Two-Player Zero-Sum Hybrid Games

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Hybrid Systems Laboratory

UC SANTA CRUZ | Baskin Engineering

Motivation

Game Theory + Control Theory

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- Multiple players with conflicting interests (noncooperative)

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- Decision making process. Optimization problem with dynamic constraints

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- Decision making process. Optimization problem with dynamic constraints
- Challenges: Both continuous and discrete behavior

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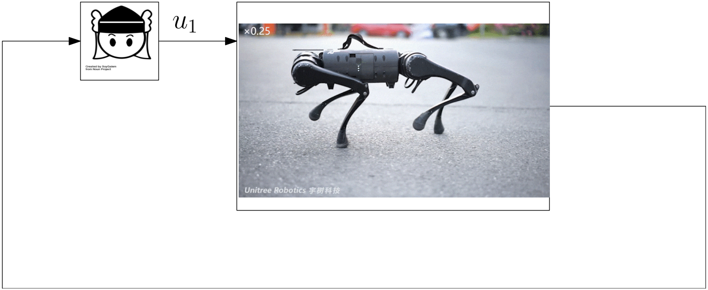

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\text{subject to} \quad & \text{hybrid dynamics}
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$$

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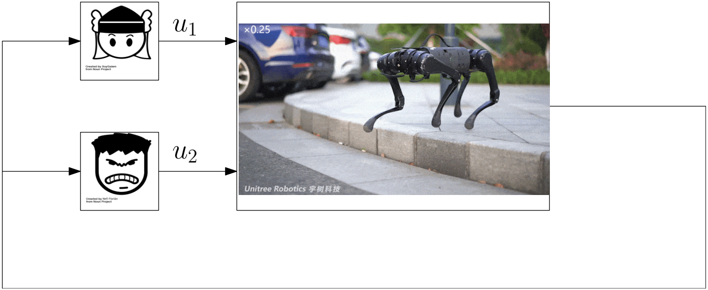

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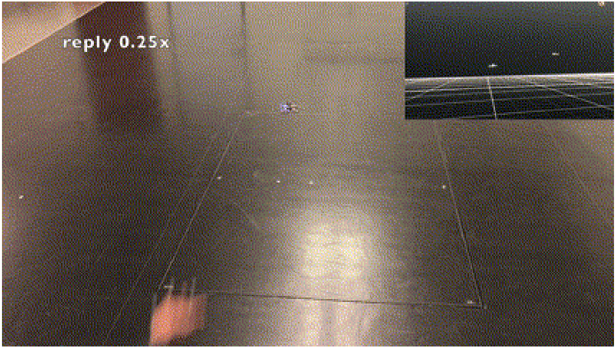

$$\underset{u_1}{\text{minimize}}\ \underset{u_2}{\text{maximize}} \quad \mathcal{J}(u_1, u_2)$$
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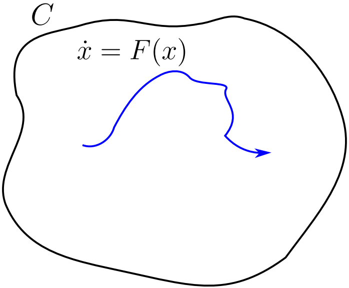
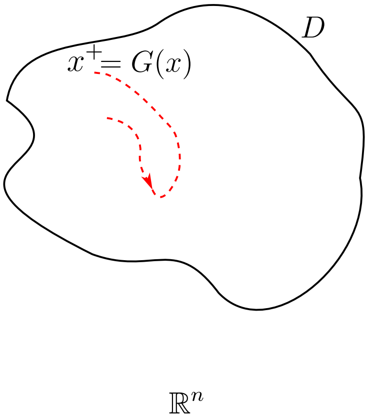
constraints

Modeling Hybrid Dynamics



$\mathbb{R}^n$

Modeling Hybrid Dynamics



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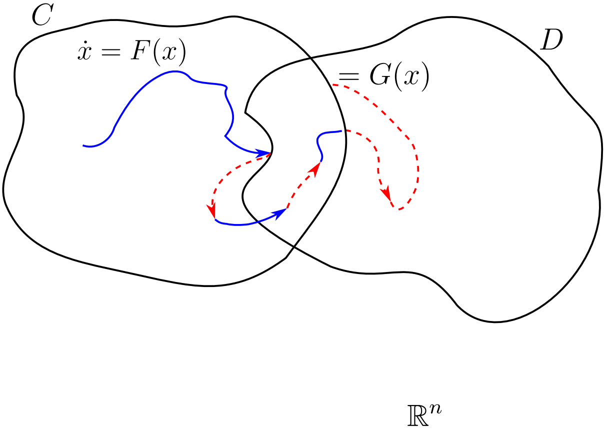

