Generalizing Formation Chain Framework with Insights from Screw Theory

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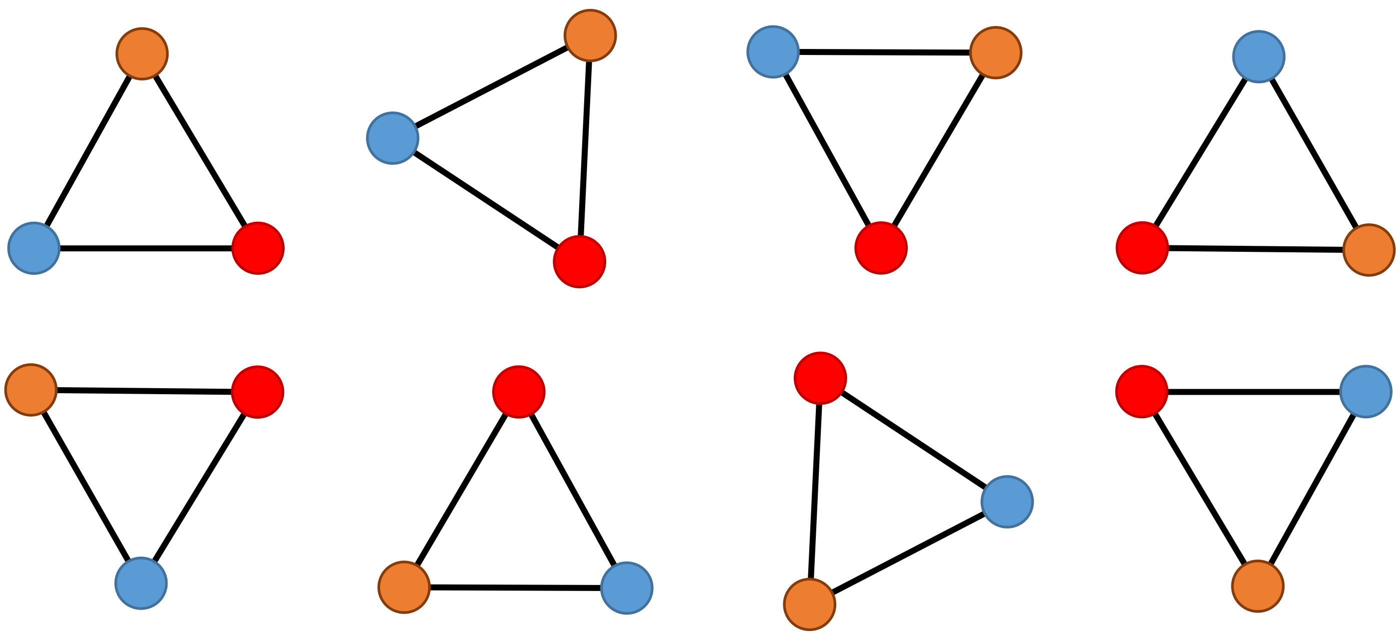
Overview

- We are seeking to solve the Swarm Initialization Problem (SIP) in the special case of a **circular swarm trajectory** (eccentricity = 0, exact).
- **Quantization of the orbit into evenly-spaced 'checkpoints'**
  - Same number of checkpoints as satellites.
  - Assume no checkpoint is 'special'; i.e., re-orienting global coordinates to set any checkpoint as 'initial' results in an identical problem.
- **Construction of formation chains** based on geometry alone.
  - Consider distinct modes of translation and rotation.
  - Utilization of homogeneous coordinates consistent with screw theory.
  - Leads to quantification of geometric swarm motion.

Swarm-Preserving Operations

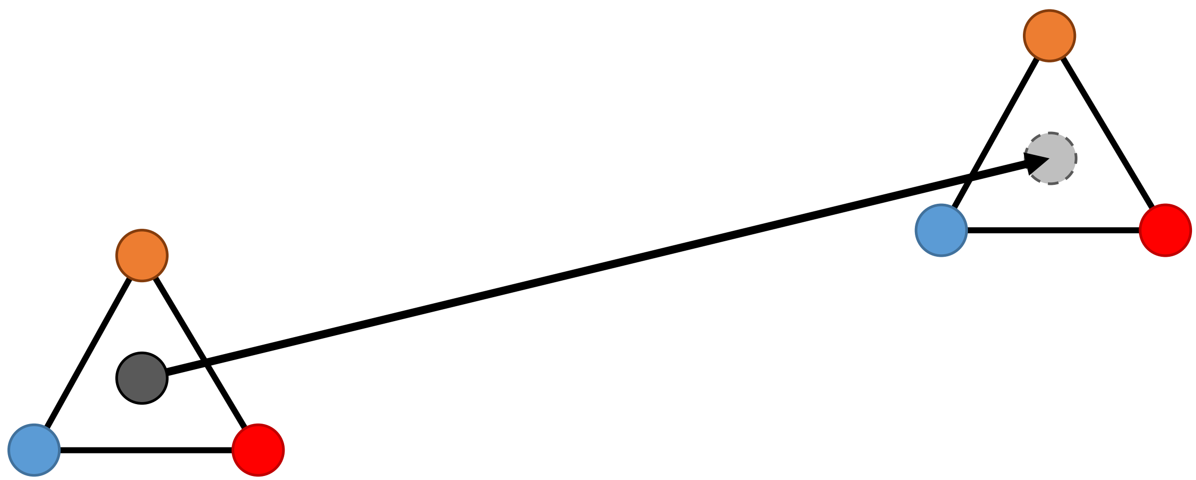
Swarm-Preserving Operations

- Invariance of swarm optimality under rotation of the swarm within a known space of valid rotations:
  - Example: Planar formations rotated about the normal of their plane.



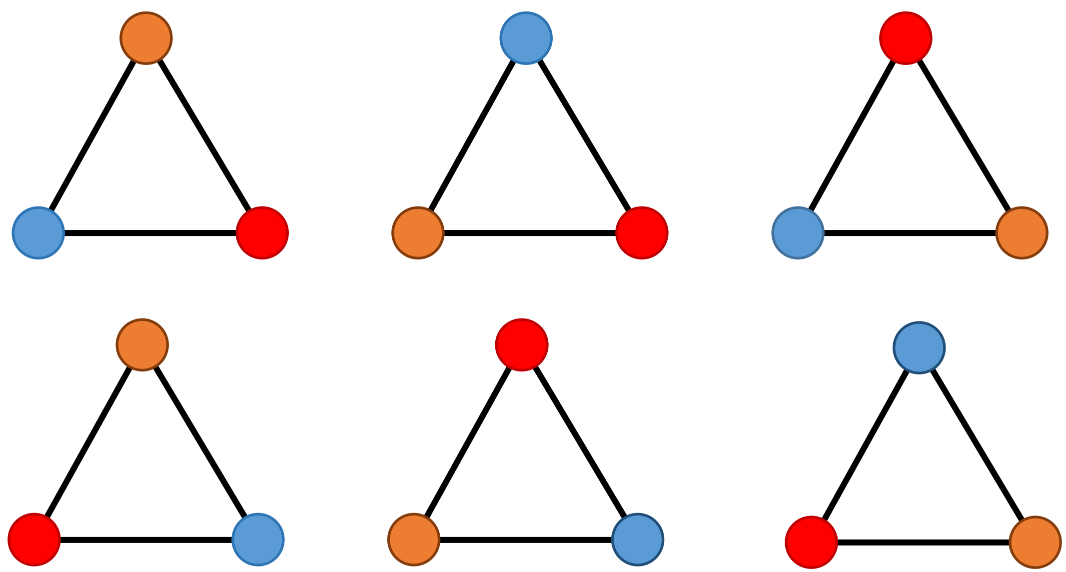
Swarm-Preserving Operations

- Invariance of swarm optimality under translation of the swarm within a known space of valid translations.
  - Complicated by requirement that no one checkpoint is ‘special’.
    - Supports three ‘modes’ of translation:
      1. Globally fixed displacement applied to all checkpoints.
      2. Rotationally fixed displacement.
      3. Swarm fixed displacement normal to axis of swarm rotation.



Swarm-Preserving Operations

- Invariance of swarm optimality under transposition, or re-labeling of any two satellites.



