Topology-Aware LTL Planning for Cooperative Navigation Tasks

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Part One

Connectivity Maintenance in General Environments

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PnP Cooperative Navigation: control objective

This is a brief update on the submitted TAC paper [1], following the previous ACC 2022 report in [2].

Provided:

- MAS with $\dot{x}_p = u_p$, $p \in \mathcal{V}$, in a compact domain $\Omega \subset \mathbb{R}^d$,
- Distance-limited comms: $p, q \in \mathcal{V}$ *may* communicate $\Leftrightarrow$ $\|x_p - x_q\| \leq R$,
- Prescribed communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$,
- *Obstacles of general shape*,
- *Solution to single-agent navigation in Ω,*

Task:

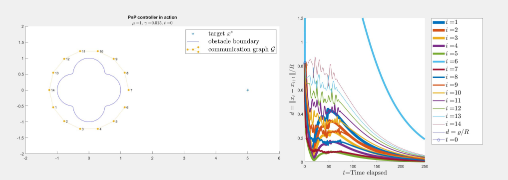
- The MAS follows a leader $\ell \in \mathcal{V}$, while maintaining $\|x_p - x_q\| \leq R$ for all $pq \in \mathcal{E}$.
- "Lazy" agents: distances between neighbors should not be contracted indefinitely.

Method:

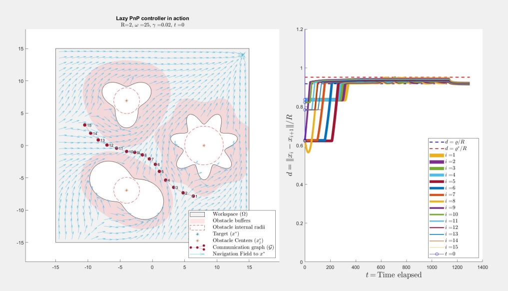
- Require a prescriptive solution—a formula—extending single-agent navigation know-how to graph-preserving MAS-navigation ('Plug-and-Play').

PnP Cooperative Navigation: control objective

Exaggerated contractive interaction:



No interaction between neighbors if close enough:



PnP Cooperative Navigation: what a single agent knows

What is an acceptable single-agent navigation solution?

Definition (Navigation Field [2], inspired by [3, 4])

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a compact domain given by $\Omega \triangleq [\beta \geq 0]$, where β is a C^∞-smooth function of $\mathbb{R}^d$ with regular value 0. A navigation field on Ω is a locally Lipschitz-continuous map $\mathfrak{n}\colon \Omega \times \Omega \to \mathbb{R}^d$ satisfying the following conditions for every $y \in \mathrm{int}(\Omega)$:

1. $\langle \mathfrak{n}(y, z), \nabla_z \beta(z) \rangle > 0$ almost everywhere on $\partial\Omega$;
2. $z = y$ is the only stable equilibrium of $\mathfrak{n}(y, -)$;
3. For almost all initial conditions $x(0) \in \Omega$, solutions $x(t)$ of $\dot{x} = \mathfrak{n}(y, x)$ converge to y as $t \to \infty$;
4. There is a continuous $\alpha : \mathrm{int}(\Omega) \to (0, \infty)$ such that $\|\mathfrak{n}(y, z)\| \geq \alpha(y)\|y - z\|$ for all z near y.

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1. $\langle \mathfrak{n}(y,z), \nabla_z \beta(z) \rangle > 0$ almost everywhere on $\partial\Omega$;
2. $z = y$ is the only stable equilibrium of $\mathfrak{n}(y,-)$;
3. For almost all initial conditions $x(0) \in \Omega$, solutions $x(t)$ of $\dot{x} = \mathfrak{n}(y,x)$ converge to y as $t \to \infty$;
4. There is a continuous $\alpha : \mathrm{int}(\Omega) \to (0,\infty)$ such that $\|\mathfrak{n}(y,z)\| \geq \alpha(y)\|y - z\|$ for all z near y.

- Removes the need for discussing specific Ω;
- All known solutions are of this form, many with $\alpha(y) \equiv 1$;
- Consistent with imposing Rantzer-type dual-Lyapunov conditions [5].

PnP Cooperative Navigation: formal objective

Configurations.