Hybrid Systems with Inputs

A hybrid system $\mathcal{H}$ with state x and input $u = (u_C, u_D)$ as in [Goebel, et.al., PUP 2012]:

$$\mathcal{H} \begin{cases} \dot{x} = F(x, u_C) & (x, u_C) \in C \\ x^+ = G(x, u_D) & (x, u_D) \in D \end{cases}$$

- C is the *flow set*
- F is the *flow map*
- D is the *jump set*
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Solutions parametrized by (t, j):

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$$([0, t_1] \times \{0\}) \cup ([t_1, t_2] \times \{1\}) \cup \ldots,$$

where $t_1 \leq t_2 \leq \ldots$ are the *jump times*.

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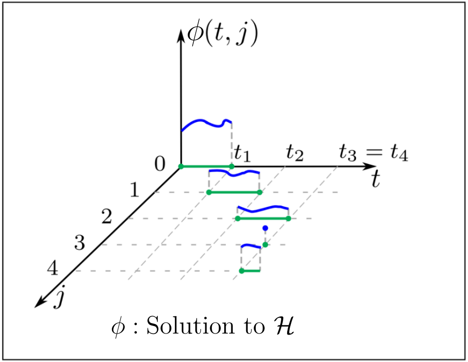

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Connections to Other Frameworks

Switched Systems

$$\begin{aligned} \dot{x} =& f_{\sigma(t)}(x) \\ \sigma(t) \in& \{1, 2, \dots\} \end{aligned}$$

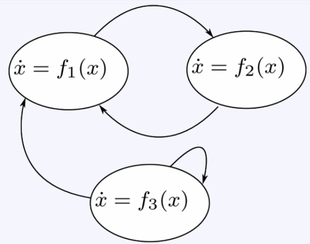
Impulsive Systems

$$\begin{aligned} \dot{x} =& f(x(t)) \\ x(t^+) =& g(x(t)) \quad t \in \{t_1, t_2, \dots\} \end{aligned}$$

Differential-Algebraic Equations

$$\begin{aligned} \dot{x} =& f(x, w) \\ 0 =& \eta(x, w) \end{aligned}$$

Hybrid Automata



Two-Player Zero-Sum Games

Two-player game: $u_C = (u_{C1}, u_{C2})$ and $u_D = (u_{D1}, u_{D2})$

- Player P_1 selects (u_{C1}, u_{D1})
- Player P_2 selects (u_{C2}, u_{D2})
  - $\mathcal{J}$ cost functional associated to the solution to $\mathcal{H}$ from ξ.

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Solve

$$\underset{(u_{C1},u_{D1})}{\text{minimize}}\ \underset{(u_{C2},u_{D2})}{\text{maximize}}\ \mathcal{J}(\xi, u_{C1}, u_{C2}, u_{D1}, u_{D2})$$

over **the set of complete input actions** as a two-player zero-sum hybrid game.

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Robust Control Problem

Find the control input (u_{C1}, u_{D1}) that **upper bounds** $\mathcal{J}$ for a disturbance (u_{C2}, u_{D2}).

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Robust Control Problem

Find the control input (u_{C1}, u_{D1}) that **upper bounds** $\mathcal{J}$ for a disturbance (u_{C2}, u_{D2}).

Security Problem

Ensure the control input (u_{C1}, u_{D1}) minimizes $\mathcal{J}$ under an attack (u_{C2}, u_{D2}) designed to **harm** $\mathcal{H}$ **as much as possible**.