Index Chains

Index Chains

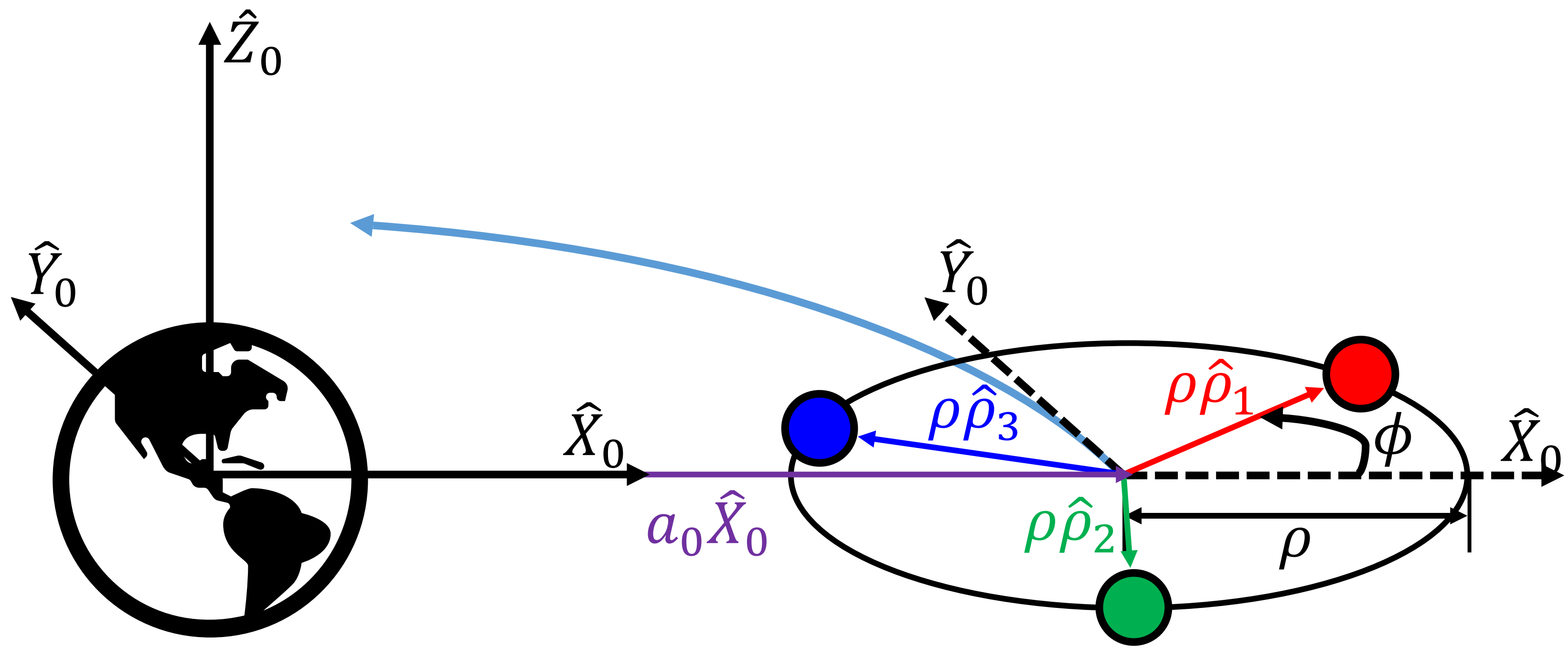
- A full index chain for each satellite is fully defined by the first two checkpoints.
  - Results in n^2 index chains for n satellites.
  - Valid assignments must have unique row and column indices.

Index Chain for 3 Satellites		Moves to Position Index		
		A	B	C
Satellite Index	1 (at A)	A, A, A	A, B, C	A, C, B
	2 (at B)	B, A, C	B, B, B	B, C, A
	3 (at C)	C, A, B	C, B, A	C, C, C

Modes of Rotation

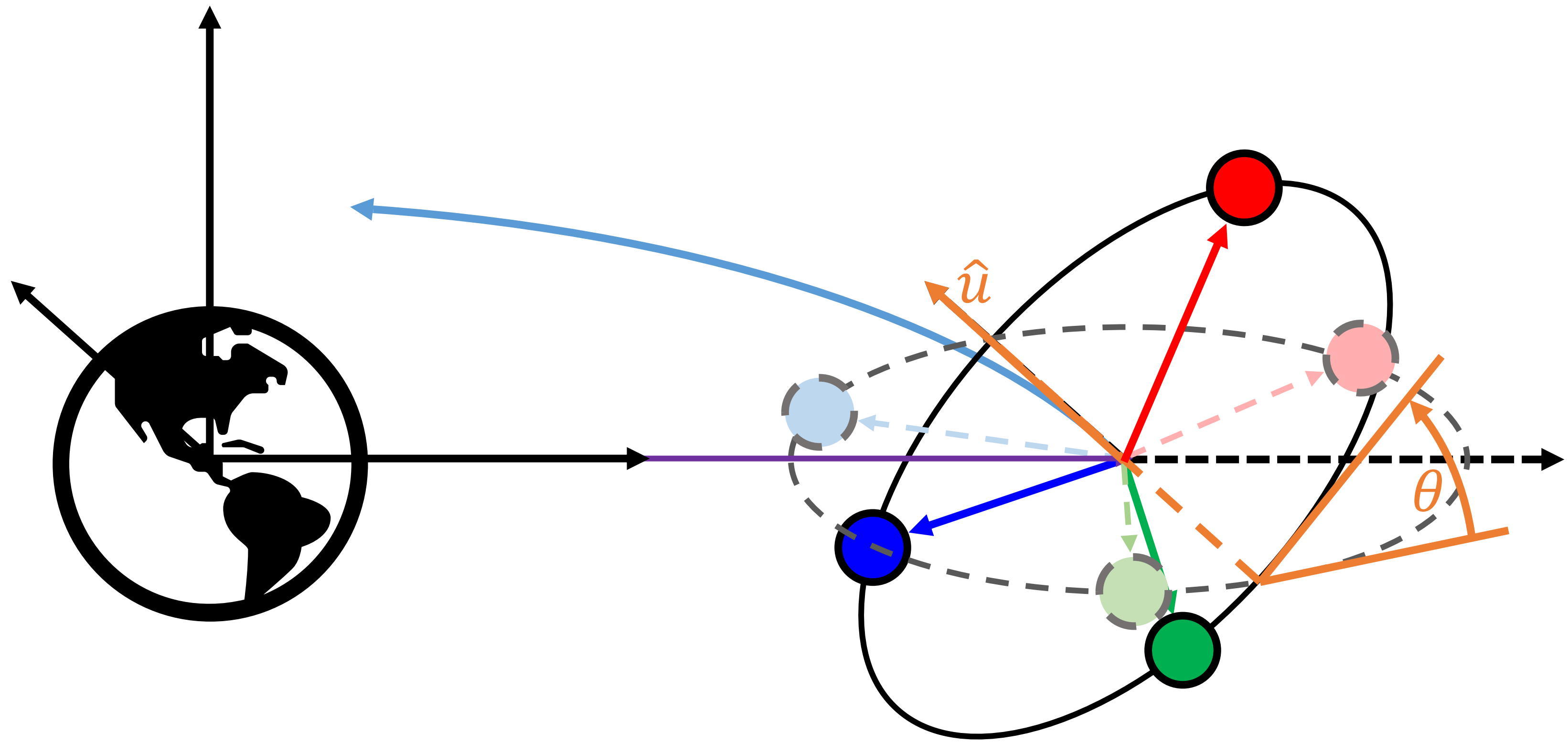
Modes of Rotation

- Fixed parameters of the chain cost function:



Modes of Rotation

- Geometric variables incorporated in the chain cost function:



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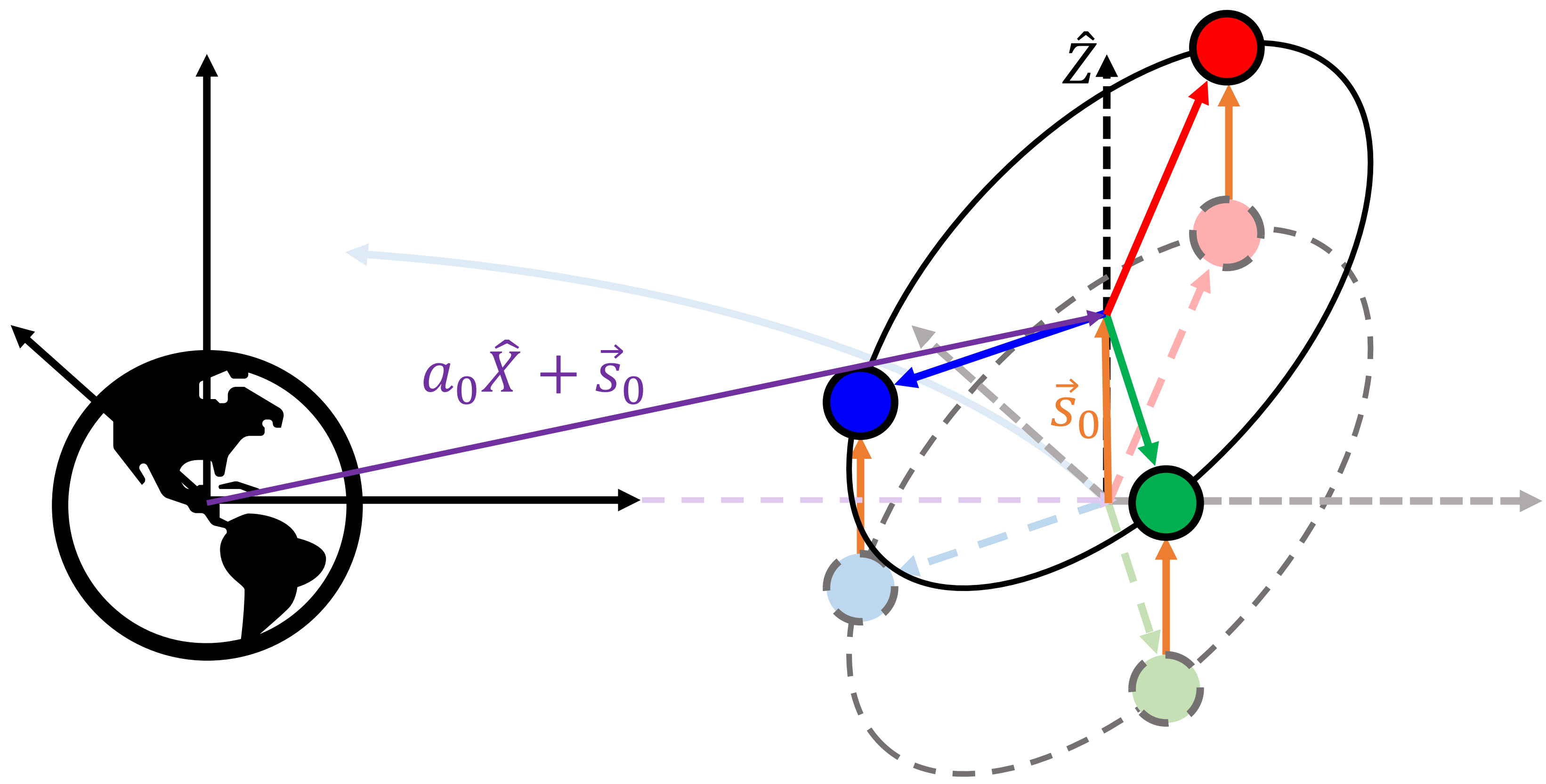

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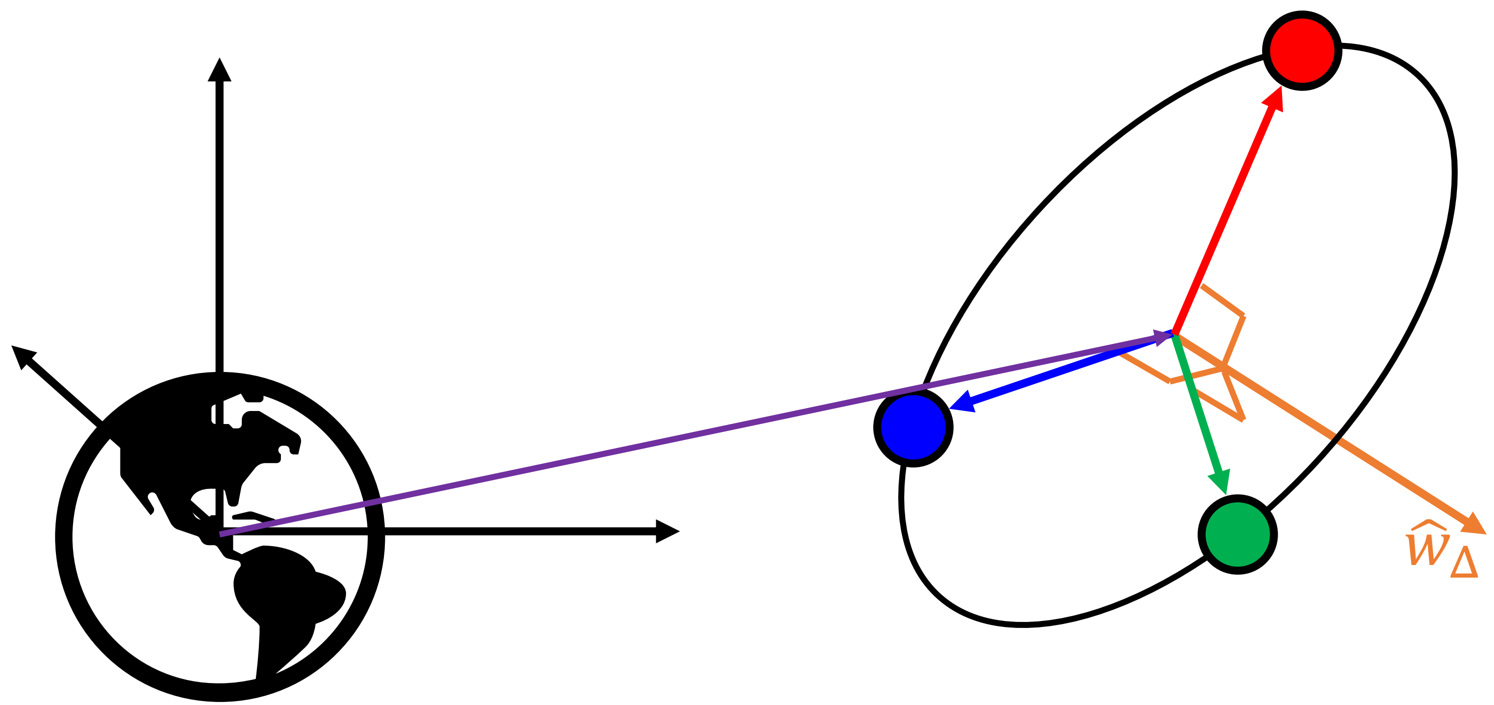

