRT finds the closest vertex in the search tree to the random state



HySST finds the vertex with the minimal cost within the neighborhood of the random state

[1] Wang, Nan, and Ricardo G. Sanfelice. "HySST: A Stable Sparse Rapidly-Exploring Random Trees Optimal Motion Planning Algorithm for Hybrid Dynamical Systems." IEEE 62nd Conference on Decision and Control (CDC). IEEE, 2023.

HySST



Sparsify the vertices by maintaining a static set of witnesses.

Theorem 2. (Asymptotic Near-Optimality of HySST [2]) *Suppose there exists an optimal motion plan* (ϕ^*, u^*) *to* $\mathcal{P}^* = (X_0, X_f, X_u, (C, f, D, g), c)$*. When HySST is used to solve* $\mathcal{P} = (X_0, X_f, X_u, (C_\delta, f_\delta, D_\delta, g_\delta), \mathrm{c})$*, the probability that HySST finds a motion plan* (ϕ', u') *such that* $c(\phi') < (1 + a\delta)c(\phi^*)$ *converges to 1 as the number of iterations approaches infinity.*

[1] Wang, Nan, and Ricardo G. Sanfelice. "HySST: A Stable Sparse Rapidly-Exploring Random Trees Optimal Motion Planning Algorithm for Hybrid Dynamical Systems." to appear in 2023 IEEE 62nd Conference on Decision and Control (CDC). IEEE, 2023.

Takeaways

- We formulate a general motion planning problem for hybrid systems.
- We propose a **probabilistically complete** HyRRT to solve feasible motion planning problems for hybrid systems.
- We propose an **asymptotically near-optimal** HySST to solve optimal motion planning problems for hybrid systems.

HyRRT

HySST

Email me at nanwang@ucsc.edu

Inverse-Optimal Safety Control for Hybrid Systems

Carlos A. Montenegro G., Santiago J. Leudo, and Ricardo G. Sanfelice

University of California, Santa Cruz, USA

AACE Program Review

April 7, 2025

Hybrid Systems Laboratory

Motivation

Control Theory

Quadruped Robot. Mutiple time domains.

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- Synthesis of controllers for safety of hybrid systems under disturbances is yet to be explored.

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- Existing methods only apply to limited classes of systems and fall short of guaranteeing safety and optimality for hybrid systems.

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Control Theory

Quadruped Robot. Mutiple time domains.

- Synthesis of controllers for safety of hybrid systems under disturbances is yet to be explored.
- Existing methods only apply to limited classes of systems and fall short of guaranteeing safety and optimality for hybrid systems.
- Challenges: Hybrid systems pose additional challenges due to interaction of discrete and continuous dynamics

Motivation

Control Theory

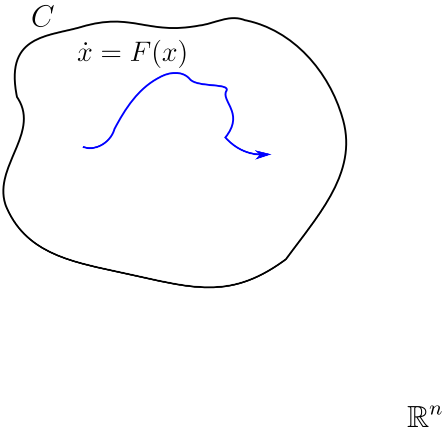
Quadruped Robot. Mutiple time domains.

- Synthesis of controllers for safety of hybrid systems under disturbances is yet to be explored.

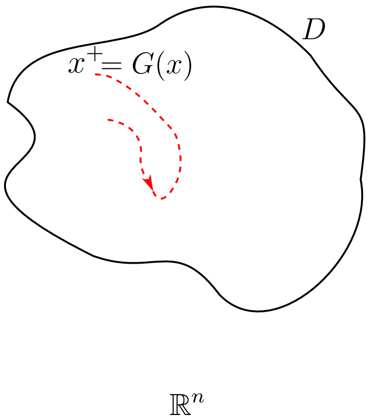
> Thus, we propose a **framework** to **certify safety and optimality** for systems with such **complex dynamics** even under **disturbances**.

