PARTIAL DIFFERENTIAL EQUATION BASED CONTROL OF NONLINEAR SYSTEMS WITH INPUT DELAYS

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2017

To the ultimate bundle of joy, my dog, Buchu and my fiancee, Aishwarya for making my life so comfortable

ACKNOWLEDGMENTS

I would like to express sincere gratitude to my advisor, Dr. Warren E. Dixon, whose experience and motivation have been instrumental in my academic success. As an advisor, he has provided guidance in my research and encouragement in developing my own ideas. As a mentor, he has helped me develop professional skills and has prepared me for the future. I would like to extend my gratitude to my committee members Dr. Carl Crane, Dr. Lawrence Ukeiley, and Dr. Zhang Lei, for the time and recommendations they provided. Also, I would like to thank my family, coworkers, friends for their support and encouragement.

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of the University of Florida in Partial Fulfillment of the
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December 2017

Chair: Warren E. Dixon
Major: Mechanical Engineering

Partial differential equation (PDE)-based control methods are developed for a class of uncertain nonlinear systems with bounded external disturbances and known/unknown time-varying input delay. Inspired by predictor-based delay compensators, a linear transformation is used to relate the control input to a spatially and time-varying function. The transformation allows the input to be expressed in a manner that separates the control into a delayed control and non-delayed control and also facilitates the ability to compensate for the time-varying aspect of the delay with less complex gain conditions than previous robust control approaches. Unlike previous predictor-based approaches, which inherently depend on the system dynamics, an auxiliary error function is introduced to facilitate a robust control structure that does not depend on known dynamics. The designed controller features gains to compensate for the delay and delay derivative independently and further robustness is achieved since the controller does not require exact model knowledge.

In Chapter 2, a tracking controller is developed for a second order system with a known time-varying input delay. A novel Lyapunov-Krasovskii functional is used in the Lyapunov-based stability analysis to prove uniform ultimate boundedness of the error signals. Chapter 3 and Chapter 4 focus on the development of a tracking controller for a generalized uncertain nonlinear systems with bounded external disturbances and unknown time-varying input delay. In Chapter 3, a nonlinear mapping is used to map the non-compact time domain to a compact spatial domain, and then a neural network (NN) is used to estimate the unknown time-varying

input delay. In Chapter 4, an accelerated gradient descent (AGD) based optimization method is demonstrated to estimate the unknown time-varying delay magnitude. Application of input time-delay for flexible system is examined in Chapter 5. Specifically an aircraft wing dynamics is considered. The NN based estimation scheme developed in Chapter 3 is combined with a boundary control method, to mitigate oscillations in the aircraft wing in Chapter 5.

CHAPTER 1
INTRODUCTION

1.1 Motivation

Various applications exhibit a delay before the controller can affect the system dynamics, i.e., the so-called input delay problem. Input delay problems are technically challenging because they heuristically require the controller to predict the future response of the system, which is difficult for nonlinear systems, especially when the dynamics are uncertain. The number of practical systems that experience input delays and the theoretical challenges associated with this problem have motivated significant research interest over the last decade. Various results have been developed in literature for the input delay problem, and most results can be primarily subdivided into two categories: known input delay and unknown time delay. There are primarily two methods developed for a nonlinear system subject to a known time-varying delay: robust control and predictor-based control. Robust strategies use an upper bound worse case effects of the input delay and use a delay dependent control gain to ensure stability. Predictor-based strategies compensate for the input delay, by predicting the system state for future time, while utilizing knowledge of model dynamics. One advantage of robust control strategies over predictor based strategies is that the former does not require system model knowledge to design the controller. However, predictor based control strategies typically yield exponential stability, unlike the uniformly ultimately bounded type stability of robust methods.

These two approaches clearly have their advantages, and this motivates the necessity of combining these two approaches to define a partial differential equation (PDE) based robust control strategy, to compensate time-varying input delay in an uncertain nonlinear dynamics. This new approach also significantly opens a new avenue, by providing an inherent way of applying different parameter estimation methods to compensate for unknown time-varying input delays. A Neural network (NN) based functional approximation method and an adaptive gradient descent based optimization method are demonstrated to be able to successfully estimate the unknown time-varying input delay for an uncertain nonlinear dynamics. To fully demonstrate the effectiveness of NN based functional approximator, a boundary controller is developed for a flexible aircraft system, subjected to an unknown time-varying input delay.

1.2 Literature Review

Various applications exhibit a delay before the controller can affect the system dynamics, i.e., the so-called input delay problem [2–20]. Input delay problems are technically challenging because they heuristically require the controller to predict the future response of the system, which is difficult for nonlinear systems, especially when subject to uncertain time-varying delays, which are the focus in this work.

In [21], a pioneering strategy is developed to compensate for the effects of input delay, followed by the finite spectrum approach in [22], and model reduction in [23]. Motivated by pioneering development such as [21–23], various results have been developed for linear systems with input delay, including results such as [24–26] for uncertain linear systems with known time-varying delays. However, such results explicitly use the linearity of the system to conclude the stability result.

More recently, researchers have focused on nonlinear systems with input delays. For example, in results such as [3, 7–11, 20, 24, 27–32], controllers are developed for nonlinear systems with known dynamics subjected to known time-varying input delay. Predictor-based controllers [24, 27–30] and robust controllers [3, 7, 8, 20, 31, 32] are two prominent strategies to compensate for input delay, for nonlinear systems. Both strategies inject delayed and delay-free control inputs in the open-loop error system. Predictor-based methods make use of a linear transformation to map the time dependent control input to a modified control input with temporal and spatial varying components (cf., [24, 27–30]). An advantage of this approach is that resulting gain conditions are generally less conservative/restrictive than robust control methods. For example, compared to robust control results such as [30, 33, 34] predictor methods often have less restrictive conditions on the magnitude of the delay, and of the few predictor-based results that consider variable delays (cf., [27, 29, 30, 35]), both the delay and the delay rate have less restrictive conditions. However, in general, based on the need to predict the state transition, such strategies have only been applied to linear systems (cf., [29, 34, 36–38]) or nonlinear systems with known dynamics (cf., [27, 28, 30, 33, 35, 39–43]). Unlike predictor based approaches, robust approaches do not require exact model knowledge (cf., [3, 7, 8, 20, 31, 32]); although, they can yield complex sufficient gain conditions, compared to predictor based methods.

In the preliminary work in [44, 45] a robust control approach was combined with the PDE-based transformation for uncertain nonlinear systems with known time-varying delays. The results in [44, 45] were extended in [46]. Specifically, in [46] a PDE-based method is developed for uncertain second order systems with unknown time-varying input delay, with an adaptive delay estimate.

Controllers for systems with unknown bounded time-varying delays have been studied in recent years [47–51]. An observer based controller is designed in [47] for a nonlinear system with unknown bounded state and input delays, which yields exponential convergence. Similarly, for a linear system with unknown time-varying input delay, a sliding mode observer based design approach is used in [48] to yield exponential convergence. In [50], for a class of an uncertain linear systems with unknown time-varying state delay, an observer and a controller is developed. An adaptive learning approach is utilized in [51], which shows error signal boundedness for a first order uncertain nonlinear system with unknown time-varying input delay, although this work is specific to first order nonlinear systems and can't be generalized for higher order systems.

Time delays are present in common engineering applications and can cause instability in an otherwise stable system, and also subsequently can cause system performance degradation. To alleviate the negative effects of delay, a control signal can be designed by predicting the future state of the system. However, an unknown and time-varying input delay adds to the complexity of the problem. For example, when the control is communicated to a plant over a network, the input delay can be unknown and time-varying due to transmission uncertainties in the communication channel. In addition to unknown variation of the input delay, unmodeled effects (e.g., exogenous disturbances) make the delay control problem more challenging for uncertain nonlinear systems. Datko et al. in [52] proved that a arbitrarily small amount of time delay, present in the boundary feedback control can destabilize an elastic system (e.g., one dimensional Euler-Bernoulli beam).

There exist literature that deals with controller design to suppress oscillation for two-dimensional airfoil system. These literature include linear-quadratic regulator [53–55], feedback linearization [56], linear reduced order model-based control approaches [57, 58], a Nissim aerodynamic energy-based control approach [59], and state-dependent Riccati equation and sliding mode control approaches [60]. Most recently, a RISE control structure was used to ensure asymptotic tracking of a two-dimensional airfoil section with modeling uncertainties in the structural and aerodynamic properties [61], and then extended to compensate for actuator saturation [62]. There are two boundary control methodologies that have been developed for a system described by a set of PDEs. The first method approximates the PDE system with a finite number of ordinary differential equations (ODE) using operator theoretic tools [63–66] or Galerkin and Rayleigh-Ritz methods [67–69]. A boundary controller is then designed using the resulting reduced-order model. The primary concern with using a reduced-order model for the control design is the potential for spillover instabilities [70, 71], in which the controller excites higher-order modes that were neglected in the approximation. In special cases, the placement of actuators and sensors can guarantee the neglected modes are not excited [72]. Specifically, placing actuators at known zero locations of the higher-order modes will alleviate spillover instabilities; however, this can conflict with the desire to place actuators away from the zeros of the controlled modes. Many PDE-based and ODE-based control strategies have been developed to stabilize the bending of a flexible beam such as [65, 66, 73, 74].