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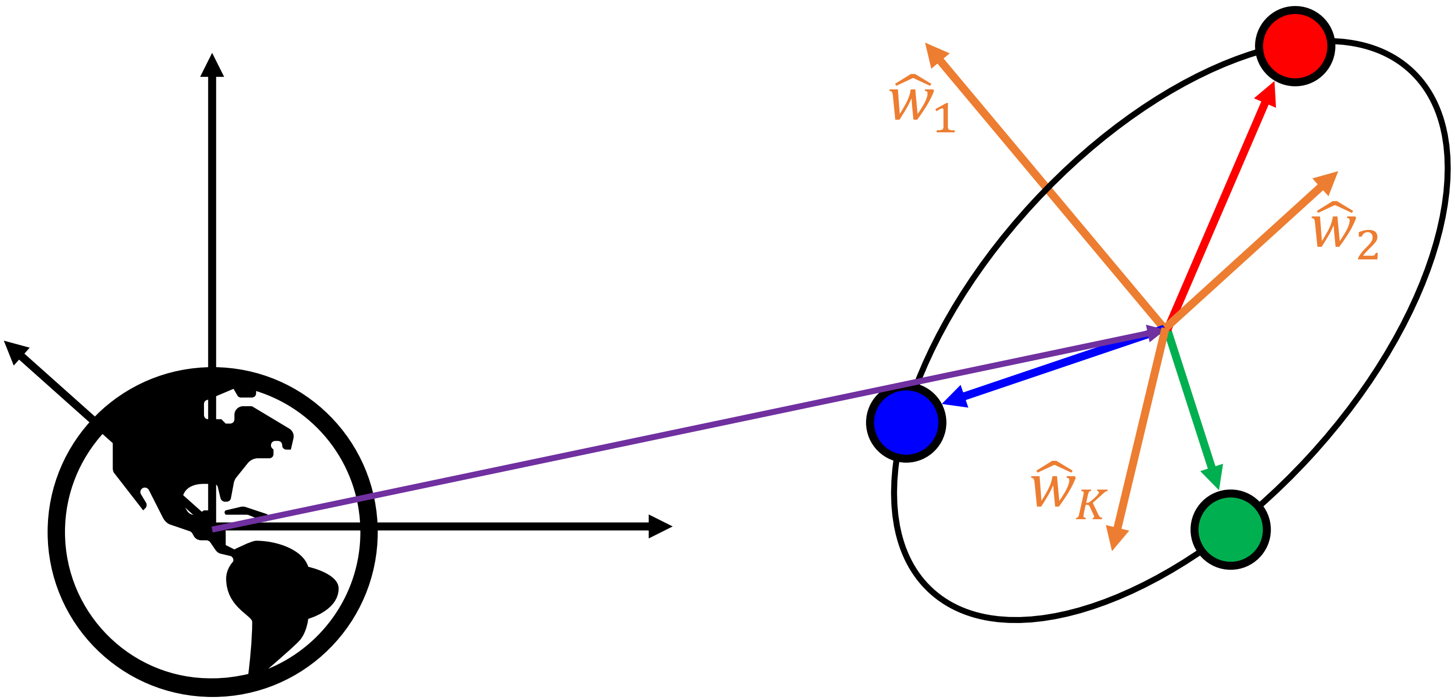
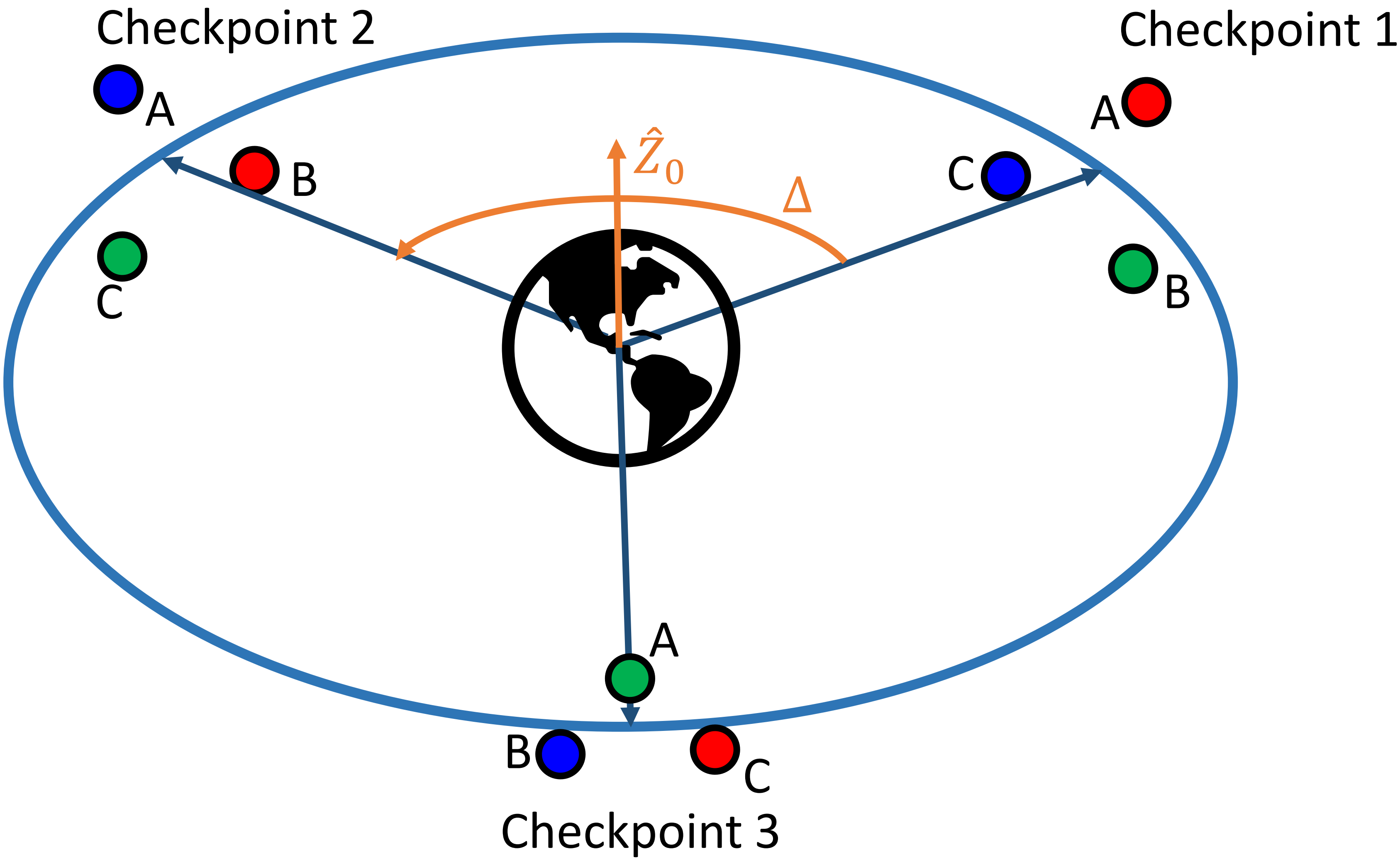
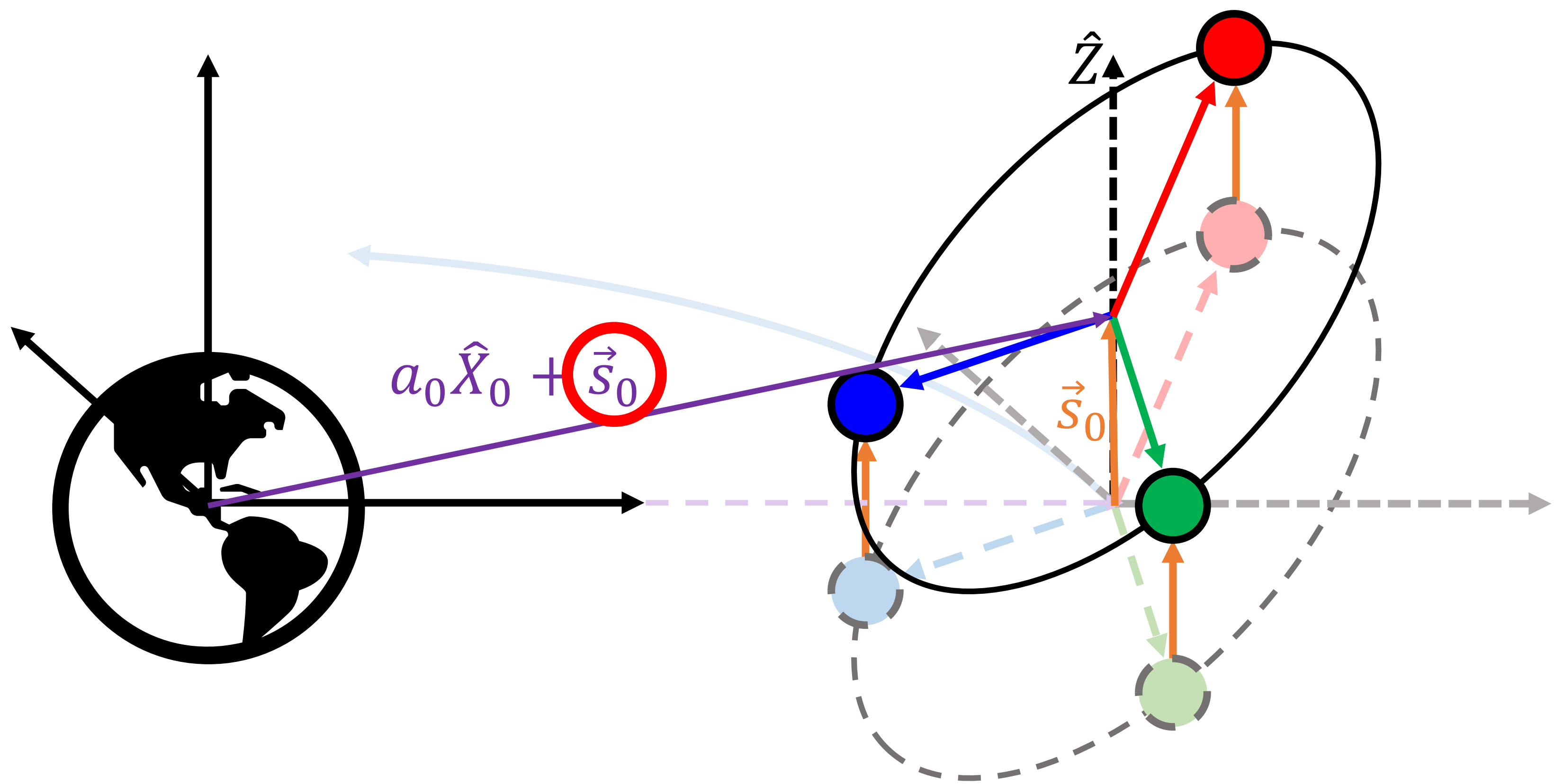