discrete and continuous dynamics

Modeling Hybrid Dynamics



Modeling Hybrid Dynamics



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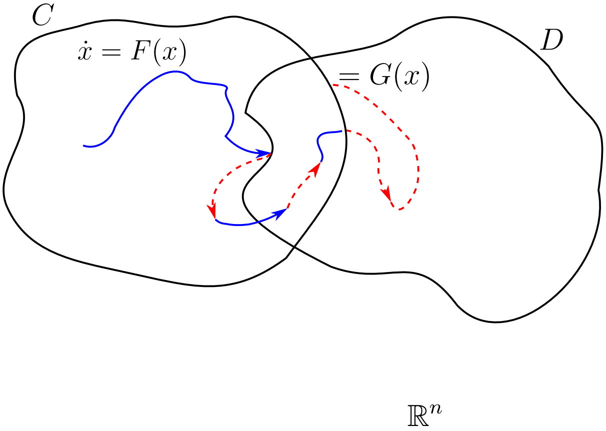

Hybrid Systems

A hybrid system $\mathcal{H}$ with state x as in [Goebel, et.al., PUP 2012]:

$$\mathcal{H}\begin{cases} \dot{x} &= F(x) \quad x \in C \\ x^+ &= G(x) \quad x \in D \end{cases}$$

- C is the *flow set*
- F is the *flow map*
- D is the *jump set*
- G is the *jump map*

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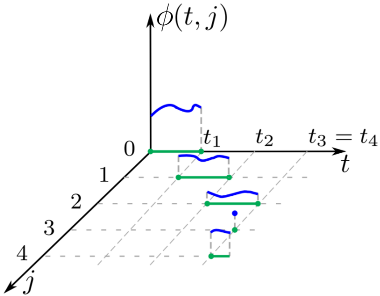

ϕ : Solution to $\mathcal{H}$

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Connections to Other Frameworks

Switched Systems

$$\begin{aligned} \dot{x} =& f_{\sigma(t)}(x) \\ \sigma(t) \in& \{1, 2, \dots\} \end{aligned}$$

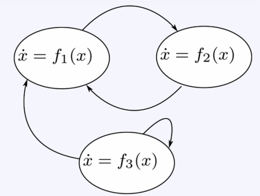
Impulsive Systems

$$\begin{aligned} \dot{x} =& f(x(t)) \\ x(t^+) =& g(x(t)) \quad t \in \{t_1, t_2, \dots\} \end{aligned}$$

Differential-Algebraic Equations

$$\begin{aligned} \dot{x} =& f(x, w) \\ 0 =& \eta(x, w) \end{aligned}$$

Hybrid Automata



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To the best of our knowledge, there does not exist previous work to synthesize **safety controllers** that are also **optimal** for **hybrid systems under disturbances** modeled in such framework.

Di

$$
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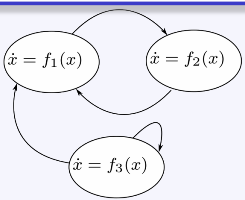

Input-to-State Safety

Consider the following hybrid system $\mathcal{H}$ on $\mathbb{R}$ affine in the input $w \in \mathbb{R}^m$:

$$\mathcal{H} \ : \begin{cases} \dot{x} \ = F(x, w_C) := f(x) + f_w(x) w_C & (x, w_C) \in C \\ x^+ = G(x, w_D) := g(x) + g_w(x) w_D & (x, w_D) \in D \end{cases}$$

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Input-to-State Safety for Hybrid Systems

Consider a closed set $K \subset \mathbb{R}^n$ defined as the 0-sublevel set of a function $B : \operatorname{dom} B \to \mathbb{R}$, and a hybrid system $\mathcal{H}$. If there exist $\bar{w} \geq 0$ and $\rho \in \mathcal{K}_\infty$ such that

$$\begin{aligned} &(\phi, w) \in \mathcal{S}_{\mathcal{H}}(K), ||w||_{\#} \leq \bar{w} \\ &\qquad\qquad \Rightarrow \; B(\phi(t, j)) \leq \rho(\bar{w}) \quad \forall (t, j) \in \operatorname{dom} \phi \end{aligned}$$

where the function ρ is referred to as the ISSf gain, then the system $\mathcal{H}$ is $\bar{w}$-small-input input-to-state safe ($\bar{w}$-small-input ISSf) with respect to the disturbance w and the set K.