Motivated by this development, the effect of an input delay has been studied for an aircraft wing, which is subjected to store induced limit cycle oscillation [1]. The controller compensates for the input delay, the magnitude of which is unknown and time-varying. The contribution of the work is to consider an example of a flexible elastic system with input delay. To replicate the claim of Datko in [52], Bialy et al. in [1] designed a adaptive boundary controller for a 2D aircraft wing in order to mitigate the effect of limit cycle oscillation on the elastic wing. This current work extends the boundary controller design to compensate for an unknown time-varying delay in boundary control feedback.

1.3 Contribution

In Chapter 2, the amalgamation of predictor based and robust based control strategies is examined. A second order Euler-Lagrange system is considered, subjected to a known time-varying input delay. The effectiveness of both predictor and robust control strategies for a time-delayed system, are taken into consideration while developing this PDE based robust control strategy. A linear PDE based transformation is developed to map the time-varying control input to a two variable control input. Doing so provides some added flexibility by keeping the time-varying delay outside of the control input term. Similar to a robust control strategy, this PDE based method has the flexibility to be applied to an uncertain system, since it does not require the system model for future system state prediction, unlike typical predictor based control strategies. Finally to demonstrate the theoretical claim of the developed PDE based robust control method, an experimental demonstration of the controller is shown for knee-shank dynamics in Chapter 2.

Chapter 3 focuses on more generalized dynamics and extends the PDE-based robust controller developed for an Euler-Lagrange system, to an uncertain $(n+1)^{th}$ order dynamic system. As previously described, because of added flexibility of the PDE based robust strategy, an unknown time-varying input delay case is considered for Chapter 3. For estimation of an unknown time-delay, a Neural Network (NN) based function approximator is designed in Chapter 3. Unlike a traditional NN function approximator, this modified version can effectively handle a explicit time dependent function such as the time-delay, by using a nonlinear mapping to convert time into a compact variable space. This method motivates the further development of a delay dependent controller design, and subsequent demonstration in Lyapunov based stability analysis, for an unknown time-varying input delayed system.

Chapter 4 starts with the same generalized dynamical system described in Chapter 3. The main contribution of Chapter 4 is the demonstration of an optimization based strategy, namely Nesterov's accelerated gradient based (AGD) strategy, for estimating the present unknown time-varying input delay. This approach utilizes the PDE based robust controller developed in Chapter 3, by adding a AGD based estimation, and demonstrating the theoretical effectiveness for a 2-link system, subjected to an unknown time-varying input delay. Simulation results show the effective estimation of the unknown time-varying input delay. Further, this AGD based estimation is extended for a m variable dynamical system, subjected to n different unknown time-varying input delays. While, most of the previous development in Chapter 4, remains unchanged, a significant modification of the estimation scheme has been demonstrated by mapping the m state variable onto a $m-1$ order hyperbolic manifold, while mapping the unknown n input delay onto a $n-1$ order hypersphere.

The development of control strategies in the previous chapters (Chapter 2-Chapter 4) is extended for a flexible nonlinear system in Chapter 5. Without loss of generality, a flexible aircraft wing dynamic is examined. A boundary controller is developed for Chapter 5 and a delay compensation term is added to the developed adaptive boundary controller. The novelty of Chapter 5 is the combination of the NN functional approximator based unknown time-varying delay compensator with a adaptive boundary controller, to successfully mitigate unnecessary oscillations of a 2D flexible aircraft wing. A Galerkin based simulation model demonstrates the controller effectiveness, through a Lyapunov stability analysis.

CHAPTER 2
CONTROL OF AN UNCERTAIN SECOND-ORDER SYSTEM WITH KNOWN TIME-VARYING INPUT DELAY: A PDE-BASED APPROACH

This chapter develops a PDE-based approach, which uses the linear transformation discussed in [30] to map the time dependent control input to a modified control input that depends on both time and a spatial variable. PDE-based approaches are generally used to solve input delay problems since they produce finite-dimensional solutions, while solutions to input delayed differential equations are infinite-dimensional in general. In addition to the transformation of infinite to finite-dimensionality, a PDE-based approach gives an added advantage of extracting the delay term from its functional, facilitating the stability analysis. Additionally, these advantages may help to reduce control effort and improve the delay compensating term, as discussed in [30]. The most prominent benefit of the approach in the current work is the amalgamation of the flexibility seen in other PDE-based approaches with the ability to robustly compensate for uncertain dynamics. This is accomplished by modifying the robust control approach in results such as [20, 32] so that it incorporates the PDE-based approach in [30]. A novel tracking error is used in this chapter to inject a delay-free control term into the closed-loop error dynamics by integrating the control states over the finite delay interval. Moreover, this robust, PDE-based design approach allows for the delay derivative gain, as well as the conventional delay magnitude gain, to be tuned independently, unlike the previous works in [20, 32]. A novel Lyapunov-Krasovskii functional is developed and used in the stability analysis to yield uniformly ultimate boundedness of the tracking error signal. Tracking experiments for the knee-shank dynamics are demonstrated to show the effectiveness of the developed controller.

2.1 Dynamic Model and Properties

Consider a class of second-order systems modeled as

$$\ddot{x}(t) = f\left(x(t), \dot{x}(t)\right) + d(t) + U(t - D(t)), \tag{2–1}$$

where $x, \dot{x}$ and $\ddot{x} \in \mathbb{R}^n$ denote the system states, $f : \mathbb{R}^n \times \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^n$ is an uncertain nonlinear function, $d : [t_0, \infty) \to \mathbb{R}^n$ is an uncertain exogenous disturbance (e.g., unmodeled effects), $U(t - D(t)) \in \mathbb{R}^n$ represents the generalized delayed input vector, and $D \in \mathbb{R}$ is a known, bounded, time-varying delay. The subsequent development is based on the assumption that $x, \dot{x}$ are measurable. Furthermore, the following conditions are assumed to be satisfied.
A linear transformation is used to represent the generalized input $U(t)$ as a function of two independent variables, i.e. p and t, where $t \in [t_0, \infty)$ and $p \in [0, 1]$. The spatial variable, p, denotes delayed and delay free control inputs at two end points of its domain. The two variable control input $u : \mathbb{R} \times [t_0, \infty) \to \mathbb{R}^n$ is analogous to $U(t)$ in the sense that [30]

$$u(p,t) \triangleq U\left(\phi(t + p\left(\phi^{-1}(t) - t\right)\right), \phi(t) \le t, \forall t \ge 0 \tag{2–2}$$

where $\phi : [t_0, \infty) \to \mathbb{R}$ is a known delay dependent invertible monotonous time function. Let $\phi(t) \triangleq t - D(t)$, s0

$$\begin{aligned}\phi\left(\phi^{-1}(t)\right) &= \phi^{-1}(t) - D\left(\phi^{-1}(t)\right), \\ t &= \phi^{-1}(t) - D\left(\phi^{-1}(t)\right).\end{aligned} \tag{2–3}$$

Based on (2–2) and (2–3), the delayed control input can be expressed as $U(t-D(t)) = u(0,t)$, and the delay-free control input can be expressed as $U(t) = u(1,t)$.

To further facilitate the subsequent development, a relationship between the spatial and temporal variation of $u(p,t)$, i.e., $u_p(p,t)$ and $u_t(p,t)$ respectively, is developed. From (2–2) and from the auxiliary function $t_d : [t_0, \infty) \times [0, 1] \to \mathbb{R}$ defined as $t_d \triangleq t + p\left(\phi^{-1}(t) - t\right)$

$$\begin{aligned}u_t(p,t) &= \frac{dU\left(\phi(t_d)\right)}{dt_d}\left(1 + p\left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)\right), \\ &= \left(\frac{\partial u(p,t)}{\partial p}\right)\left(\frac{\partial p}{\partial t_d}\right)\left(1 + p\left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)\right), \\ &= \frac{1 + p\left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)}{\phi^{-1}(t) - t} u_p(p,t), \\ &= \delta(p,t) u_p(p,t),\end{aligned} \tag{2–4}$$

where the auxiliary function $\delta : \mathbb{R} \times (0, \infty) \in \mathbb{R}^n$ is defined as[1]

$$\delta(p,t) \triangleq \frac{1 + p\left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)}{\phi^{-1}(t) - t}. \tag{2–5}$$