- *Configurations/Ensemble States*

$$\mathbf{x} \triangleq (x_p)_{p\in\mathcal{V}} \in (\mathbb{R}^d)^{\mathcal{V}}, \ \Delta\mathbf{x} \triangleq (x_q - x_p)_{pq\in\mathcal{E}} \in (\mathbb{R}^d)^{\mathcal{E}} \tag{1}$$

$\leadsto$ *need to be careful about edge orientation*

- *s-Available edges* of a configuration $\mathbf{x}$, for $s > 0$, are

$$\mathcal{E}_s(\mathbf{x}) \triangleq \{pq \in \tbinom{\mathcal{V}}{2} : \|x_q - x_p\| \leq s\}. \tag{2}$$

- *s-Valid Configurations for $\mathcal{G}$* are the ones in $\mathscr{C}_s(\mathcal{G})$, where

$$\mathscr{C}_s(\mathcal{G}) \triangleq \{\mathbf{x} \in \Omega^{\mathcal{V}} : \mathcal{E} \subseteq \mathcal{E}_s(\mathbf{x})\}. \tag{3}$$

Problem (Weak Invariance Problem for Graph Maintenance (WIP))

For any $\varrho^* \in (0, R)$, construct controllers $\mathbf{u}$ such that every solution of $\dot{\mathbf{x}} = \mathbf{u}$ emanating from $\mathbf{x}(0) \in \mathscr{C}_{\varrho^*}(\mathcal{G})$ remains in $\mathscr{C}_R(\mathcal{G})$ for all time.

PnP Cooperative Navigation: holonomic solution

- *The PnP field* is a superposition of navigation fields aimed at moving MAS neighbors,

$$u_p \triangleq \sum_{q\sim p} \xi_q^p \mathfrak{n}_q^p + v_p\,, \quad \mathfrak{n}_q^p(\mathbf{x}) \triangleq \mathfrak{n}(x_q, x_p). \tag{4}$$

- *Asymmetric Rescaling Factors,* $\xi_q^p(\mathbf{x}) \triangleq \xi(x_q, x_p)$ given by

$$\xi(y,z) \triangleq \frac{r(\|y-z\|)\|y-z\|^2}{\langle \mathfrak{n}(y,z), y-z\rangle}. \tag{5}$$

- *Edge Tension Function.* $r : [0,\infty) \to [0,\infty)$, $\mu \geq 0$, $\omega > 0$, $\alpha \in [0,1]$,

$$r(s) \triangleq \begin{cases} \mu, & \text{if } s \in [0,\varrho], \\ \mu + \omega(s-\varrho)^{1+\alpha}, & \text{if } \sigma \in [\varrho, R] \\ 0, & \text{if } \sigma \in (R,\infty]. \end{cases} \tag{6}$$

  Where $\varrho \in (0,R)$ is a safety distance, $m \triangleq \frac{R}{\varrho}$ and $M \triangleq \frac{r(R)}{r(\varrho)}$ are characteristics of the tension.

- *The task component,* v_p is zero for all p but the leader ℓ, with

$$v_\ell \triangleq \gamma\, \mathfrak{n}(x^*, x_\ell) - \sum_{q\sim\ell} \xi_q^\ell \mathfrak{n}_q^\ell, \tag{7}$$

  to keep the leader navigate to x^* while unaffected by the network.

PnP Cooperative Navigation: quality requirements of $\mathfrak{n}(y,z)$

So why the particular rescaling from (5)?

- $\mathbf{u} - \mathbf{v}$ is related to the consensus controller, $\mathbf{u}_w \triangleq -(\mathbf{L}_w \otimes \mathbf{I}_d)\mathbf{x}$ with $w_{pq} \triangleq r(\|x_q - x_p\|)$.

$\rightsquigarrow$ *note how $\|\mathbf{u}_w\|$ may drop to zero, with r not bounded away from zero, even when $\mathbf{x}$ is not in consensus!*

Definition

Let $\delta \in (0,1]$. A navigation field $\mathfrak{n}$ on Ω is (R,δ)-good, if for all $y, z \in \Omega$ with $\|y - z\| \leq R$ one has

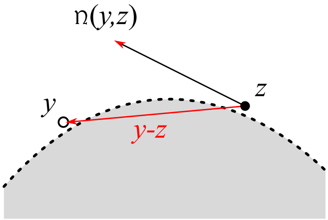
$$\langle \mathfrak{n}(y,z), y - z \rangle \geq \delta \|\mathfrak{n}(y,z)\| \|y - z\|. \tag{8}$$

- $\mathfrak{n}$ is "well-aligned" with the radial field for nearby targets:

$$\cos \angle(\mathfrak{n}(y,z), y - z) \geq \delta,$$

imposing a tradeoff between obstacle curvature and communication radius.