Modes of Rotation



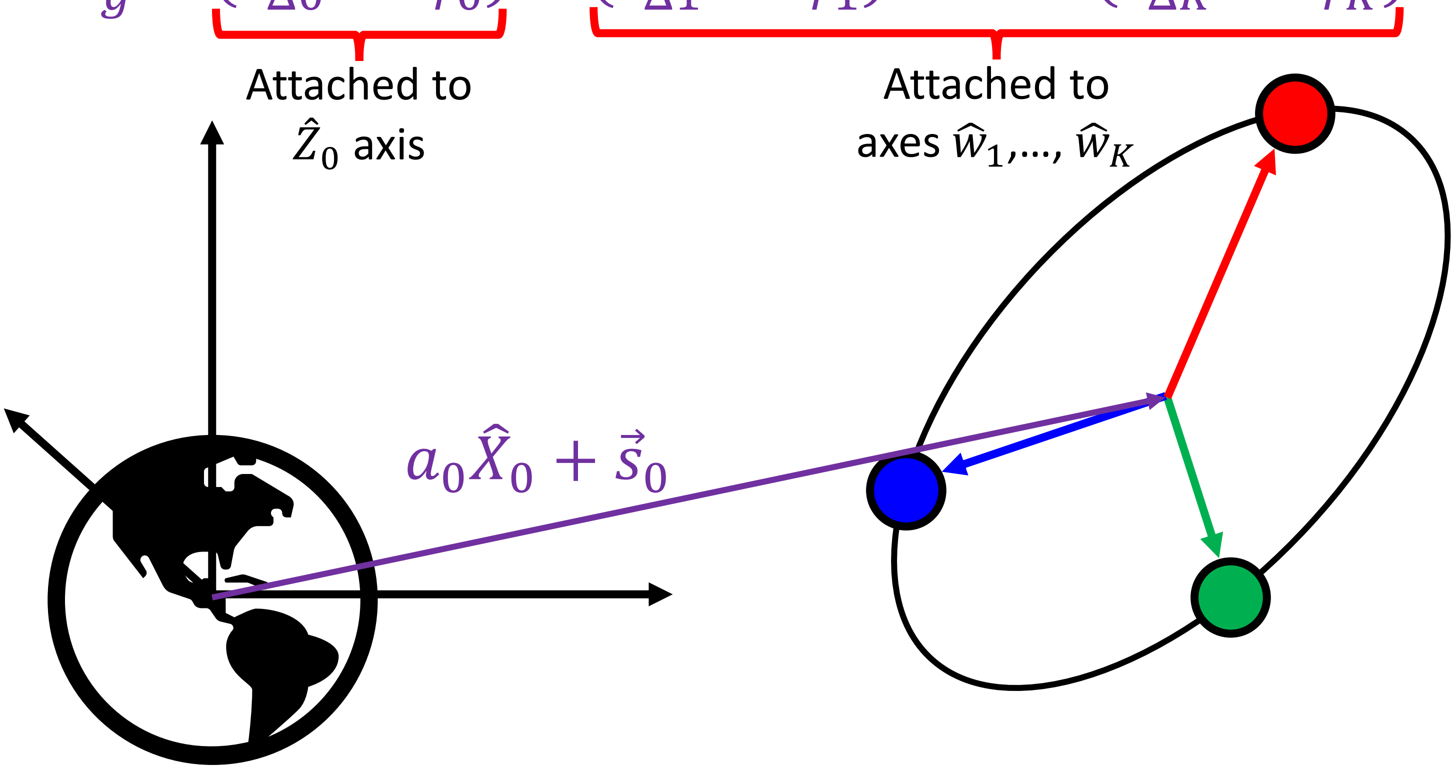
Modes of Translation

- Geometric variables incorporated in the chain cost function:



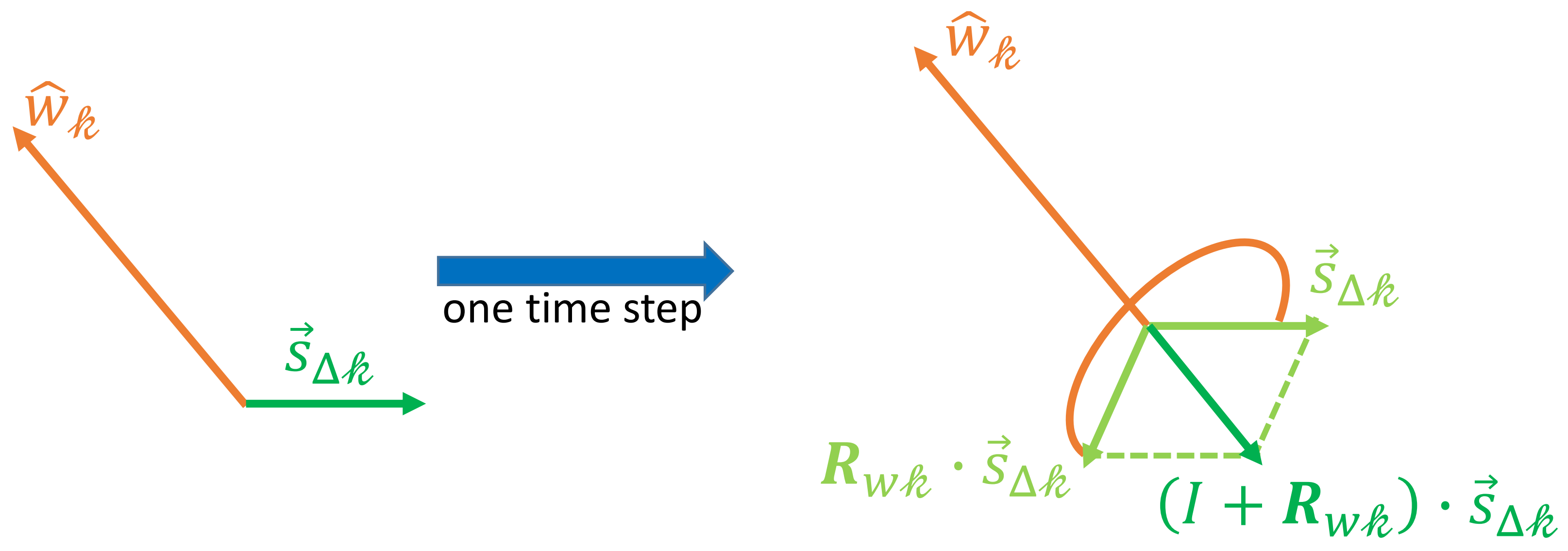
Modes of Translation

$$\vec{s}_0 = \vec{s}_g + (\vec{s}_{\Delta 0} + \vec{s}_{r0}) + (\vec{s}_{\Delta 1} + \vec{s}_{r1}) + \cdots + (\vec{s}_{\Delta K} + \vec{s}_{rK})$$