Since D is a known time-varying function, $\phi^{-1}(t)$ is calculated by solving (2–3). To facilitate the subsequent stability analysis the time derivative of $\phi^{-1}(t)$ can be calculated by taking the time derivative of (2–3) as

$$\begin{aligned} t &= \phi^{-1}(t) - D\left(\phi^{-1}(t)\right), \\ 1 &= \frac{d}{dt}\left(\phi^{-1}(t)\right) - \frac{dD\left(\phi^{-1}(t)\right)}{d\left(\phi^{-1}(t)\right)} \frac{d\left(\phi^{-1}(t)\right)}{dt}, \\ &= \frac{d\left(\phi^{-1}(t)\right)}{dt}\left(1 - D^*(\phi^{-1}(t))\right), \end{aligned}$$

and then rearranging to yield

$$\frac{d\left(\phi^{-1}(t)\right)}{dt} = \frac{1}{1 - D^*(\phi^{-1}(t))}, \tag{2–6}$$

where $D^*(\phi^{-1}(t)) \triangleq \frac{dD\left(\phi^{-1}(t)\right)}{d(\phi^{-1}(t))}$. Using (2–3), (2–5), and (2–6), the p variation of $\delta(p,t)$, i.e, $\delta_p(p,t)$ can be calculated as

$$\delta_p = \frac{D^*(\phi^{-1}(t))}{\left(1 - D^*(\phi^{-1}(t))\right)} \frac{1}{D\left(\phi^{-1}(t)\right)}. \tag{2–7}$$

From (2–3), (2–5), and (2–7), $\delta(0,t) \triangleq \delta_0 = \frac{1}{D(\phi^{-1}(t))}$, $\delta(1,t) \triangleq \delta_1 = \frac{1}{(1-D^*(\phi^{-1}(t)))}\delta_0$, and $\delta_p(p,t) \triangleq \delta_p = \frac{D^*(\phi^{-1}(t))}{(1-D^*(\phi^{-1}(t)))}\delta_0$. Using Assumptions 2.1 and 2.3, $\underline{\delta_0} \leq \delta_0 \leq \bar{\delta}_0$, $\underline{\delta_1} \leq \delta_1 \leq \bar{\delta}_1$, $\underline{|\delta_p|} \leq |\delta_p| \leq |\bar{\delta}_p|$, where $\underline{\delta_0}, \bar{\delta}_0, \underline{\delta_1}, \bar{\delta}_1, \underline{|\delta_p|}, |\bar{\delta}_p| \in \mathbb{R}$ are known positive constants. [2]

[1] $\phi^{-1} - t = D\left(\phi^{-1}(t)\right)$ and $D(t)$ is assumed to be non-zero.

[2] The upper bounds of $D(\phi^{-1}(t))$, and $\dot{D}(\phi^{-1}(t))$ are the same as $D(t)$ and $\dot{D}(t)$, respectively.

Remark 2.1. From (2–7), singularities can occur in δ_0, δ_1 and δ_p when $D(\phi^{-1}(t)) = 0$ and when $D^*(\phi^{-1}(t)) = 1$. When $D(\phi^{-1}(t)) = 0$ there is no delay in the system. While δ_0, δ_1 and δ_p will be singular in this case, these terms will vanish from the control and stability analysis. From (2–4), when $D^*(\phi^{-1}(t)) = 1$, for a nonzero delay magnitude $D(\phi^{-1}(t))$, $u_p(1,t) = 0 \iff u(1,t) = u(0,t) \iff \delta_p = 0 \iff \delta_0 = \delta_1 = \frac{1}{D(\phi^{-1}(t))}$.

Assumption 2.1. *The nonlinear exogenous disturbance term and its first time derivative (i.e., d, $\dot{d}$) exist and are bounded by known positive constants [75–77].*

Assumption 2.2. *The desired trajectory $x_d \in \mathbb{R}^n$ is designed such that $x_d, \dot{x}_d, \ddot{x}_d$ exist and are bounded by known positive constants.*

Assumption 2.3. *The time-varying delay $D(t) \in \mathbb{R}$ is bounded by known positive constants $\underline{D}$ and $\bar{D}$, as $\underline{D} \leq D(t) \leq \bar{D}$. The system remains bounded in the interval $[t_0, t_0 + \bar{D}]$.*

Assumption 2.4. *The delay rate $\dot{D}(t) \in \mathbb{R}$ is bounded by known constants $\underline{\dot{D}}$ and $\bar{\dot{D}}$, as $\underline{\dot{D}} \leq \dot{D}(t) \leq \bar{\dot{D}}$.*

2.2 Control Objective

The objective is to develop a continuous controller which ensures that the generalized state x of the input-delayed system in (2–1) tracks a desired trajectory, x_d, despite uncertainties and additive disturbances in the dynamics. To quantify the control objective, a tracking error, denoted by $e \in \mathbb{R}^n$, is defined as

$$e \triangleq x_d - x. \tag{2–8}$$

To facilitate the subsequent analysis, a measurable auxiliary tracking error, denoted by $r \in \mathbb{R}^n$, is defined as

$$r \triangleq \dot{e} + \alpha e - \beta e_u, \tag{2–9}$$

where $\alpha, \beta \in \mathbb{R}$ are known, positive constants. In (2–9), $e_u \in \mathbb{R}^n$ is an auxiliary signal that is used to obtain a delay dependent control signal to negate the effect of the delayed input in

(2–1), defined as

$$e_u \triangleq \int_0^1 u\left(p,t\right)dp. \tag{2–10}$$

By using the Leibnitz integral rule and (2–4), the time derivative of e_u can be determined as

$$\dot{e}_u = \int_0^1 u_t\left(p,t\right)dp = \int_0^1 \delta(p,t)d\left(u\left(p,t\right)\right). \tag{2–11}$$

Integrating (2–11) by parts yields[3]

$$\begin{aligned}\dot{e}_u &= \delta(1,t)u(1,t) - \delta(0,t)u(0,t) - \int_0^1 \delta_p u\left(p,t\right)dp, \\ &= \delta(1,t)u(1,t) - \delta(0,t)u(0,t) - \delta_p e_u. \end{aligned} \tag{2–12}$$

Since $e, \dot{e}$ are assumed to be measurable e_u can be obtained from (2–12) and r can be computed for all time and used as feedback.

2.3 Control Development

The open-loop error system for r is obtained by taking the time derivative of (2–9) and using the expressions in (2–1), (2–8), and (2–12) as

$$\begin{aligned}\dot{r} = \ddot{x}_d - f\left(x(t), \dot{x}(t)\right) - d(t) - u(0,t) + \alpha\left(r - \alpha e + \beta e_u\right) - \delta_1 \beta u(1,t) \\ + \delta_0 \beta u(0,t) + \delta_p \beta e_u. \end{aligned} \tag{2–13}$$

The open-loop error system in (2–13) contains both a delayed and delay-free control input resulting from the time derivative of e_u. Based on the subsequent stability analysis, the control input is designed as

$$U(t) \triangleq u(1,t) \triangleq \frac{1}{k}r, \tag{2–14}$$

[3] δ_p is independent of p.

where $k \in \mathbb{R}^{+}$ is constant, adjustable control gain. To facilitate the subsequent stability analysis, the terms in (2–13) can be segregated into terms upper bounded by a state-dependent function and terms upper bounded by a known constant as

$$\dot{r} = \tilde{N} + N_d - e + \alpha\beta e_u - \delta_1 \beta u(1,t) - (1-\delta_0\beta)\, u(0,t) + \delta_p \beta e_u, \tag{2–15}$$

where the terms $\tilde{N}, N_d \in \mathbb{R}^n$ are defined as

$$\tilde{N} \triangleq f\left(x_d\left(t\right), \dot{x}_d\left(t\right)\right) - f\left(x\left(t\right), \dot{x}\left(t\right)\right) + \alpha r - \alpha^2 e + e, \tag{2–16}$$

and

$$N_d \triangleq -f\left(x_d\left(t\right), \dot{x}_d\left(t\right)\right) + \ddot{x}_d - d\left(t\right). \tag{2–17}$$

By substituting (2–14) into (2–15), the closed-loop error system for r is

$$\dot{r} = \tilde{N} + N_d - e + \alpha\beta e_u - \frac{\delta_1 \beta r}{k} - (1-\delta_0\beta)\, u(0,t) + \delta_p \beta e_u. \tag{2–18}$$

Remark 2.2. Using the Mean Value Theorem and Assumption 2.2, the expression in (2–16) can be upper bounded as

$$\left\|\tilde{N}\right\| \leq \rho\left(\|z\|\right)\|z\|, \tag{2–19}$$

where $\rho : \mathbb{R} \to \mathbb{R}$ is positive definite, non-decreasing, radially unbounded function, and $z \in \mathbb{R}^{3n}$ is a vector of error signals, defined as

$$z \triangleq \begin{bmatrix} e^T & r^T & e_u^T \end{bmatrix}^T. \tag{2–20}$$

Remark 2.3. Using Assumptions 2.1 and 2.2, N_d can be upper bounded as

$$\sup_{t\in[0,\infty)} \|N_d\| \leq \Theta, \tag{2–21}$$

where $\Theta \in \mathbb{R}$ is a known positive constant.

2.4 Stability Analysis

To facilitate the subsequent stability analysis, let $y \in \mathbb{R}^{3n+1}$ be defined as

$$y \triangleq \left[\begin{array}{cc} z^T & \sqrt{Q} \end{array} \right]^T, \tag{2–22}$$

where $Q \in \mathbb{R}$ is an LK functional defined as

$$Q \triangleq \lambda_Q \int_0^1 e^{\omega_2 p} \|u(p,t)\|^2 dp, \tag{2–23}$$

where $\omega_2, \lambda_Q \in \mathbb{R}^+$ are constants. Let $\mathscr{D}$ be an open and connected set, and $\mathcal{S}_{\mathscr{D}} \subset \mathscr{D}$ is defined as

$$\mathcal{S}_{\mathscr{D}} \triangleq \left\{ y \in \mathbb{R}^{3n+1} |\ \|y\| < \inf \left\{ \rho^{-1} \left([\sqrt{\frac{\lambda_1 \epsilon_1}{2}}, \infty) \right) \right\} \right\}, \tag{2–24}$$

where ϵ_1 and $\lambda_1 \in \mathbb{R}$ are known, positive constants.