- This also means that $U_y(z) \triangleq \|z - y\|^2$ is a strict Lyapunov function for $\mathfrak{n}(y,z)$ at y in $y + R\mathbb{B}$.



PnP Cooperative Navigation: results

We have already proved:

- In the contractive case, $\mu > 0$, The PnP controller $\mathbf{u}$ solves the WIP, for appropriate parameter choices, and sufficient slow-down of the leader.

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- This was based on a property of the *total tension potential*,

$$V_{\mathcal{G}}(\mathbf{x}) \triangleq \sum_{pq\in\mathcal{E}} V_{pq}(\mathbf{x}) = \frac{1}{2}\sum_{p\in\mathcal{V}}\sum_{q\sim p} P(\|x_q - x_p\|), \tag{9}$$

where each edge contributes

$$V_{pq}(\mathbf{x}) \triangleq P(\|x_q - x_p\|),\ P(\rho) \triangleq \int_0^{\rho} r(s)s\mathtt{d}s. \tag{10}$$

One shows that if $|\mathcal{E}|\,P(\varrho^*) < P(R)$, then any controller satisfying $\dot{V}_{\mathcal{G}} \leq 0$ over $\mathscr{C}_R(\mathcal{G}) \setminus \mathscr{C}_{\varrho^*}(\mathcal{G})$ is a solution to the WIP.

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- To prove that $\mathbf{u}$ satisfies this, one bounds $\dot{V}_{\mathcal{G}}$ as

$$\dot{V}_{\mathcal{G}}(\mathbf{x}) \leq -\lambda_2(G,w)^2\|\Delta\mathbf{x}\|_\infty^2 + 4\sqrt{N}\Delta(\mathcal{G})Rr(R) \times \{\text{stuff we can handle}\} \tag{11}$$

by decomposing all the $\xi_q^p \mathfrak{n}_q^p$ orthogonally into $w_{pq}(x_q - x_p)$ and an orthogonal vector of bounded length. When $\mu = 0$ this bound is worthless: from where $\mathbf{L}_w$ stands, the weights w_{pq} disconnect $\mathcal{G}$ even if the distances don't!

PnP Cooperative Navigation: results

Main Observation for the case of $\mu = 0$:

- Replacing the graph $\mathcal{G}$ with the collection $\mathscr{C}$ of its connected components taking into account null weights yields

$$\dot{V}_{\mathcal{G}}(\mathbf{x}) \leq -\sum_{G^* \in \mathcal{C}} \lambda_2(G^*, w|_{G^*})^2 \|\mathtt{proj}_{G^*} \Delta \mathbf{x}\|_\infty^2 + 4\sqrt{N}\Delta(\mathcal{G}) R r(R) \times \{\text{stuff we can handle}\}, \quad (9)$$

- leading to similar inequalities allowing to select parameter values satisfying the WIP criterion.
- This was THE most important case to handle, since there is little point in sequentially composing MAS controllers which individually tend to bring the MAS to near-rendezvous.

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Additional contributions:

- (R, δ)-goodness bounds on SOTA navigation fields;
- Working MATLAB implementations of SOTA navigation fields tested with PnP in challenging environments (multiple star-convex obstacles).

PnP Cooperative Navigation: future work

Applications:

- Sequential and parallel compositions framework for connecting/disconnecting distance-limited networks in the presence of large obstacles.
- Obstacle-aware LTL-based MAS-planning (some news on laying down the foundations from Yu Wang in Part 2)

Further Development:

- Non-holonomic extensions, e.g. differential drive (with Patrick Amy and Ishan Agrawal);

  $\rightsquigarrow$ *Run PnP on robot dogs and huskies!*
- Improved bounds on $\dot{V}_{\mathcal{G}}$ for less conservative control / adaptation;
- PnP extensions for other problems, e.g. optimal controllers???

Part Two

Topologically-Aware Planning

Dan P. Guralnik, Yu Wang and Warren E. Dixon

Overview

- **Motivation:** How to systematically solve planning and control problems for complex tasks in spaces and with atomic propositions that are not convex or even contractible Euclidean domains?
- **Topology offers a paradigm:** Present the workspace as the union of multiple contractible sub-spaces; then, patching local controllers together results in a global controller.
- **Open question:** How to plan for complex objectives over this “patchified” topological space?