Two-Player Zero-Sum Games

- General framework

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- Sufficient conditions for optimality to evaluate the value function

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- Sufficient conditions for optimality to evaluate the value function
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- Application in a security scenario

Formulation of Two-Player Zero-Sum Hybrid Games

Following the formulation in [Başar and Olsder, SIAM 1999], for each $i \in \{1, 2\}$, the i-th player P_i

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- Dynamics $\mathcal{H}_i$ with data (C_i, F_i, D_i, G_i)
- State $x_i \in \mathbb{R}^{n_i}$
- Hybrid input $u_i = (u_{Ci}, u_{Di}) \in \mathbb{R}^{m_{Ci}} \times \mathbb{R}^{m_{Di}}$
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Elements of a two-player zero-sum hybrid game

1. The state $x = (x_1, x_2) \in \mathbb{R}^n$.
2. The set of joint input actions $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ with elements $u = (u_1, u_2)$.
   Each player selects its action independently from the action of the other player.

cont'd

3. *The dynamics of the game, denoted by $\mathcal{H}$, with data*

$$\begin{aligned}
C &:= C_1 \times C_2 \\
F(x,u_C) &:= (F_1(x,u_C), F_2(x,u_C)) \qquad \forall (x,u_C) \in C \\
D &:= \{(x,u_D) \in \mathbb{R}^n \times \mathbb{R}^{m_D} : (x_i,u_{Di}) \in D_i, i \in \{1,2\}\} \\
G(x,u_D) &:= \{\hat{G}_i(x,u_D) : (x_i,u_{Di}) \in D_i, i \in \{1,2\}\} \qquad \forall (x,u_D) \in D
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where $\hat{G}_1(x,u_D) = (G_1(x,u_D), I_{n_2})$, *and* $\hat{G}_2(x,u_D) = (I_{n_1}, G_2(x,u_D))$.

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4. *The strategy space of the game $K = K_1 \times K_2$. Collection of mappings $\kappa = (\kappa_1, \kappa_2)$. Each $\kappa_i \in K_i$ is said to be a permissible strategy for P_i.*

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5. *The cost associated to P_i, $(\xi, u) \mapsto \mathcal{J}_i(\xi, u)$.*
   Single cost functional $\mathcal{J} = \mathcal{J}_1 = -\mathcal{J}_2$ associated to the **unique** *solution to $\mathcal{H}$ from ξ for u.*

Saddle-Point Equilibrium

Solution of a zero-sum hybrid game [Başar and Olsder, SIAM 1999]

Consider a two-player zero-sum game with dynamics $\mathcal{H}$ and $\mathcal{J}_1 = \mathcal{J}$, $\mathcal{J}_2 = -\mathcal{J}$.

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A strategy $\kappa = (\kappa_1, \kappa_2) \in K$ is **a saddle-point equilibrium** if for each $\xi \in \Pi(\overline{C} \cup D)$, every u^* rendering a maximal response ϕ^* to $\mathcal{H}$ from ξ, with

$$u^* = (u_1^*, u_2^*) = (\kappa_1(\phi^*), \kappa_2(\phi^*))$$

satisfies

$$\mathcal{J}(\xi, (u_1^*, u_2)) \leq \mathcal{J}(\xi, u^*) \leq \mathcal{J}(\xi, (u_1, u_2^*)) \tag{1}$$

for all u_1 and all u_2 that render maximal solutions.

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$\Pi(C)$ denotes the projection of the set C onto $\mathbb{R}^n$.
A equilibrium solution to the zero-sum two-player game is a strategy in K.
A solution to a hybrid system $\mathcal{H}$ is a hybrid arc, and it is maximal if it cannot be extended.

Problem Statement

Consider a two-player zero-sum hybrid game with dynamics $\mathcal{H}$. Given $\xi \in \mathbb{R}^n$ and a joint input action $u = (u_C, u_D) \in \mathcal{U}$ rendering a **unique** maximal complete solution (ϕ, u) to $\mathcal{H}$ from ξ, the cost associated to it

$$\mathcal{J}(\xi, u) := \overbrace{\sum_{j=0}^{\sup_j \operatorname{dom} \phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j)) dt}^{\text{Cost-to-flow}} + \underbrace{\sum_{j=0}^{\sup_j \operatorname{dom} \phi - 1} L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j))}_{\text{Cost-to-jump}} + \underbrace{\limsup_{\substack{t+j \to \infty \\ (t,j) \in \operatorname{dom} \phi}} q(\phi(t,j))}_{\text{Terminal cost}}$$

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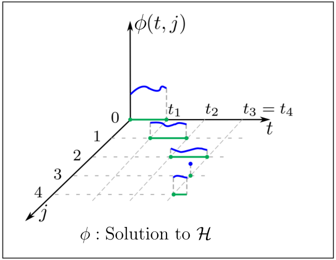

Cost-to-flow

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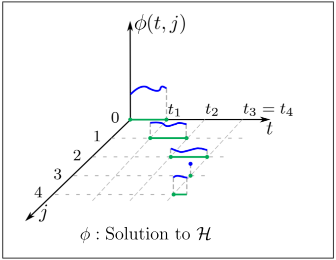

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Cost-to-jump

Terminal cost

Problem ($\diamond$): Given $\xi \in \mathbb{R}^n$, solve

$$\underset{u_1}{\text{minimize}} \ \underset{u_2}{\text{maximize}} \quad \mathcal{J}(\xi, (u_1, u_2))$$

over the set of input actions yielding **complete** solutions to $\mathcal{H}$.