Theorem 2.1. *Given the open-loop error system in (2–13), the controller in (2–14) ensures UUB tracking in the sense that*

$$\|e\| \leq \mathit{\Gamma}_0 \exp\left(-\mathit{\Gamma}_1 t\right) + \mathit{\Gamma}_2, \tag{2–25}$$

provided that $y(\eta) \in \mathcal{S}_{\mathscr{D}}, \forall \eta \in [t_0, t_0 + D_{max}]$, the control gains satisfy sufficient gain conditions (see Section 2.6), and Assumptions 2.1-2.3 are satisfied, where Γ_0, Γ_1 and Γ_2 are known positive constants, where Γ_2 can be made arbitrarily small.

Proof. Let $V : \mathscr{D} \times [t_0,\ \infty) \to \mathbb{R}$ be a continuously differentiable, positive-definite functional defined as

$$V \triangleq \frac{1}{2} e^T e + \frac{1}{2} r^T r + \frac{\omega_1}{2} e_u^T e_u + Q, \tag{2–26}$$

where $\Phi_1 \|y\|^2 \leq V \leq \Phi_2 \|y\|^2$. The time derivative of (2–26) can be obtained after applying the Leibniz integral rule to obtain the time derivative of (2–23) and utilizing (2–9), (2–12), and (2–18) as

$$\dot{V} = r^T \left(\tilde{N} + N_d + \alpha \beta e_u - \delta_1 \beta \frac{1}{k} r \right) - r^T \left((1 - \delta_0 \beta) u(0,t) - \delta_p \beta e_u \right) - \omega_1 \delta_p e_u^T e_u$$

$$-e^T\alpha e+e^T\beta e_u+\frac{\omega_1\delta_1 e_u^T r}{k}-\omega_1\delta_0 e_u^T u(0,t)+\lambda_Q\delta_1 e^{\omega_2}\|u(1,t)\|^2-\lambda_Q\delta_0\|u(0,t)\|^2$$
$$-\lambda_Q\omega_2\int_0^1\delta(p,t)e^{\omega_2 p}\|u(p,t)\|^2dp-\lambda_Q\delta_p\int_0^1 e^{\omega_2 p}\|u(p,t)\|^2dp. \tag{2–27}$$

By using (2–14), (2–17), and (2–19), and canceling common terms in (2–27), an upper bound can be obtained for (2–28) as

$$\dot V\le\|r\|\rho\left(\|z\|\right)\|z\|+\|r\|\Theta+\omega_1|\bar\delta_p|\|e_u\|^2-\frac{\beta\underline{\delta_1}||r||^2}{k}-\alpha||e||^2+\beta||e^Te_u||$$
$$+\left(\bar\delta_0\beta+1\right)||r^Tu(0,t)||+\beta\left(\alpha+|\bar\delta_p|\right)\|r^Te_u\|+\omega_1\bar\delta_1||\frac{e_u^Tr}{k}||$$
$$+\frac{\lambda_Q\bar\delta_1e^{\omega_2}||r||^2}{k^2}-\lambda_Q\underline{\delta_0}||u(0,t)||^2+\omega_1\bar\delta_0||e_u^Tu(0,t)||$$
$$-\lambda_Q\omega_2\int_0^1\delta(p,t)e^{\omega_2p}\|u(p,t)\|^2dp+\lambda_Q|\bar\delta_p|\int_0^1e^{\omega_2p}\|u(p,t)\|^2dp. \tag{2–28}$$

From (2–5) and (2–6), $\delta(p,t)$ can be lower bounded as

$$\delta(p,t)=\delta_0+\left(\delta_1-\delta_0\right)p\ge\min\left\{\delta_0,\delta_1\right\}, \tag{2–29}$$

for $p\in[0,1]$. Using the Cauchy-Schwarz inequality,

$$\|e_u\|^2\le\int_0^1\|u^2\left(p,t\right)\|dp\int_0^1 1dp,$$
$$\|e_u\|^2\le\int_0^1\|u\left(p,t\right)\|^2dp. \tag{2–30}$$

After using Young's Inequality and the inequalities in (2–29) and (2–30), (2–28) can be upper bounded as

$$\dot V\le\frac{1}{\epsilon_1}\rho^2\left(\|z\|\right)\|z\|^2+\frac{1}{\epsilon_2}\Theta^2-\left(\alpha-\frac{\beta}{2k}\right)||e||^2-\frac{\lambda_Q\omega_2}{2}\min\left\{\delta_0,\delta_1\right\}\|e_u\|^2$$
$$-\left(\frac{\beta\underline{\delta_1}}{k}-\frac{\bar\delta_0}{2k\epsilon}-\frac{(\epsilon_1+\epsilon_2)}{4}\right)\|r\|^2+\left(\frac{\epsilon}{2k}+\frac{\beta|\bar\delta_p|\alpha^2\epsilon}{2k}+\frac{\alpha^2\beta\epsilon}{2}\right)\|r\|^2$$
$$+\left(\frac{k\beta|\bar\delta_p|}{2\epsilon\alpha^2}+\omega_1|\bar\delta_p|+\frac{k\beta}{2}\right)\|e_u\|^2+\left(\frac{\omega_1\bar\delta_1k}{2\epsilon}+\frac{\epsilon\omega_1\bar\delta_0}{2k}\right)\|e_u\|^2$$
$$+\left(\frac{\lambda_Q\bar\delta_1e^{\omega_2}}{k^2}+\frac{\omega_1\bar\delta_1\epsilon}{2k^3}\right)\|r\|^2-\left(\lambda_Q\underline{\delta_0}-k\left(\frac{\bar\delta_0\beta^2\epsilon}{2}+\frac{\omega_1\bar\delta_0+1}{2\epsilon}\right)\right)\|u(0,t)\|^2$$
$$-\frac{\lambda_Q\omega_2}{2}\min\left\{\delta_0,\delta_1\right\}\int_0^1e^{\omega_2p}\|u(p,t)\|^2dp+\lambda_Q|\bar\delta_p|\int_0^1e^{\omega_2p}\|u(p,t)\|^2dp. \tag{2–31}$$

Since by definition $\|y\| \geq \|z\|$, the following upper bound can be obtained

$$\dot{V} \leq -\left(\frac{\lambda_1}{2} - \frac{1}{\epsilon_1}\rho^2\left(\|y\|\right)\right)\|z\|^2 - \frac{\lambda_1}{2}||z||^2 - \lambda_{Q_{res}}\|u(0,t)\|^2 + \frac{1}{\epsilon_2}\Theta^2$$
$$-\left[\frac{\omega_2}{2}\min\left\{\delta_0, \delta_1\right\} - |\bar{\delta}_p|\right]Q, \tag{2–32}$$

where $\lambda_1, \lambda_r, \lambda_{e_u}, \lambda_{Q_{res}} \in \mathbb{R}$ are defined as

$$\lambda_1 \triangleq \min\left\{\left(\alpha - \frac{\beta}{2k}\right), \lambda_r, \lambda_{e_u}\right\}, \tag{2–33}$$

$$\lambda_r = \frac{\beta\underline{\delta_1}}{k} - \frac{(\epsilon_1+\epsilon_2)}{4} - \frac{\bar{\delta}_0}{2k\epsilon} - \frac{\lambda_Q\bar{\delta}_1 e^{\omega_2}}{k^2} - \frac{\epsilon}{2k} - \frac{\beta|\bar{\delta}_p|\alpha^2\epsilon}{2k} - \frac{\alpha^2\beta\epsilon}{2} - \frac{\omega_1\bar{\delta}_1\epsilon}{2k^3}, \tag{2–34}$$

$$\lambda_{e_u} = \frac{\lambda_Q\omega_2}{2}\min\left\{\delta_0, \delta_1\right\} - \omega_1|\bar{\delta}_p| - \frac{k\beta}{2} - \frac{\beta|\bar{\delta}_p|k}{2\epsilon\alpha^2} - \frac{\omega_1\bar{\delta}_1 k}{2\epsilon} - \frac{\epsilon\omega_1\bar{\delta}_0}{2k}, \tag{2–35}$$

$$\lambda_{Q_{res}} = \lambda_Q\underline{\delta_0} - k\left(\frac{\bar{\delta}_0\beta^2\epsilon}{2} + \frac{\omega_1\bar{\delta}_0+1}{2\epsilon}\right). \tag{2–36}$$