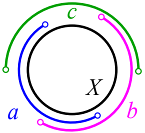
Recap: Linear temporal logic (LTL)

- LTL formulae can include two temporal operators, next ($\bigcirc$) and until (U), and any recursive combinations of the operators captured by the syntax

$$\varphi := \mathrm{true} \mid a \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \bigcirc\varphi \mid \varphi_1 \mathsf{U} \varphi_2$$

  where a is a boolean variable.

- Example: $X = \mathbb{S}^1$ with an open cover indexed by $\mathrm{AP} = \{a, b, c\}$.



  - Problem: Plan for $a\mathsf{U}b$.

    $\rightsquigarrow$ *$c \wedge \neg a \wedge \neg b$ is an implicit obstacle for this task*

- Planning and control for LTL objectives can be solved algorithmically on finite-state discrete transition systems.
- **Question.** Assuming holonomic dynamics, how to systematically generalize these methods to spaces that are not necessarily copies of Euclidean space, while avoiding rigid methods such as polyhedral decompositions [6]?

The Nerve Simplicial Complex (SCX) and Good Covers

Let $(X, \mathscr{T})$ be a nice[1] topological space.

- **Indexed Covers** are maps $\mathbb{U} : \mathrm{AP} \to \mathscr{T}$ such that $X = \bigcup_{\alpha \in \mathrm{AP}} \mathbb{U}(\alpha)$.

[1] e.g., $(X, \mathscr{T})$ is completely regular, *II*-countable, connected, and locally contractible.

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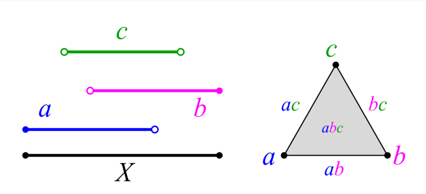
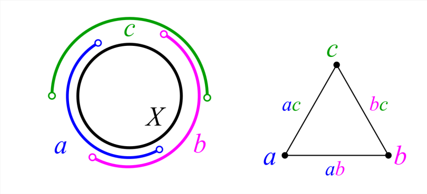
- **The Nerve of** $\mathbb{U}$ is the scx $N(\mathbb{U})$ of all $\mathbb{U}$-consistent sets $\sigma \subset \mathrm{AP}$.





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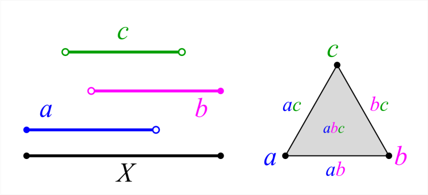

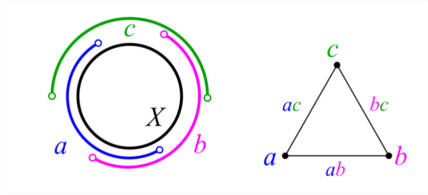

Theorem (Nerve Lemma)

If every $\widetilde{\mathbb{U}}(\sigma)$, $\sigma \in N(\mathbb{U})$ is contractible, then X is homotopy-equivalent to the geometric realization of $N(\mathbb{U})$. An open cover with this property is called a good cover.

[1] e.g., $(X, \mathscr{T})$ is completely regular, *II*-countable, connected, and locally contractible.

The Nerve vs. 2^{AP}: the Shtan'ko-Shtogrin map [7]

- The geometric realization $|N(\mathbb{U})|$ of the nerve is constructed in $\mathbb{R}^{\mathrm{AP}}$, as a union of geometric simplices spanned by the e_α, $\alpha \in \mathrm{AP}$

$$|N(\mathbb{U})| \triangleq \bigcup_{\sigma \in N(\mathbb{U})} \dot{\Delta}^\sigma, \quad \dot{\Delta}^\sigma \triangleq \left\{ \sum_{\alpha\in\sigma} t_\alpha e_\alpha \in \mathbb{R}^{\mathrm{AP}} \colon \sum_{\alpha\in\sigma} t_\alpha = 1, (\forall_{\alpha\in\sigma})(t_\alpha > 0) \right\}$$

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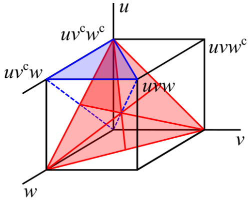
- The nerve is mapped *homeomorphically* into the positive boundary of the unit cube:

$$\square^{\mathrm{AP}} \triangleq [0,1]^{\mathrm{AP}} \subset \mathbb{R}^{\mathrm{AP}}, \quad \square_+^{\mathrm{AP}} \triangleq \{\xi \in \square^{\mathrm{AP}} \colon \exists_\alpha\, \xi(\alpha) = 1\},$$