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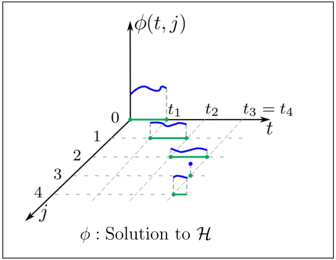

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Cost-to-jump

Terminal cost

Value Function

Given $\xi \in \mathbb{R}^n$, the *value function* at ξ is given by

$$\mathcal{J}^*(\xi) := \min_{u_1} \max_{u_2} \mathcal{J}(\xi, (u_1, u_2)) = \max_{u_2} \min_{u_1} \mathcal{J}(\xi, (u_1, u_2))$$

over the set of joint input actions yielding complete solutions to $\mathcal{H}$

Design of Saddle-Point Equilibrium

Theorem [J. Leudo and Sanfelice, HSCC 2022]

Given a two-player zero-sum hybrid game with

- dynamics $\mathcal{H}$,
- costs L_C, L_D and q,

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\begin{aligned}
0 &= \min_{u_{C1}} \max_{u_{C2}} \{ L_C(x,(u_{C1},u_{C2})) + \langle \nabla V(x), F(x,(u_{C1},u_{C2})) \rangle \} \\
&= \max_{u_{C2}} \min_{u_{C1}} \{ L_C(x,(u_{C1},u_{C2})) + \langle \nabla V(x), F(x,(u_{C1},u_{C2})) \rangle \} \qquad \forall x \in \Pi(C)
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$$= \max_{u_{C2}} \min_{u_{C1}} \{L_C(x,(u_{C1},u_{C2})) + \langle \nabla V(x), F(x,(u_{C1},u_{C2}))\rangle\} \qquad \forall x \in \Pi(C)$$

Optimizer: (u_{C1}^*, u_{C2}^*)

$$0 = \underbrace{L_C(x,(u_{C1}^*,u_{C2}^*))}_{\text{Cost of flowing}} + \underbrace{\nabla V(x) F(x,(u_{C1}^*,u_{C2}^*))}_{\text{Change of } V \text{ along flow}}$$

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$$= \max_{u_{C2}} \min_{u_{C1}} \{L_C(x, (u_{C1}, u_{C2})) + \langle \nabla V(x), F(x, (u_{C1}, u_{C2})) \rangle\} \qquad \forall x \in \Pi(C)$$

$$V(x) = \min_{u_{D1}} \max_{u_{D2}} \{L_D(x, (u_{D1}, u_{D2})) + V(G(x, (u_{D1}, u_{D2})))\}$$
$$= \max_{u_{D2}} \min_{u_{D1}} \{L_D(x, (u_{D1}, u_{D2})) + V(G(x, (u_{D1}, u_{D2})))\} \qquad \forall x \in \Pi(D)$$

(Hamilton-Jacobi-Isaacs hybrid equations)

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if there exists a function V satisfying regularity conditions (see paper) and

Optimizer: (u_{D1}^*, u_{D2}^*)

$$0 = \underbrace{L_D(x, (u_{D1}^*, u_{D2}^*))}_{\text{Cost of jumping}} + \underbrace{V(G(x, (u_{D1}^*, u_{D2}^*))) - V(x)}_{\text{Change of } V \text{ along jump}}$$

$$\begin{aligned} V(x) &= \min_{u_{D1}} \max_{u_{D2}} \{L_D(x, (u_{D1}, u_{D2})) + V(G(x, (u_{D1}, u_{D2})))\} \\ &= \max_{u_{D2}} \min_{u_{D1}} \{L_D(x, (u_{D1}, u_{D2})) + V(G(x, (u_{D1}, u_{D2})))\} \qquad \forall x \in \Pi(D) \end{aligned}$$

(Hamilton-Jacobi-Isaacs hybrid equations)