Provided $y(\eta) \in \mathcal{S}_{\mathscr{D}}, \forall\eta \in [t-\bar{D}, t]$ and all the gain conditions are satisfied sufficiently (see Section 2.6), the expression in (2–32) reduces to

$$\dot{V} \leq -\lambda_2\|y\|^2 + \frac{1}{\epsilon_2}\Theta^2, \tag{2–37}$$

where $\lambda_2 \in \mathbb{R}$ is defined as

$$\lambda_2 \triangleq \min\left\{\frac{\lambda_1}{2}, \frac{\omega_2}{2}\min\left\{\delta_0, \delta_1\right\} - |\bar{\delta}_p|\right\}. \tag{2–38}$$

The inequality in (2–37) can be further upper bounded as

$$\dot{V} \leq -\frac{\lambda_2}{\Phi_2}V + \frac{1}{\epsilon_2}\Theta^2. \tag{2–39}$$

The solution of the inequality in (2–39) can be obtained as

$$V(t) \leq V(t_0)\exp\left(-\frac{\lambda_2}{\Phi_2}(t-t_0)\right) + \frac{\Phi_2\Theta^2}{\lambda_2\epsilon_2}\left(1 - \exp\left(-\frac{\lambda_2}{\Phi_2}(t-t_0)\right)\right), \tag{2–40}$$

where $\epsilon_2 \in \mathbb{R}$ can be made arbitrarily large to make Γ_2, introduced in (2–25), arbitrarily small. Using (2–26) and (2–40), the inequality in (2–25) can be obtained, and $e(t)$, $r(t)$, $e_u(t) \in \mathcal{L}_\infty$. Hence, from (2–14), $u(t) \in \mathcal{L}_\infty$. □

2.5 Experimental Results

Neuromuscular electrical stimulation (NMES) (or functional electrical stimulation (FES) for functional exercise tasks) is the application of electrical current across muscle fibers to produce a muscle contraction. The presence of an electromechanical delay (EMD) in the muscle response to the control input results in performance degradation in the tracking of a human limb via NMES, including potentially destabilizing effects (cf., [13, 14, 19, 78, 79]).

Motivated by the nature of the EMD in NMES systems, the performance of the developed controller in (2–14) was examined through a series of dynamic tracking experiments of the knee-joint dynamics. The nonlinear dynamics of the knee-shank can be modeled as in [80] as

$$\begin{aligned} B_m(q,\dot{q})U(t-D(t)) = J\ddot{q} + mgl\sin(q) + M_e + \kappa_1\exp(-\kappa_2 q)(q-\kappa_3) \\ -\beta_1\tanh(-\beta_2\dot{q}) + \beta_3\dot{q} + d(t), \end{aligned} \tag{2–41}$$

where $q, \dot{q}, \ddot{q} \in \mathbb{R}$ denote the angular position, velocity and acceleration of the shank about the knee-joint, respectively, $\kappa_i, \beta_i \in \mathbb{R}, i = (1, 2, 3)$ are uncertain positive constants, $d : [0, \infty) \to \mathbb{R}$ is a unknown exogenous disturbance assumed to be bounded, $B_m(q, \dot{q}) : \mathbb{R} \to \mathbb{R}$ denotes the control effectiveness of the quadriceps muscle group, which is a uncertain function dependent on the muscle's moment arm and the map between stimulation intensity to muscle force, $U : [0, \infty) \to \mathbb{R}$ is the voltage potential applied across the quadriceps muscle group by the electrical stimulation, and $D : [0, \infty) \to \mathbb{R}$ denotes the EMD. The uncertain positive constants J, m, g, $l \in \mathbb{R}$ symbolize the inertia of shank and foot, the combined mass of the shank and foot, the gravitational acceleration, and the distance between the knee-joint and the lumped center of mass of the shank and foot, respectively.

Six healthy individuals (5 male, 1 female, aged 21-31) participated in the experiments. Prior to participation, written informed consent was obtained from all participants, in accordance with the institutional review board at the University of Florida. The experimental apparatus, illustrated in [81], consisted of the following: 1) a modified leg extension machine equipped with orthotic boots to fix the ankle and securely fasten the shank and the foot, 2) optical encoders (BEI technologies) to measure the leg angle (i.e., the angle between the vertical and the shank), 3) a current-controlled 8-channel stimulator (RehaStim, Hasomed GmbH), 4) a data acquisition board (Quanser QPIDe) with QUARC software, 5) a desktop computer running Matlab/Simulink, and 6) a pair of 3" by 5" Valutrode® surface electrodes placed proximally and distally over the quadriceps muscle group.[4] Surface electrical stimulation was applied to quadriceps muscle group using a single conventional channel during knee-joint tracking experiments with a testing duration of 60 seconds.

During the experiments, electrical pulses were delivered at a constant stimulation frequency of 35 Hz, the pulsewidth was fixed to a constant value (i.e., between 300 and 400 μs), and the controller in (2–14) was used to modulate the amplitude. The main factors to determine the pulsewidth value for an individual were the muscle sensitivity to stimulation, tracking accuracy, and stimulation sensitivity. The control gains were adjusted during pretrial tests to achieve satisfactory trajectory tracking where the desired angular trajectory of the knee joint was selected as a sinusoid with a range of 5° to 50° and a period of 2 seconds [81]. The time-varying nature of the EMD was modeled as a continuously differentiable sigmoid function ranging between $[80 - 140]$ ms. This choice of EMD is based on recent results reported in NMES studies such as [19, 78]. The root-mean-square (RMS) tracking error was computed over the entire trial as a control performance metric. Table 2-1 presents the mean RMS error over the entire experiment duration in all the tracking trials. An illustrative example of a complete dynamic tracking trial is shown in Figure 2-1. To further illustrate the impact of compensating for the delay, Figure 2-2 shows a brief trial where the control gain multiplying the delay compensation term e_u in (2–9) was set to zero (i.e., $\beta = 0$), while keeping the other gains the same. The resulting performance depicted in Figure 2-2 shows unsatisfactory performance.

[4] Surface electrodes for the study were provided compliments of Axelgaard Manufacturing Co., Ltd.

Remark 2.4. The gain conditions in Section 2.6, although only sufficient conditions, were used as a guide to determine the gains during the experiments. Specifically, for the sets of experiments $\alpha \in [3.5, 12]$, $k \in [0.02, 0.06]$, $\beta \in [0.4, 0.5]$, where $\epsilon \in [0.0008, 0.005]$, $\omega_1 = 0.01$, $\omega_2 = 2$ and $\lambda_Q \in [0.01, 0.025]$, based on the approximate delay model from [19, 78]. For example, for subject 4 (S4) right leg (R), the gains were selected as $k = 0.03$, $\alpha = 7.5$ and $\beta = 0.5$, which satisfy the gain conditions in Section 2.6. Note that, in addition to gain selection, various factors also contribute to the experimental results including: electrode placement, level of muscle fatigue, sensitivity to stimulation, etc. The experiments were performed in healthy normal volunteers, as in results such as [14, 78, 80–87], to illustrate the robustness to uncertainty in the dynamics and the input delay. Different neurological conditions can impact the results (e.g., increased/decreased sensitivity to stimulation, more susceptible to fatigue, etc.). Hence, further experiments would be required via clinical trials in specific populations of individuals with neurological conditions to gauge the impact of the developed controller for specific rehabilitation outcomes.

2.6 Control Gain Selection

The stability analysis in Section 2.4 requires that λ_1 in (2–33), λ_r in (2–34), λ_{e_u} in (2–35), $\lambda_{Q_{res}}$ in (2–36), and λ_2 in (2–38) be positive constants. For some given lower and upper bounds on the delay and delay rate $\bar{D}$, $\underline{\dot{D}}$ and $\bar{\dot{D}}$, (i.e., $\underline{\delta_0}$, $\bar{\delta}_0$, $\underline{\delta_1}$, $\bar{\delta}_1$, $|\bar{\delta}_p|$), this section develops sufficient gain conditions to ensure λ_1, λ_r, λ_{e_u}, $\lambda_{Q_{res}}$ and $\lambda_2 > 0$. Using the definitions of λ_1 in (2–33) and λ_2 in (2–38), and using the inequality in (2–32), sufficient lower bounds for α and ω_2 can be obtained as

$$\alpha > \frac{\beta}{2k}, \tag{2–42}$$

$$\omega_2 > \frac{2|\bar{\delta}_p|}{\min\{\delta_0, \delta_1\}}. \tag{2–43}$$

From the definition of $\lambda_{Q_{res}}$ in (2–36), it is clear that for an arbitrarily large k and β and arbitrarily small ω_1 and ϵ, that λ_Q can be selected sufficiently large as

$$\lambda_Q > \frac{k}{\underline{\delta_0}}\left(\frac{\bar{\delta}_0\beta^2\epsilon}{2}+\frac{\omega_1\bar{\delta}_0+1}{2\epsilon}\right), \tag{2–44}$$

to ensure that $\lambda_{Q_{res}} > 0$. Also from the definition of λ_{e_u} in (2–35), λ_Q also needs to satisfy the following inequality

$$\lambda_Q > \frac{2k\epsilon\omega_1|\bar{\delta}_p| + k^2\epsilon\beta + \frac{\beta|\bar{\delta}_p|k^2}{\alpha^2} + \omega_1\bar{\delta}_1 k^2 + \epsilon^2\omega_1\bar{\delta}_0}{k\epsilon\omega_2\min\{\delta_0,\delta_1\}}, \tag{2–45}$$

to ensure $\lambda_{e_u} > 0$. By selecting α on the order of k^2, β on the order of k^3, ϵ on the order of $\frac{1}{k^4}$, and ω_1 on the order of $\frac{1}{k^5}$, the lower bound in (2–44) can be proven to be larger than (2–45). For example, if $\alpha = k^2$, $\beta = k^3$, $\epsilon = \frac{1}{k^4}$ and $\omega_1 = \frac{1}{k^5}$ then

$$\left(\frac{\bar{\delta}_0 k^3}{2\underline{\delta_0}}+\frac{\bar{\delta}_0+k^5}{2\underline{\delta_0}}\right) > \frac{1}{\omega_2\min\{\delta_0,\delta_1\}}\left(\frac{2|\bar{\delta}_p|}{k^5}+k^4+k^4|\bar{\delta}_p|+\bar{\delta}_1+\frac{\bar{\delta}_0}{k^{10}}\right), \tag{2–46}$$

for large values of $k > 1$ and ω_2.