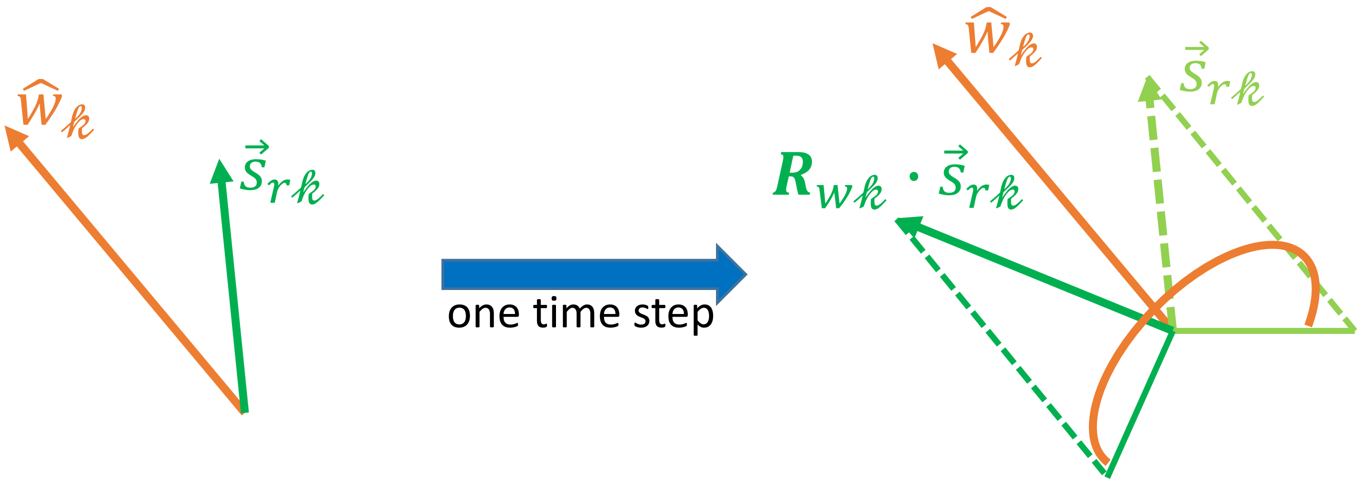
Modes of Translation

- For $\mathcal{k} \in \{1, \ldots, K\}$, we have $\vec{s}_{\Delta\mathcal{k}}$ and $\vec{s}_{r\mathcal{k}}$, where
  - $\vec{s}_{\Delta\mathcal{k}}$ is incremental – that is, at each time step, a rotated value is added to the overall displacement,
  - $\vec{s}_{r\mathcal{k}}$ is fixed to $\hat{w}_{\mathcal{k}}$ in such a way that it rotates by some integer multiple of Δ with each time step.



Modes of Translation

- For $k \in \{1, \ldots, K\}$, we have $\vec{s}_{\Delta k}$ and $\vec{s}_{rk}$, where
  - $\vec{s}_{\Delta k}$ is incremental – that is, at each time step, a rotated value is added to the overall displacement,
  - $\vec{s}_{rk}$ is fixed to $\hat{w}_k$ in such a way that it rotates by some integer multiple of Δ with each time step.



The Formation Chain Function

The Formation Chain Function

Let us consider each mode of translation separately.

- The $\mathscr{k}$th mode of displacement at time step 0 is

$$\vec{s}_{\mathscr{k},0} = \vec{s}_{r\mathscr{k}} + \vec{s}_{\Delta\mathscr{k}}.$$

Per the previous two slides, the displacement at time step 1 is

$$\vec{s}_{\mathscr{k},1} = \boldsymbol{R}_{w\mathscr{k}} \cdot \vec{s}_{r\mathscr{k}} + (\boldsymbol{I} + \boldsymbol{R}_{w\mathscr{k}}) \cdot \vec{s}_{\Delta\mathscr{k}}.$$

Rearranging, we obtain

$$\begin{aligned} \vec{s}_{\mathscr{k},1} &= \boldsymbol{R}_{w\mathscr{k}} \cdot (\vec{s}_{r\mathscr{k}} + \vec{s}_{\Delta\mathscr{k}}) + \vec{s}_{\Delta\mathscr{k}} \\ &= \boldsymbol{R}_{w\mathscr{k}} \cdot \vec{s}_{\mathscr{k},0} + \vec{s}_{\Delta\mathscr{k}} \end{aligned}$$

We see that $\vec{s}_{\mathscr{k},0}$ is transformed by a mode of rotation *and* a mode of translation, as described by screw theory.

  - For the sake of compactification, we take $\boldsymbol{R}_{w0} \equiv \boldsymbol{R}_{Z0}$.

The Formation Chain Function

Consequently, we investigate the position of the swarm centroid

- The position of the swarm centroid at time step 0 is

$$\vec{r}^{*}_{0,0} = a_0 \hat{X}_0 + \vec{s}_g + \vec{s}_{0,0} + \vec{s}_{1,0} + \vec{s}_{2,0} + \cdots + \vec{s}_{K,0}.$$

Consequently, the displacement at time step 1 is

$$\vec{r}^{*}_{0,1} = \boldsymbol{R}_{Z0} \cdot a_0 \hat{X}_0 + \vec{s}_g + \vec{s}_{0,1} + \vec{s}_{1,1} + \vec{s}_{2,1} + \cdots + \vec{s}_{K,1}.$$

It can be shown that

$$\vec{r}^{*}_{0,1} = \boldsymbol{R}_{Z0} \cdot \vec{r}^{*}_{0,0} + \sum_{k=1}^{K} \left[(\boldsymbol{R}_{wk} - \boldsymbol{R}_{Z0}) \cdot \vec{s}_{k,0} \right] + \vec{s}_{\Delta}$$

where

$$\vec{s}_{\Delta} = (\boldsymbol{I} - \boldsymbol{R}_{Z0}) \cdot \vec{s}_g + \sum_{k=0}^{K} \vec{s}_{\Delta k}$$

The Formation Chain Function

Note that the dependence on $\vec{s}_{0,0}$ drops out, so we may summarize the propagation of all displacement modes in matrix form as such

$$\begin{bmatrix} \vec{r}^{\,*}_{0,1} \\ \vec{s}_{1,1} \\ \vdots \\ \vec{s}_{K,1} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{Z0} & \boldsymbol{R}_{w1} - \boldsymbol{R}_{Z0} & \cdots & \boldsymbol{R}_{wK} - \boldsymbol{R}_{Z0} & \vec{s}_{\Delta} \\ & \boldsymbol{R}_{w1} & & & \vec{s}_{\Delta 1} \\ & & \ddots & & \vdots \\ & & & \boldsymbol{R}_{wK} & \vec{s}_{\Delta K} \\ & & & & 1 \end{bmatrix} \begin{bmatrix} \vec{r}^{\,*}_{0,0} \\ \vec{s}_{1,0} \\ \vdots \\ \vec{s}_{K,0} \\ 1 \end{bmatrix}$$

With suitable definitions, we may express this as

$$\begin{bmatrix} \vec{r}^{\,*}_{0,1} \\ \boldsymbol{S}_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{Z0} & \boldsymbol{D}_W & \vec{s}_{\Delta} \\ & \boldsymbol{R}_W & \boldsymbol{S}_{\Delta} \\ & & 1 \end{bmatrix} \begin{bmatrix} \vec{r}^{\,*}_{0,0} \\ \boldsymbol{S}_0 \\ 1 \end{bmatrix}$$

where

$$\boldsymbol{D}_W = \boldsymbol{\Sigma}_K \cdot \boldsymbol{R}_W - \boldsymbol{R}_{Z0} \cdot \boldsymbol{\Sigma}_K, \qquad \boldsymbol{\Sigma}_P = \underbrace{\begin{bmatrix} \boldsymbol{I} & \cdots & \boldsymbol{I} \end{bmatrix}}_{\text{blockwise } 1\times P}$$

$$\boldsymbol{R}_W = \begin{bmatrix} \boldsymbol{R}_{w1} & & \\ & \ddots & \\ & & \boldsymbol{R}_{wK} \end{bmatrix}, \qquad \boldsymbol{S}_\Delta = \begin{bmatrix} \vec{s}_{\Delta,1} \\ \vdots \\ \vec{s}_{\Delta,K} \end{bmatrix}, \qquad \boldsymbol{S}_j = \begin{bmatrix} \vec{s}_{1,j} \\ \vdots \\ \vec{s}_{K,j} \end{bmatrix}$$