Design of Saddle-Point Equilibrium

cont'd

and each complete solution (ϕ, u) satisfies

$$\limsup_{\substack{t+j\to\infty \\ (t,j)\in \text{dom}\,\phi}} V(\phi(t,j)) = \limsup_{\substack{t+j\to\infty \\ (t,j)\in \text{dom}\,\phi}} q(\phi(t,j)) \quad (2)$$

then

Design of Saddle-Point Equilibrium

cont'd

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then

$$\mathcal{J}^*(\xi) = V(\xi) \qquad \forall \xi \in \Pi(\overline{C} \cup D), \tag{3}$$

Design of Saddle-Point Equilibrium

cont'd

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and any stationary feedback law $\kappa := (\kappa_C, \kappa_D)$ with values

$$\kappa_C(x) \in \arg\min_{u_{C1}} \max_{u_{C2}} \{L_C(x, u_C) + \langle \nabla V(x), F(x, u_C)\rangle\} \qquad \forall x \in \Pi(C)$$

$$\kappa_D(x) \in \arg\min_{u_{D1}} \max_{u_{D2}} \{L_D(x, u_D) + V(G(x, u_D))\} \quad \forall x \in \Pi(D)$$

is a **pure strategy saddle-point equilibrium** for the two-player infinite-horizon hybrid game with $\mathcal{J}_1 = \mathcal{J}$, $\mathcal{J}_2 = -\mathcal{J}$.

Example: Security of Juggling System

Goal: Optimally stabilize a bouncing ball actuated at jumps under attacks

$$
\begin{cases}
(\dot{x}_1, \dot{x}_2) = (x_2, -1) & x_1 \geq 0 \\
(x_1^+, x_2^+) = (0, \lambda x_2 + u_{D1} + u_{D2}) & x_1 = 0 \text{ and } x_2 \leq 0
\end{cases}
$$

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- x_1 height of the ball
- x_2 velocity of the ball
- $\lambda \in [0, 1)$ coefficient of restitution
- u

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- $\lambda \in [0, 1)$ coefficient of restitution
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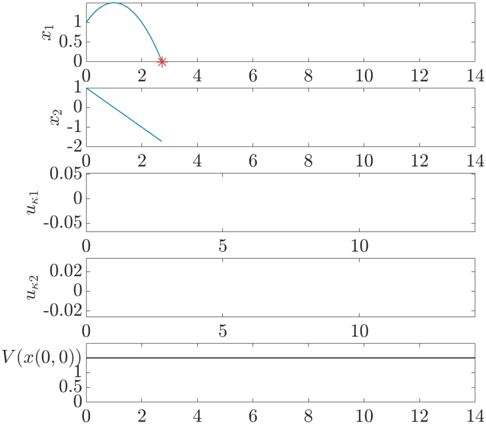

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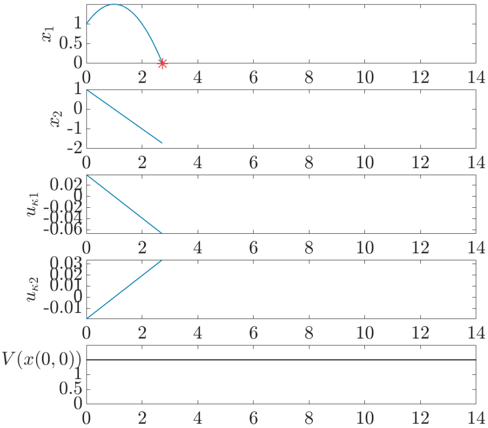

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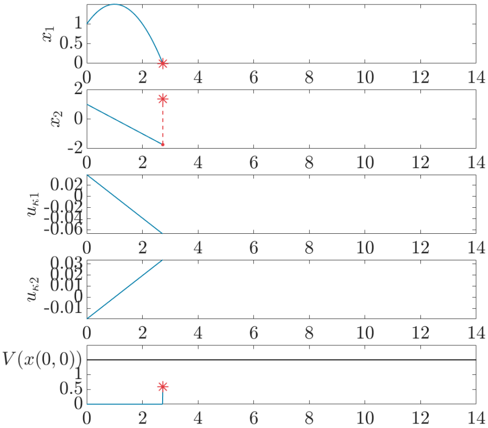