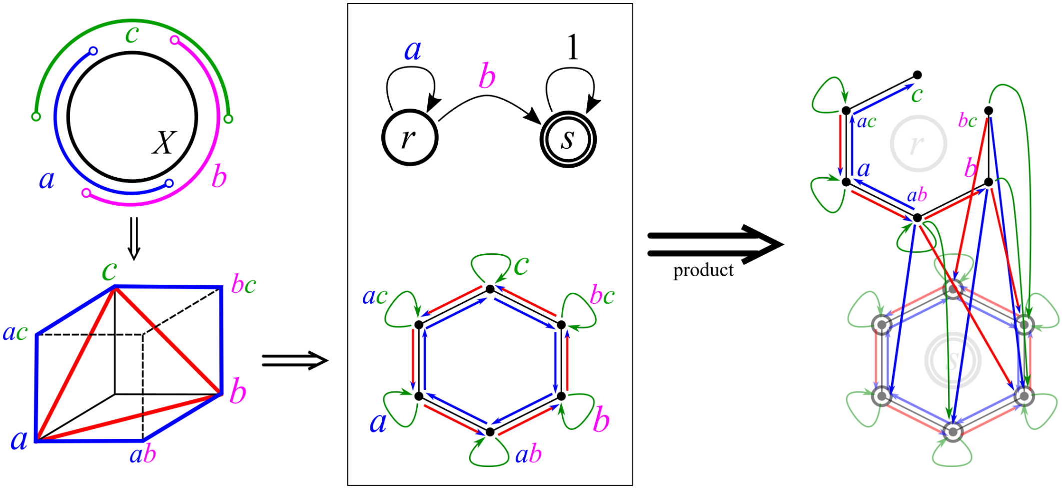
$$c \colon \begin{cases} \Delta^{\mathrm{AP}} & \to \quad \square_+^{\mathrm{AP}} \\ \xi & \mapsto \quad \frac{\xi}{\|\xi\|_\infty}, \end{cases}$$

realizing the natural map of $N(\mathbb{U})$ into $\mathbf{2}^{\mathrm{AP}}$.

$\rightsquigarrow$ *each d-simplex is made of $(d+1)$ d-cubes meeting in its barycenter and creating a 'corner'*

An Example LTL Planning Problem

LTL-based planning in discrete transition systems is done in the product of an appropriate Büchi automaton with the transition system. We solve the problem from the preceding slide:



Realizability as a Challenge to Nerve-Based Planning

- **Main Challenge:** Not all $\sigma \in N(\mathbb{U})$ are witnessed by a point of X.

Definition (Realizability)

For $x \in X$, one has $\sigma(x) \triangleq \{\alpha \in \mathrm{AP} \colon x \in \mathbb{U}(\alpha)\} \in N(\mathbb{U})$. A simplex $\sigma \in N(\mathbb{U})$ is said to be *$\mathbb{U}$-realized*, if $\sigma = \sigma(x)$ for some $x \in X$.

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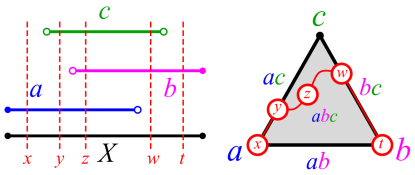
- $\sigma \in N(\mathbb{U})$ is realized $\iff \bigcap_{\alpha \in \sigma} \mathbb{U}(\alpha) \setminus \bigcup_{\beta \in \mathrm{AP} \setminus \sigma} \mathbb{U}(\beta) \neq \varnothing$.



- $\{c\}$ is not realized;
- $\{a, b\}$ is not realized.

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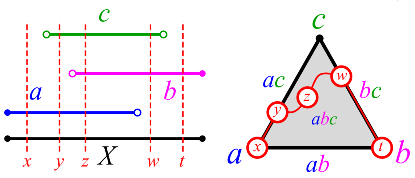
- **Main Challenge:** Not all $\sigma \in N(\mathbb{U})$ are witnessed by a point of X.

Definition (Realizability)

For $x \in X$, one has $\sigma(x) \triangleq \{\alpha \in \mathrm{AP} \colon x \in \mathbb{U}(\alpha)\} \in N(\mathbb{U})$. A simplex $\sigma \in N(\mathbb{U})$ is said to be *$\mathbb{U}$-realized*, if $\sigma = \sigma(x)$ for some $x \in X$.