Because the relationship between time steps is consistent, we may state generally that

$$\begin{bmatrix} \vec{r}^{\,*}_{0,j} \\ \boldsymbol{S}_j \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{Z0} & \boldsymbol{D}_W & \vec{s}_\Delta \\ & \boldsymbol{R}_W & \boldsymbol{S}_\Delta \\ & & 1 \end{bmatrix} \begin{bmatrix} \vec{r}^{\,*}_{0,j-1} \\ \boldsymbol{S}_{j-1} \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} \vec{r}^{\,*}_{0,0} \\ \boldsymbol{S}_0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_0 \hat{X}_0 + \vec{s}_0 \\ \boldsymbol{S}_0 \\ 1 \end{bmatrix}$$

Lastly, we determine the evolution of the satellite positions relative to the swarm centroid. Consider that

$$\vec{r}^{\,*}_{i,j} = \vec{\rho}^{\,*}_{i,j} + \vec{r}^{\,*}_{0,j}$$

where

$$\vec{\rho}^{\,*}_{i,j} = \boldsymbol{R}_{\Delta} \cdot \vec{\rho}^{\,*}_{i,j-1},$$

and

$$\vec{r}^{\,*}_{0,j} = \boldsymbol{R}_{Z0} \cdot \vec{r}^{\,*}_{0,j-1} + \boldsymbol{D}_{W} \cdot \boldsymbol{S}_{j-1} + \vec{s}_{\Delta}.$$

Since

$$\vec{\rho}^{\,*}_{i,j-1} = \vec{r}^{\,*}_{i,j-1} - \vec{r}^{\,*}_{0,j-1},$$

it follows that

$$\vec{r}^{\,*}_{i,j} = \boldsymbol{R}_{\Delta} \cdot \vec{r}^{\,*}_{i,j-1} + (\boldsymbol{R}_{Z0} - \boldsymbol{R}_{\Delta}) \cdot \vec{r}^{\,*}_{0,j-1} + \boldsymbol{D}_{W} \cdot \boldsymbol{S}_{j-1} + \vec{s}_{\Delta}.$$

Finally, let us define

$$\boldsymbol{c}_{Z0} = \boldsymbol{\Sigma}_n^T \cdot (\boldsymbol{R}_{Z0} - \boldsymbol{R}_\Delta), \qquad \boldsymbol{c}_W = \boldsymbol{\Sigma}_n^T \cdot \boldsymbol{D}_W, \qquad \boldsymbol{c}_\Delta = \boldsymbol{\Sigma}_n^T \cdot \vec{s}_\Delta$$

$$\mathcal{D}_n(\boldsymbol{R}_\Delta) = \underbrace{\begin{bmatrix} \boldsymbol{R}_\Delta & & \\ & \ddots & \\ & & \boldsymbol{R}_\Delta \end{bmatrix}}_{\text{blockwise } n\times n}, \qquad \boldsymbol{r}_j^* = \begin{bmatrix} \vec{r}_{1,j}^{\,*} \\ \vdots \\ \vec{r}_{n,j}^{\,*} \end{bmatrix}, \qquad \boldsymbol{\rho}_0^* = \begin{bmatrix} \vec{\rho}_1^{\,*} \\ \vdots \\ \vec{\rho}_n^{\,*} \end{bmatrix}$$

We at last obtain

$$\begin{bmatrix} \boldsymbol{r}_j^* \\ \vec{r}_{0,j}^{\,*} \\ \boldsymbol{S}_j \\ 1 \end{bmatrix} = \begin{bmatrix} \mathcal{D}_n(\boldsymbol{R}_\Delta) & \boldsymbol{c}_{Z0} & \boldsymbol{c}_W & \boldsymbol{c}_\Delta \\ & \boldsymbol{R}_{Z0} & \boldsymbol{D}_W & \vec{s}_\Delta \\ & & \boldsymbol{R}_W & \boldsymbol{S}_\Delta \\ & & & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_{j-1}^* \\ \vec{r}_{0,j-1}^{\,*} \\ \boldsymbol{S}_{j-1} \\ 1 \end{bmatrix}$$

where $\boldsymbol{r}_0^* = \mathcal{D}_n(\boldsymbol{R}_u) \cdot \boldsymbol{\rho}_0^* + \boldsymbol{\Sigma}_n^T \cdot \vec{r}_{0,0}^{\,*}$.

The Formation Chain Function

To account for the effect of re-indexing, we introduce the assignment matrix $\boldsymbol{A}_{\mathscr{p}}$, where $\mathscr{p}$ is an index to distinguish different assignment matrices from one another.

- Assignment applies to the first row only.
  - Columns 3, 4 and 5 all contain the same element, so that

$$\boldsymbol{A}_{\mathscr{p}} \cdot \boldsymbol{c}_{Z0} = \boldsymbol{c}_{Z0}, \qquad \boldsymbol{A}_{\mathscr{p}} \cdot \boldsymbol{c}_{W} = \boldsymbol{c}_{W}, \qquad \boldsymbol{A}_{\mathscr{p}} \cdot \boldsymbol{c}_{\Delta} = \boldsymbol{c}_{\Delta}$$

Thus, we obtain

$$\begin{bmatrix} \boldsymbol{r}_{\jmath}^{*} \\ \vec{r}_{0,\jmath}^{*} \\ \boldsymbol{S}_{\jmath} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{\mathscr{p}} \cdot \mathcal{D}_{n}(\boldsymbol{R}_{\Delta}) & \boldsymbol{c}_{Z0} & \boldsymbol{c}_{W} & \boldsymbol{c}_{\Delta} \\ & \boldsymbol{R}_{Z0} & \boldsymbol{D}_{W} & \vec{s}_{\Delta} \\ & & \boldsymbol{R}_{W} & \boldsymbol{S}_{\Delta} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_{\jmath-1}^{*} \\ \vec{r}_{0,\jmath-1}^{*} \\ \boldsymbol{S}_{\jmath-1} \\ 1 \end{bmatrix}.$$

The Formation Chain Function

The final value of the Formation Chain Function is as follows:

$$\begin{bmatrix} \boldsymbol{r}_{j}^{*} \\ \vec{r}_{0,j}^{\,*} \\ \boldsymbol{S}_{j} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{p} \cdot \mathcal{D}_{n}(\boldsymbol{R}_{\Delta}) & \boldsymbol{c}_{Z0} & \boldsymbol{c}_{W} & \boldsymbol{c}_{\Delta} \\ & \boldsymbol{R}_{Z0} & \boldsymbol{D}_{W} & \vec{s}_{\Delta} \\ & & \boldsymbol{R}_{W} & \boldsymbol{S}_{\Delta} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_{j-1}^{*} \\ \vec{r}_{0,j-1}^{\,*} \\ \boldsymbol{S}_{j-1} \\ 1 \end{bmatrix}.$$

Using a time-invariant state transition matrix, we obtain

$$\boldsymbol{z}_{j} = \boldsymbol{\Phi} \cdot \boldsymbol{z}_{j-1}.$$

Note, it can be shown that

$$\boldsymbol{z}_{j} = \boldsymbol{\Phi}^{j} \cdot \boldsymbol{z}_{0}$$

and

$$\boldsymbol{z}_{n} = \boldsymbol{\Phi}^{n} \cdot \boldsymbol{z}_{0} = \boldsymbol{z}_{0}.$$

Reproducing LISA

Reproducing LISA

- Process is intended for swarms with any number of satellites.
- Process must produce results of LISA swarm to be considered viable.
- Consider a LISA-like configuration with the following parameters.