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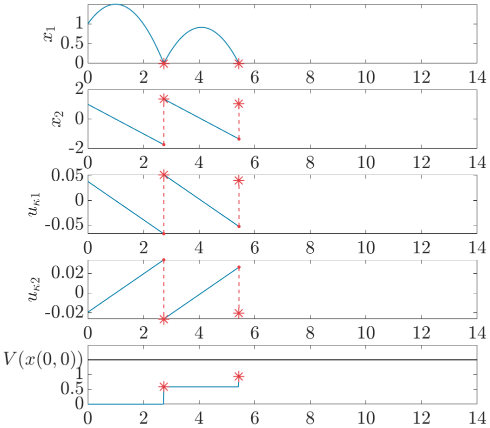

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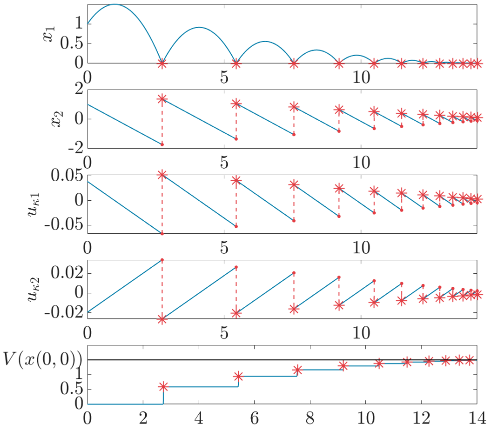

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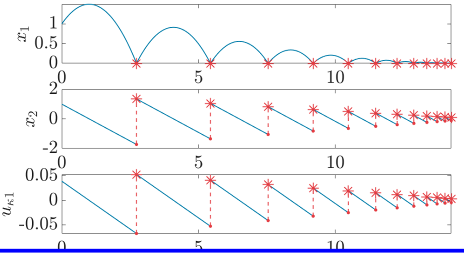

Under the worst-case attack at jumps, Player P_1 selects the strategy that minimizes the energy and regulates the ball as time increases. The optimal cost is computed by evaluating the function V at the initial state.

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Security Problem as a Zero-Sum Game

- u_{D1}: P_1 minimizes a cost functional $\mathcal{J}$
- u_{D2}: the worst-case attack by P_2
- No cost to flow $L_C(x, u_C) := 0$
- $L_D(x, u_D) := x_2^2 Q_D + u_D^\top R_D u_D$
- Terminal cost $q(x) := \frac{1}{2}x_2^2 + x_1$

$V(x) := \frac{1}{2}x_2^2 + x_1$ solves HJI hybrid equations.

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Using our Theorem, the value function at $\xi = (\xi_1, \xi_2)$ is

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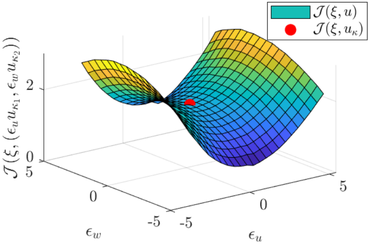
and attained by

$$\kappa_{D1}(x) = \frac{R_{D2}\lambda}{R_{D1} + R_{D2} + 2R_{D1}R_{D2}} x_2$$

$$\kappa_{D2}(x) = \frac{R_{D1}\lambda}{R_{D1} + R_{D2} + 2R_{D1}R_{D2}} x_2$$

then, κ_D is the saddle-point equilibrium.



Conclusion

- General framework to model hybrid games
- Sufficient conditions for optimality to evaluate value function
- Sufficient conditions to attain saddle-point equilibrium
- Application in security scenario

- S. J. Leudo, K. Garg, R.G. Sanfelice, A. Cardenas. **An Observer-based Switching Algorithm for Safety under Sensor Denial-of-Service Attacks**, to appear in the 2023 American Control Conference.
- S. J. Leudo, and R.G. Sanfelice. **Sufficient Conditions for Optimality in Finite-Horizon Two-Player Zero-Sum Hybrid Games**, 2022 IEEE Conference on Decision and Control, December 2022.
- S.J. Leudo, and R.G. Sanfelice. **Sufficient Conditions for Optimality and Asymptotic Stability in Two-Player Zero-Sum Hybrid Games**, the ACM International Conference on Hybrid Systems: Computation and Control, 2022.
- S.J. Leudo, F. Ferrante, and R.G. Sanfelice. **Upper Bounds and Cost Evaluation in Dynamic Two-player Zero-Sum Games**, IEEE Conference on Decision and Control, December, 2020.

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