Input-to-State Safety

Safety and invariance

It is immediate that the system $\mathcal{H}$ is $\bar{w}$-small-input ISSf with respect to w and K if and only if there exist $\bar{w} \geq 0$ and $\rho \in \mathcal{K}_\infty$ such that the set $K_d(\bar{w}) \supset K$ defined as

$$K_d(\bar{w}) := \{x \in \Pi(C) \cup \Pi(D) : B(x) - \rho(\bar{w}) \leq 0\}$$

is conditionally pre-invariant for $\mathcal{H}$ with respect to w and K.

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Theorem. ISSf under a barrier function candidate.

Given a hybrid system $\mathcal{H}$ and a closed set $K \subset \mathbb{R}^n$, suppose B is an ISSf-BF candidate for $\mathcal{H}$ with respect to $(K, K_d(\bar{w}))$, where $K_d(\bar{w})$ is defined for some $\rho \in \mathcal{K}_\infty$ and $\bar{w} \geq 0$.

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$$\langle \nabla B(x), F(x, w_C) \rangle \leq -\alpha_C B(x) \qquad \forall (x, w_C) \in C : x \in \mathcal{V} \backslash K_d(\bar{w}), |w_C| \leq \bar{w}$$

$$B(G(x, w_D)) - B(x) \leq -\alpha_D (B(x) - \rho(\bar{w})) \qquad \forall (x, w_D) \in D : x \in K_d(\bar{w}), |w_D| \leq \bar{w}$$

Input-to-State Safety QP Filter

But we love control! Let's bring it back:

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Assumption. Control-affine jumps.

Consider a hybrid system $\mathcal{H}$, a feedback law $\kappa := (\kappa_C, \kappa_D) = (\bar{\kappa}_C + \widehat{\kappa}_C, \bar{\kappa}_D + \widehat{\kappa}_D)$, and a scalar function B. Suppose that there exist functions $\widehat{B}_{Lu}$ and $\widehat{B}_{Lw}$ such that, for all $(x, (\kappa_D(x), w_D)) \in D$,

$$\begin{aligned} B\left(G(x, (\kappa_D(x), w_D))\right) &= B\big(g(x) + g_u(x)\kappa_D(x) + g_w(x)w_D\big) \\ &\leq B\big(g(x) + g_u(x)\bar{\kappa}_D(x)\big) + \widehat{B}_{Lu}(x)\widehat{\kappa}_D(x) + \widehat{B}_{Lw}(x)w_D. \end{aligned}$$

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Given $\bar{\kappa}_C$ and $\alpha_C \geq 0$, we define

$$\begin{aligned} \omega_C(x) &:= L_{f+f_u\bar{\kappa}_C} B(x) \\ &\quad + |L_{f_w} B(x)| \rho^{-1}(B(x)) + \alpha_C B(x) \end{aligned}$$

and introduce the following QP

$$\begin{aligned} \widehat{\kappa}_{C_{QP}}(x) = \arg\min_{v \in \mathbb{R}^{m_{C_u}}} & \quad |v|^2 \\ \text{subject to} & \quad L_{f_u} B(x) v \leq -\omega_C(x) \end{aligned}$$

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Gi

> **Theorem.** The QP filter $\widehat{\kappa}_{QP} := (\widehat{\kappa}_{C_{QP}}, \widehat{\kappa}_{D_{QP}})$ renders the hybrid closed-loop system $\mathcal{H}_\kappa$ $\bar{w}$-small-input ISSf w.r.t. (K, w).

$$+ |L_{f_w} B(x)| \rho^{-1}(B(x)) + \alpha_C B(x)$$

$$+ |B_{Lw}(x)| \bar{w} + \alpha_D (B(x) - \rho(\bar{w}))$$

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Two-Player Zero-Sum Hybrid Game

Given $\xi \in K$, an input action $(u, w) = ((u_C, u_D), (w_C, w_D))$, the stage cost for flows L_C, the stage cost for jumps L_D, and the terminal cost q, we define the cost associated to the solution $(\phi, (u, w))$ to $\mathcal{H}$ from ξ as