From (2–34), λ_Q is multiplied by a negative term in the definition of λ_r. To develop a sufficient condition to ensure $\lambda_r > 0$, the lower bound for λ_Q in (2–44) is substituted into (2–34) to develop the following sufficient inequality:

$$\begin{aligned}\beta\left(2\underline{\delta_1}-\frac{\bar{\delta}_0\alpha k\epsilon}{2\underline{\delta_0}}e^{\omega_2}-|\bar{\delta}_p|\epsilon\alpha^2-\alpha^2\epsilon\right) &> \frac{(\epsilon_1+\epsilon_2)\,k}{2}\\ &+\left(\frac{1}{\epsilon\underline{\delta_0}}+\frac{\omega_1\bar{\delta}_0}{\epsilon\underline{\delta_0}}\right)\bar{\delta}_1 e^{\omega_2}+\epsilon+\frac{\omega_1\bar{\delta}_1\epsilon}{k^2}+\frac{\bar{\delta}_0}{\epsilon}.\end{aligned} \tag{2–47}$$

Based on (2–47), a sufficient condition for the upper bound on ϵ can be established as

$$\epsilon < \frac{2\underline{\delta_1}}{|\bar{\delta}_p|\alpha^2+\alpha^2+\frac{\bar{\delta}_0\alpha k}{2\underline{\delta_0}}e^{\omega_2}}, \tag{2–48}$$

to ensure the parenthetical terms on the left side of (2–47) are positive. Based on (2–47) and (2–48) a sufficient lower bound for β can be established for an arbitrarily large k, ϵ_2, ω_2 and arbitrarily small ω_1 and ϵ, to ensure $\lambda_r > 0$ as

$$\beta > \frac{\frac{(\epsilon_1+\epsilon_2)k}{2} + \left(\frac{1}{\epsilon\underline{\delta_0}} + \frac{\omega_1\bar{\delta}_0}{\epsilon\underline{\delta_0}}\right)\bar{\delta}_1 e^{\omega_2} + \epsilon + \frac{\omega_1\bar{\delta}_1\epsilon}{k^2} + \frac{\bar{\delta}_0}{\epsilon}}{2\underline{\delta_1} - \frac{\bar{\delta}_0\alpha\epsilon}{2\underline{\delta_0}}e^{\omega_2} - |\bar{\delta}_p|\epsilon\alpha^2 - \alpha^2\epsilon}. \tag{2–49}$$

To yield (2–46) and to satisfy (2–44), (2–48), and (2–49), ϵ_2, α, λ_Q and β are selected sufficiently large and ϵ and ω_1 are selected sufficiently small. For example, selecting $\alpha = k^2$, $\beta = k^3$ and $\epsilon = \frac{1}{k^4}$, as previously, then (2–49) can be written as

$$k^6\left(\left(\frac{1}{\underline{\delta_0}} + \frac{\omega_1\bar{\delta}_0}{\underline{\delta_0}}\right)\bar{\delta}_1 e^{\omega_2} + \bar{\delta}_0\right) + k^5\left(\frac{\bar{\delta}_0}{2\underline{\delta_0}}e^{\omega_2} + 1 + \frac{(\epsilon_1+\epsilon_2)}{2} - 2\underline{\delta_1}\right)$$
$$+k^3|\bar{\delta}_p| + k^2 + \omega_1\bar{\delta}_1 > 0,$$

which clearly holds for any $k > 1$.

2.7 Conclusion

In this work, a robust controller was developed for an uncertain nonlinear second-order system with an additive disturbance subject to time-varying input delays. A filtered tracking error signal was designed to facilitate the control design and stability analysis. A novel Lyapunov-Krasovskii functional was used in the Lyapunov-based stability analysis to show UUB of the tracking error. The designed controller is a novel, continuous, robust controller which has explicit delay magnitude and delay derivative dependent control gain terms. Dynamic tracking experiments for the knee-shank dynamics are performed to demonstrate the applicability and the effectiveness of the PDE robust control approach. Motivated by the present results, future work will focus on extending the input delay method developed in this paper to compensate for uncertain time-varying delays.

Table 2-1. Mean RMS Error (Degrees) for Subject 1 (S1) to Subject 6 (S6) for both Right (R) and Left (L) legs.

Subject	**Leg**	**RMS Error (deg.)**
S1	R	3.44
	L	4.52
S2	R	5.03
	L	6.75
S3	R	5.86
	L	5.85
S4	R	4.72
	L	3.72
S5	R	5.27
	L	5.10
S6	R	5.73
	L	5.26
Mean		**5.10**
SD		**0.93**

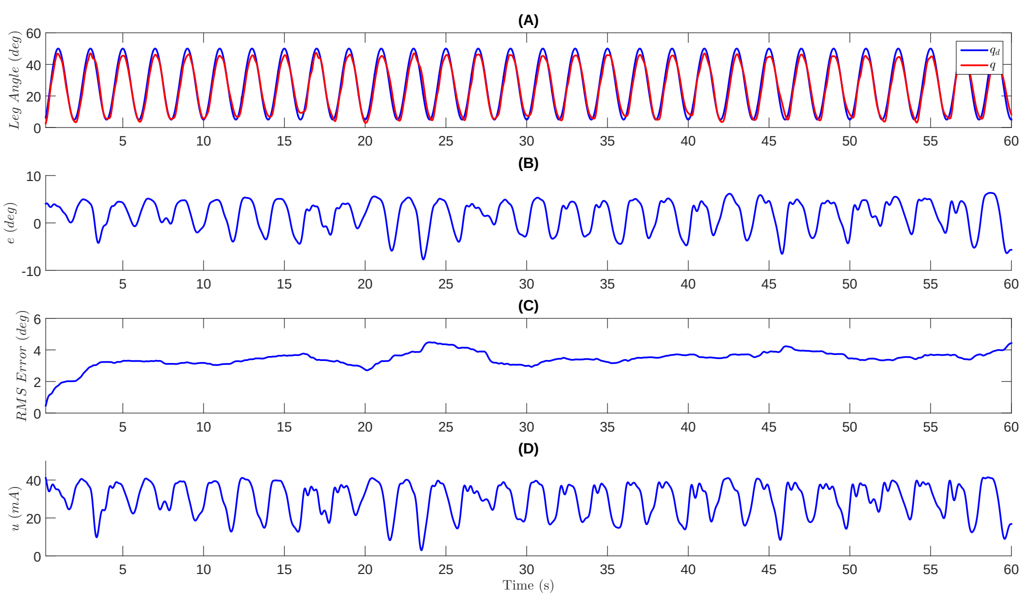

Figure 2-1. Tracking performance example taken from the right leg of subject 1 (S1-Right). Plot A includes the desired trajectory (blue solid line) and the actual leg angle (red line). Plot B illustrates the angle tracking error. Plot C depicts the RMS tracking error. Plot D depicts the control input (current amplitude in mA).

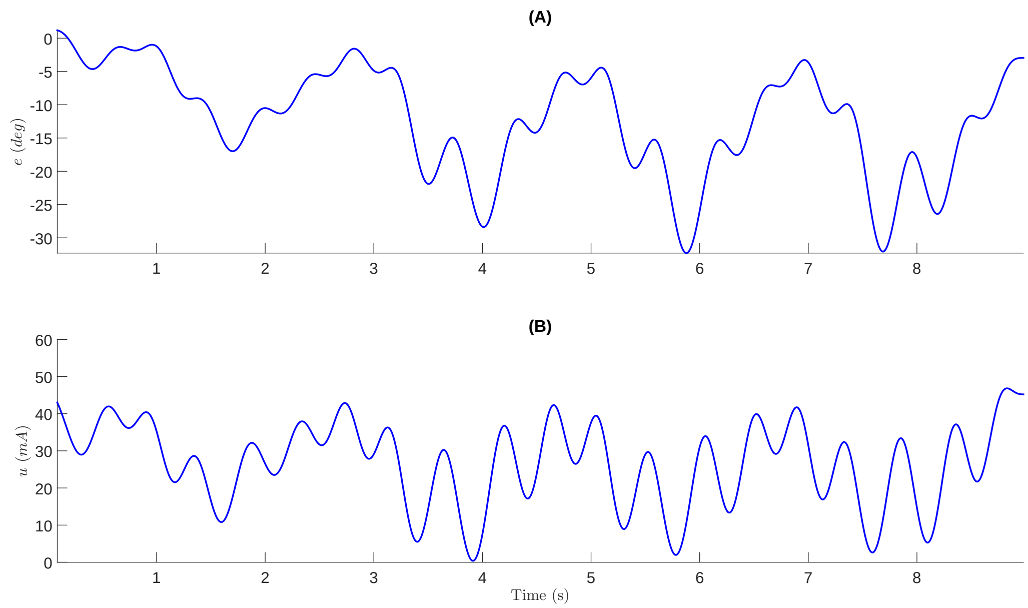

Figure 2-2. Tracking performance example taken from the right leg of subject 5 (S5-Right). Plot A illustrates the angle tracking error when $\beta = 0$ in (2–9). Plot B depicts the control input (current amplitude in mA).