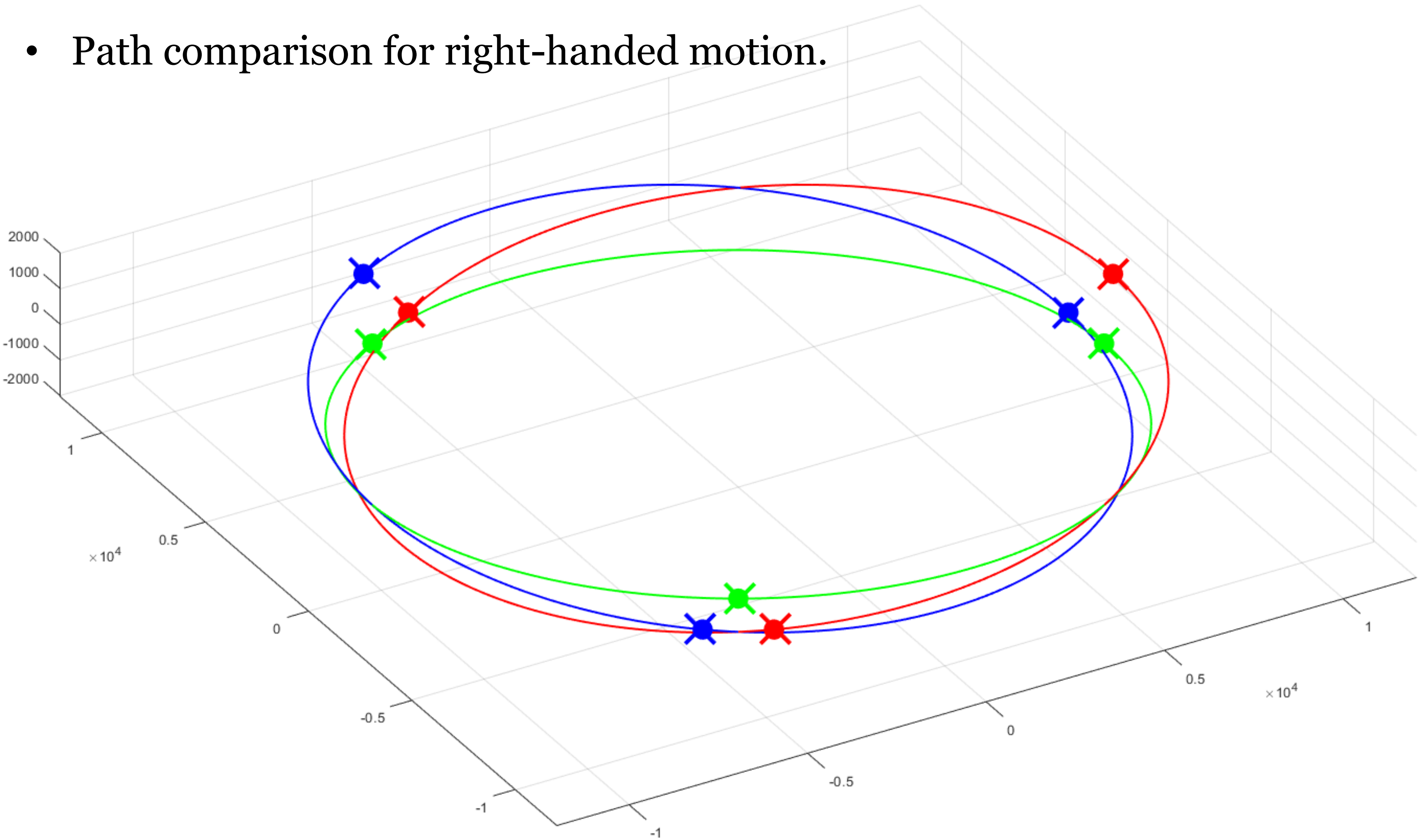
Parameter / Variable	Value
$\hat{\rho}_i$	$\cos\big((i-1)\cdot 120°\big)\,\hat{X} - \sin\big((i-1)\cdot 120°\big)\,\hat{Y}$
ρ	1,000 km
a_0	10,000 km
ϕ	$0°$
$\hat{u}$	$\hat{Y}$
θ	$-60°$
$\hat{w}$	$\sin(120°)\,\hat{X} + \cos(120°)\,\hat{Z}$

- Left-handed motion ⟹ path A, C, B; right-handed motion ⟹ path A, B, C.

Parameter / Variable	Left-Handed Motion	Right-Handed Motion
$\vec{s}_g$	$-(20.14\text{ km})\,\hat{Z}$	$(66.62\text{ km})\,\hat{Z}$
$\vec{s}_{rk}, \vec{s}_{\Delta k}\ \forall\ k \geq 0$	$\vec{0}$ km	$\vec{0}$ km
a	10,076 km	10,032 km
e_1	0.0454	0.0508
e_2	0.0454	0.0508
e_3	0.0454	0.0508
ν_1	180°	180°
ν_2	305°	65°
ν_3	55°	295°

Reproducing LISA

- Path comparison for right-handed motion.



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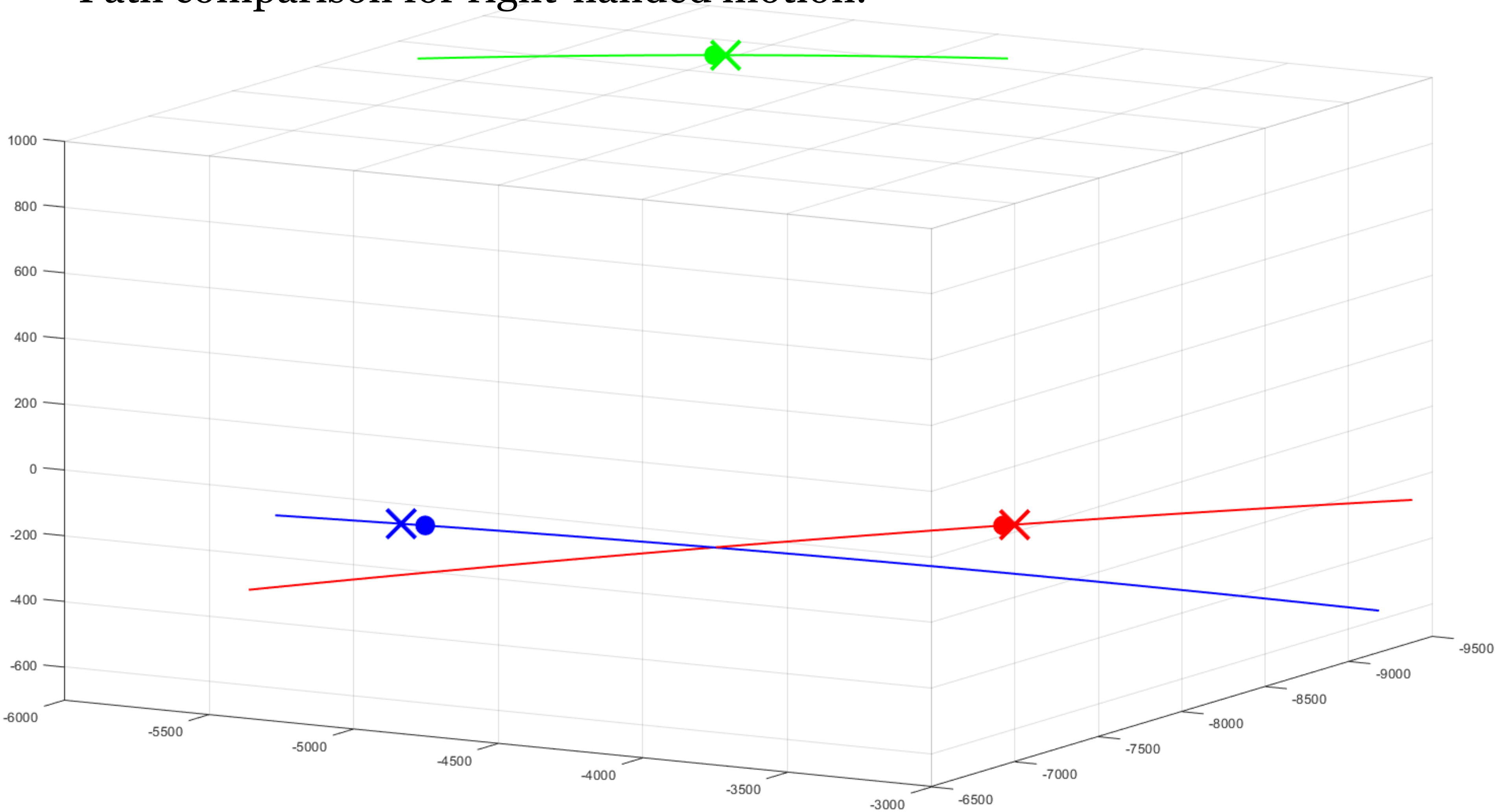

Conclusions

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- Summary:
  - Extended the number of degrees of freedom.
  - Uncovered dynamics consistent with screw theory, which pair rotational and translational movement.
  - Drafted a journal article which is now in the early stages of publication.
- Next Steps:
  - Utilize these results to create satellite swarms of various sizes and shapes.
  - Move past the swarm initialization problem towards other engineering challenges surrounding the concept.

Questions