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$$\mathcal{J}(\xi, (u, w)) := \sum_{j=0}^{\sup_j \text{dom}\, \phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t, j), (u_C(t, j), w_C(t, j)))dt$$

$$+ \sum_{j=0}^{\sup_j \text{dom}\, \phi - 1} L_D(\phi(t_{j+1}, j), (u_D(t_{j+1}, j), w_D(t_{j+1}, j)))$$

$$+ \limsup_{\substack{t+j \to \sup_t \text{dom}\, \phi + \sup_j \text{dom}\, \phi \\ (t,j) \in \text{dom}\, \phi}} q(\phi(t, j))$$

Two-Player Zero-Sum Hybrid Game

Given $\xi \in K$, an input action $(u, w) = ((u_C, u_D), (w_C, w_D))$, the stage cost for flows

Definition. (Value function). Given $\xi \in K$ and a nominal feedback law $\bar{\kappa}$, the value function at ξ is given by

$$\mathcal{J}^*(\xi) := \min_{u} \max_{w} \underset{u=(u,w)\in\mathcal{U}_{\mathcal{H}}(\bar{\kappa},\bar{w})}{} \mathcal{J}(\xi, (u, w)).$$

$$\sum_{j=0} \int_{t_j}$$

$$+ \sum_{j=0}^{\sup_j \text{dom}\, \phi - 1} L_D(\phi(t_{j+1}, j), (u_D(t_{j+1}, j), w_D(t_{j+1}, j)))$$

$$+ \limsup_{\substack{t+j\to\sup_t \text{dom}\,\phi + \sup_j \text{dom}\,\phi \\ (t,j)\in \text{dom}\,\phi}} q(\phi(t, j))$$

Two-Player Zero-Sum Hybrid Game

Given $\xi \in K$, an input action $(u, w) = ((u_C, u_D), (w_C, w_D))$, the stage cost for flows

Definition. (Value function). Given $\xi \in K$ and a nominal feedback law $\bar{\kappa}$, the value function at ξ is given by

$$\mathcal{J}^*(\xi) := \min_{\substack{u \\ u=(u,w)\in\mathcal{U}_{\mathcal{H}}(\bar{\kappa},\bar{w})}} \max_{w} \mathcal{J}(\xi, (u, w)).$$

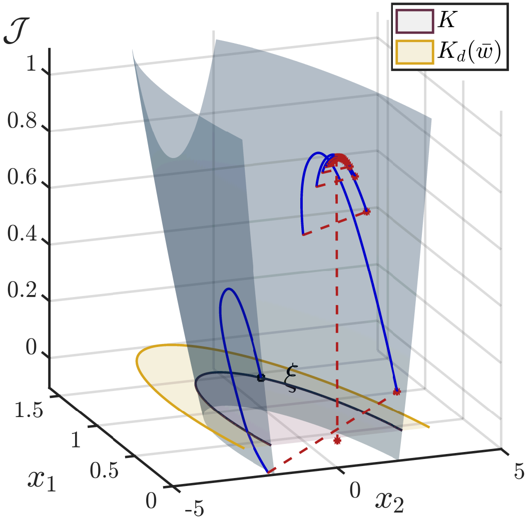
$j=0$

$\sup_j \text{dom}\, \phi - 1$

Theorem. The QP filter $\widehat{\kappa}_{QP} := (\widehat{\kappa}_{C_{QP}}, \widehat{\kappa}_{D_{QP}})$ minimizes $\mathcal{J}$, under the worst case disturbacen, for some L_C, L_D, and q. In addition, the CBF B is the value function for the hybrid game.

$(t,j)\in\text{dom}\, \phi$

Two-Player Zero-Sum Hybrid Game



Two-Player Zero-Sum Hybrid Game

For more details:

C. A. Montenegro G., S. Leudo, and R. G. Sanfelice, "**Inverse-Optimal Safety Control for Hybrid Systems**", in Proceedings of the 28th ACM International Conference on Hybrid Systems: Computation and Control, Irvine, CA, USA, 2025.

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