CHAPTER 3
CONTROL OF AN INPUT DELAYED UNCERTAIN NONLINEAR SYSTEM WITH AN ADAPTIVE DELAY ESTIMATE

The focus of this chapter is a robust controller for an uncertain nonlinear system with bounded disturbances and an unknown time-varying input delay. The developed controller uses the linear mapping approach inspired by predictor-based approaches such as [30] to map the time dependent control input to a modified control input that depends both on time and a spatial variable. Similar to predictor-based approaches, the modified input can be segregated into delayed and delay-free components. This segregation impacts the stability analysis in a way that allows for arbitrarily large delay rates, unlike existing results (cf., [3, 8, 31, 32]). While benefiting from the added flexibility in the stability analysis resulting from the linear mapping, the controller maintains robustness to unmodeled effects. Due to challenges associated with stability analysis, previous approaches (e.g., [88]) have relied on a constant estimate of the delay, despite the fact that the delay is known to vary in time. Motivated by this fact, another contribution of this result is that a neural network (NN) estimation scheme is introduced to estimate the unknown time-varying delay magnitude. Since the universal functional approximation theorem only holds for continuous functions whose domain is compact, a nonlinear mapping is introduced to map the non-compact time domain to a compact domain. Lyapunov-Krasovskii functionals (LK) are used in the Lyapunov-based analysis to prove that the tracking errors exponentially converge to a steady-state residual that is a function of the system uncertainty (i.e., uniform ultimately bounded (UUB) tracking).

3.1 Dynamic Model

Consider a class of nonlienar systems expressed in Brunovsky canonical form, described as

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad i = 1, \ldots, n, \\ \dot{x}_{n+1} &= f\left(X\right) + d(t) + U\left(t - D(t)\right), \end{aligned} \tag{3–1}$$

where $x_i : [0, \infty) \to \mathbb{R}^m \quad i = 1, \ldots, n$ denote the system states, $X \triangleq \left[x_1^T \ldots x_n^T\right]^T$: $[0, \infty) \to \mathbb{R}^{m \times n}$, $f : \mathbb{R}^{m \times n} \to \mathbb{R}^m$ is an uncertain nonlinear function, uniformly bounded in t, $d : [0,\infty) \to \mathbb{R}^m$ is an unknown exogenous disturbance (e.g., unmodeled effects), $U : [0,\infty) \to \mathbb{R}^m$ represents the generalized input vector, and $D : [0,\infty) \to \mathbb{R}$ is an unknown, bounded, time-varying delay. A linear transformation is used to represent the generalized input $U(t)$ as a function of two independent variables, i.e. p and t, where $t \in [0,\infty)$ and $p \in [0,1]$ [30]. The spatial variable, p, denotes delayed and delay free control inputs at $p=0$ and $p=1$, respectively. The dynamic model in (3–1) can be written as

$$x_1^{(n+1)} = f\left(X\right) + d(t) + U\left(t - D(t)\right), \tag{3–2}$$

where the superscript (i) denotes the i^{th} time derivative. The two variable control input $u : [0,1] \times [t_0,\infty) \to \mathbb{R}^m$ is analogous to $U(t)$ in the sense that [30]

$$u(p,t) \triangleq U\left(\phi(t + p\left(\phi^{-1}(t) - t\right)\right) \quad \phi(t) \leq t, \forall t \geq 0, \tag{3–3}$$

where $\phi : [0,\infty) \to \mathbb{R}$ is a known delay dependent invertible monotonic time function, defined as $\phi(t) \triangleq t - \hat{D}(t)$, where $\hat{D}(t)$ represents a subsequently designed time-varying delay estimate[1] , where $\phi^{-1}(t)$ exists for all time. The transformation defined in (3–3), is used to express the delayed control input as $U(t - \hat{\tau}(t)) = u(0,t)$, and the delay-free control input as $U(t) = u(1,t)$. The spatial and time variation of $u(p,t)$, denoted by $u_p(p,t)$ and $u_t(p,t)$ respectively, can be related as

$$u_t(p,t) = \delta(p,t)u_p(p,t), \tag{3–4}$$

where the auxiliary function $\delta : [0,1] \times (0,\infty) \in \mathbb{R}$ is defined as

$$\delta(p,t) \triangleq \frac{1 + p\left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)}{\phi^{-1}(t) - t} = \frac{1 + p\left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)}{\hat{D}\left(\phi^{-1}(t)\right)}. \tag{3–5}$$

[1] The upper bounds of $\hat{D}(\phi^{-1}(t))$, and $\dot{\hat{D}}(\phi^{-1}(t))$ are the same as of $\hat{D}(t)$ and $\dot{\hat{D}}(t)$, respectively.

To facilitate the subsequent stability analysis the time derivative of $\phi^{-1}(t)$ can be determined by first substituting $t = \phi^{-1}(t)$ in the definition of $\phi(t)$ and taking the time derivative of the resulting expression

$$\begin{aligned}
t &= \phi^{-1}(t) - \hat{D}\left(\phi^{-1}(t)\right), \\
1 &= \frac{d}{dt}\left(\phi^{-1}(t)\right) - \frac{d\hat{D}\left(\phi^{-1}(t)\right)}{d\left(\phi^{-1}(t)\right)}\frac{d\left(\phi^{-1}(t)\right)}{dt}, \\
&= \frac{d\left(\phi^{-1}(t)\right)}{dt}\left(1 - \hat{D}^{*}(\phi^{-1}(t))\right), \\
\frac{d\left(\phi^{-1}(t)\right)}{dt} &= \frac{1}{1 - \hat{D}^{*}(\phi^{-1}(t))},
\end{aligned} \tag{3–6}$$

where $\hat{D}^{*}(\phi^{-1}(t)) \triangleq \frac{d\hat{D}\left(\phi^{-1}(t)\right)}{d(\phi^{-1}(t))}$. Using (3–5) and (3–6), the p variation of $\delta(p,t)$, i.e, $\delta_p(p,t)$ can be calculated as

$$\delta_p = \frac{\hat{D}^{*}(\phi^{-1}(t))}{\left(1 - \hat{D}^{*}(\phi^{-1}(t))\right)}\frac{1}{\hat{D}\left(\phi^{-1}(t)\right)}. \tag{3–7}$$

Evaluating $\delta(p,t)$ at $p = 0, 1$ and using (3–6), yields

$$\delta(0,t) \triangleq \delta_0 = \frac{1}{\hat{D}\left(\phi^{-1}(t)\right)}, \tag{3–8}$$

$$\delta(1,t) \triangleq \delta_1 = \frac{1}{\left(1 - \hat{D}^{*}(\phi^{-1}(t))\right)}\delta_0. \tag{3–9}$$

Using Assumptions 3.1 and the projection law discussed in Section 3.7, following bounds have been developed for δ_0, δ_1 and δ_p, as $\underline{\delta_0} \leq \delta_0 \leq \bar{\delta}_0$, $\underline{\delta_1} \leq \delta_1 \leq \bar{\delta}_1$, $\underline{|\delta_p|} \leq |\delta_p| \leq |\bar{\delta}_p|$, where $\underline{\delta_0}$, $\bar{\delta}_0$, $\underline{\delta_1}$, $\bar{\delta}_1$, $\underline{|\delta_p|}$ and $|\bar{\delta}_p|$ are known positive constants.

Remark 3.1. From (3–7)-(3–9) singularities can occur in δ_0, δ_1 and δ_p when $\hat{D}^{*}(\phi^{-1}(t)) = 1$, keeping in mind the singularity due to $\hat{D}(\phi^{-1}(t)) = 0$ is avoided by designing projection law as discussed in Section 3.7. From (3–4), when $\hat{D}^{*}(\phi^{-1}(t)) = 1$, for a nonzero delay magnitude $\hat{D}(\phi^{-1}(t))$, $u_p(1,t) = 0 \iff u(1,t) = u(0,t) \iff \delta_p = 0 \iff \delta_0 = \delta_1 = \frac{1}{\hat{D}(\phi^{-1}(t))}$.