- $\sigma \in N(\mathbb{U})$ is realized $\iff \bigcap_{\alpha \in \sigma} \mathbb{U}(\alpha) \setminus \bigcup_{\beta \in \mathrm{AP} \setminus \sigma} \mathbb{U}(\beta) \neq \varnothing$.

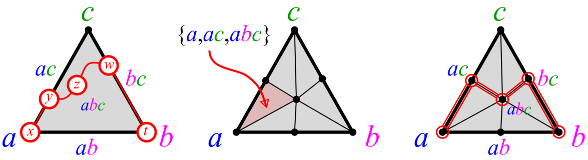


- $\{c\}$ is not realized;
- $\{a, b\}$ is not realized.

- Unrealized simplices are an obstruction to planning using $N(\mathbb{U})$:
  - Not every path in $|N(\mathbb{U})|$ is realizable as a path in X;
  - Homotoping an unrealizable plan to a realizable one may violate task constraints.

The Reduced Nerve

- Recall, if K is a scx, then $\mathrm{Sd}(K)$ is the scx of all $T \subset K$ that are ($\subseteq$)-chains.[2]



There is no way to access a from abc except via ac, so the red simplex of $\mathrm{Sd}(N(\mathbb{U}))$ in the center should not be deemed realizable, yielding a "reduced nerve" as in the diagram on the right (red).

[2] $T \in \mathrm{Sd}(K)$ iff, for all $\sigma, \tau \in T$ one has $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$.

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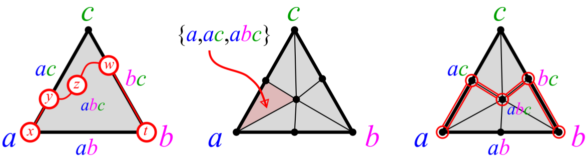

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Definition (Realized Simplex in $\mathrm{Sd}(N(\mathbb{U}))$)

Let $T = \{\sigma_0, \ldots, \sigma_d\}$ be a d-simplex in $\mathrm{Sd}(N(\mathbb{U}))$, where $\sigma_{i-1} \subset \sigma_i$ for all $i = 1, \ldots, d$. T is *realized* if all the σ_i in T are realized, and there exists a continuous map $s : \Delta^T \to X$ such that

$$\sigma\left(s\left(\sum_{i=1}^{d} \xi_i e_{\sigma_i}\right)\right) = \sigma_j \iff \xi_j > 0 \wedge (\forall_{i>j})\ \xi_i = 0,$$

where $(\xi_0, \ldots, \xi_d)$ are the barycentric coordinates on Δ^T.

---

[2] $T \in \mathrm{Sd}(K)$ iff, for all $\sigma, \tau \in T$ one has $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$.

The Reduced Nerve

- We are after the following result (strong form):

Theorem (Nerve Lemma for Reduced Nerve?)

Let $N_{red}(\mathbb{U})$ be the sub-complex of $\mathrm{Sd}(N(\mathbb{U}))$ consisting of its realized simplices. Let $Y = |\mathrm{Sd}(N(\mathbb{U}))|$, $Y_{red} = |N_{red}(\mathbb{U})|$. If $\mathbb{U}$ is a good cover, then Y_{red} is a strong deformation retract of Y — there is a continuous deformation $H : Y \times [0,1] \to Y$ such that:

1. $H(y,t) = y$ *for all* $y \in Y_{red}$*;* $\rightsquigarrow$ *the deformation fixes the real nerve pointwise*
2. $H(y,1) \in Y_{red}$ *for all* $y \in Y$*;* $\rightsquigarrow$ *the target of the deformation is the real nerve*
3. $H(y,0) = y$ *for all* $y \in Y$*.* $\rightsquigarrow$ *the full nerve gets deformed to the real nerve*

In particular, $Y_{red} = |N_{red}(\mathbb{U})|$ *has the homotopy type of* X*.*

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In particular, $Y_{red} = |N_{red}(\mathbb{U})|$ has the homotopy type of X.

- It would suffice to obtain the weaker ("In particular...") statement.
- APPLICATIONS:
  - LTL planning as discussed above: paths in the real nerve form realizable plans in X;
  - Enable the use of computational algebraic topology (CAT) tools in LTL planning;
  - Access to problems with complex combinatorial structure (e.g. coordinated navigation);
  - An avenue for unification with discrete Conley theory [8]?

Thank You!

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[2] D. Guralnik, P. Stiller, F. Zegers, and W. E. Dixon, "Distributed cooperative navigation with communication graph maintenance using single-agent navigation fields," in *Proc. Am. Control Conf.*, June 2022.

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