Assumption 3.1. *The unknown time-varying delay $D(t) \in \mathbb{R}$ is upper and lower bounded by known positive constants $\bar{D}$ and $\underline{D}$ respectively, as $\underline{D} \leq D(t) \leq \bar{D}, \forall t$.*

Assumption 3.2. *The desired trajectory* $x_d(t) \in \mathbb{R}^m$ *is designed such that* $x_d^{(i)}(t) \in \mathbb{R}^m, \forall i = 0, 1, \ldots, (n+2)$ *exist and are bounded by known positive constants.*

3.2 Control Objective

The control objective is to develop a controller which ensures that the state x_1 of (3–2) tracks x_d, despite uncertainties and additive disturbances in the dynamics. To quantify the control objective, a tracking error, $e_1 : [0, \infty) \to \mathbb{R}^m$, is defined as

$$e_1 \triangleq x_d - x_1. \tag{3–10}$$

To facilitate the subsequent analysis, measurable auxiliary tracking errors, denoted by $e_i(t) \in \mathbb{R}^m, i = 2, 3, \ldots, n$, are defined as

$$\begin{aligned} e_2 &\triangleq \dot{e}_1 + e_1, \\ e_3 &\triangleq \dot{e}_2 + e_2 + e_1, \\ &\vdots \\ e_n &\triangleq \dot{e}_{n-1} + e_{n-1} + e_{n-2}. \end{aligned} \tag{3–11}$$

A general expression for $e_i(t), \quad i = 2, 3, \ldots, n$ can be written as

$$e_i = \sum_{j=0}^{i-1} a_{i,j} e_1^{(j)}, \tag{3–12}$$

where $a_{i,j} \in \mathbb{R}$, are known coefficients, calculated using the definition of Fibonacci sequences, with $a_{n,(n-1)} = 1$. An auxiliary tracking error signal, $r : [0, \infty) \to \mathbb{R}^m$, is defined as

$$r \triangleq \dot{e}_n + \alpha e_n - \beta e_u, \tag{3–13}$$

where $\alpha, \beta \in \mathbb{R}$ are known, positive, constant gains. In (3–13), $e_u : [0, \infty) \to \mathbb{R}^m$ is an auxiliary error term, introduced to obtain a delay-free control expression for the input in the closed-loop error system and is defined as

$$e_u \triangleq \int_0^1 u(p,t)\, dp. \tag{3–14}$$

To calculate e_u, $\dot{e}_u$ needs to be measured and integrated over the time domain $[0,t]$. Using the Leibnitz integral rule, and differentiating (3–14) with respect to time gives $\dot{e}_u$ as

$$\dot{e}_u = \delta_1 u(1,t) - \delta_0 u(0,t) - \delta_p e_u. \tag{3–15}$$

Given an initial condition for e_u, (3–15) can be used to compute e_u and $\dot{e}_u$.

3.3 Development of Error Signals

The open-loop error system for $r(t)$ can be obtained by taking the time derivative of (3–13) and using (3–2), (3–10), (3–12) and (3–15) as

$$\dot{r} = x_d^{(n+1)} - f(X) - d(t) - U(t - D(t)) + \alpha \dot{e}_n + \delta_p \beta e_u + \sum_{j=0}^{n-2} a_{n,j} e_1^{(j+2)}$$
$$-\delta_1 \beta u(1,t) + \delta_0 \beta u(0,t). \tag{3–16}$$

The open-loop error system in (3–16) contains both a delayed and delay-free control input, since the time derivative of (3–14) is used in (3–16). Based on the subsequent stability analysis, the delay-free control input is designed as

$$U(t) = u(1,t) \triangleq \frac{1}{k} r, \tag{3–17}$$

where $k \in \mathbb{R}^+$ is a constant, adjustable control gain. To facilitate the subsequent stability analysis, (3–16) can be segregated into terms that can be upper bounded by a state-dependent function and terms which can be upper bounded by a known constant as

$$\dot{r} = \tilde{N} + N_d - e_n - \delta_1 \beta u(1,t) + \delta_0 \beta u(0,t) - U(t - D(t)) + \delta_p \beta e_u. \tag{3–18}$$

In (3–18), $\tilde{N}\left(X, X_d, e_n, \dot{e}_n, e_1^{(1)}, e_1^{(2)}, \ldots, e_1^{(n)}\right) \in \mathbb{R}^m$ is an auxiliary term defined as

$$\tilde{N} \triangleq -f(X) + f(X_d) + e_n + \alpha \dot{e}_n + \sum_{j=0}^{n-2} a_{n,j} e_1^{(j+2)}, \tag{3–19}$$

where $X_d(t) \triangleq [x_d^T, 0, \ldots 0]^T$. Using the Mean Value Theorem, and Assumption 3.2, $\tilde{N}\left(X, X_d, e_n, \dot{e}_n, e_1^{(1)}, e_1^{(2)}, \ldots, e_1^{(n)}\right)$ in (3–19) can be upper bounded as

$$\left\|\tilde{N}\right\| \leq \rho\left(\|z\|\right)\|z\|, \tag{3–20}$$

where $\rho : \mathbb{R} \to \mathbb{R}$ is a known positive definite, non-decreasing, radially unbounded function, and $z \in \mathbb{R}^{m(n+2)}$ is a vector of error signals, defined as

$$z \triangleq \left[\begin{array}{ccccc} e_1^T & e_2^T \ldots e_n^T & r^T & e_u^T \end{array}\right]^T. \tag{3–21}$$

Also in (3–16), $N_d\left(X_d, x_d^{(n+1)}, d\right) \in \mathbb{R}^m$ is an auxiliary term defined as

$$N_d \triangleq -f(X_d) + x_d^{(n+1)} - d(t). \tag{3–22}$$

Using Assumptions 3.1 and 3.2, $N_d\left(X_d, x_d^{(n+1)}, d\right)$ can be upper bounded as

$$\sup_{t \in [0,\infty)} \|N_d\| \leq \Theta, \tag{3–23}$$

where $\Theta \in \mathbb{R}^+$ is a known constant. Substituting (3–17) into the open-loop error system in (3–18), the closed-loop error system can be obtained as

$$\dot{r} = \tilde{N} + N_d - e_n - \frac{\delta_1 \beta r}{k} + \delta_0 \beta u(0,t) - U\left(t - D(t)\right) + \delta_p \beta e_u. \tag{3–24}$$

3.4 Estimation of Delay

A neural network (NN) based function approximator is used to estimate the unknown delay magnitude. The universal function approximation theorem only holds over a compact domain. Therefore, to approximate the unknown delay function, a nonlinear mapping is defined to map the non-compact domain to a compact spatial domain. Let $f_L : t \to \xi$ be defined as

$$f_L \triangleq \frac{\kappa t}{1+\kappa t}, \quad t \in [0,\infty), \quad \xi \in [0,1], \tag{3–25}$$

where $\kappa \in \mathbb{R}^+$ is a user defined saturation coefficient. Using (3–25), $D(t)$ can be mapped into the domain ξ as

$$D(t) = D\left(f_L^{-1}(\xi)\right) \triangleq D_{f_L}(\xi). \tag{3–26}$$

The universal functional approximation theorem can be used to represent $D_{f_L}(\xi)$ by a three-layer NN as

$$D_{f_L}(\xi) \triangleq W^T \sigma\left(V^T \Xi\right) + \epsilon, \tag{3–27}$$

where $V \in \mathbb{R}^{2\times L}$ and $W \in \mathbb{R}^{(L+1)\times 1}$ are the unknown bounded unknown constant ideal weights for the first-to-second and second-to-third layers, respectively, L is the number of neurons in the hidden layer, $\sigma \in \mathbb{R}^{(L+1)}$ is activation function, ϵ is the functional reconstruction error, and $\Xi = [1 \quad \xi]^T$ denotes the input to the NN. Based on (3–26), the NN estimation for $\hat{\tau}(t)$ is given by

$$\hat{D}(t) = \hat{W}^T \sigma\left(\hat{V}^T \Xi\right), \tag{3–28}$$

where $\hat{W}$ and $\hat{V}$ are estimates of the ideal weights. In (3–28), σ is selected as a saturated activation function (i.e., $\log\mathrm{sig}$, $\tanh$), in order to simplify the development of projection law, discussed in Section 3.7. Using (3–27) and (3–28), the mismatch between $D(t)$ and $\hat{D}(t)$ can be obtained using a Taylor's series approximation, which after some algebraic manipulation, can be expressed as

$$\begin{aligned} D(t) - \hat{D}(t) &= W^T \sigma\left(V^T \Xi\right) - \hat{W}^T \sigma\left(\hat{V}^T \Xi\right) + \epsilon, \\ &= \tilde{W}^T \sigma\left(\hat{V}^T \Xi\right) + \hat{W}^T \sigma'\left(\hat{V}^T \Xi\right) \tilde{V}^T \Xi \\ &+ W^T \mathcal{O}\left(\tilde{V}^T \Xi\right)^2 + \epsilon + \tilde{W} \sigma'\left(\hat{V}^T \Xi\right) \tilde{V}^T \Xi, \end{aligned} \tag{3–29}$$

where $\tilde{W} = W - \hat{W} \in \mathbb{R}^{(L+1)\times 1}$ and $\tilde{V} = V - \hat{V} \in \mathbb{R}^{2\times L}$, are the estimate mismatch for the ideal weight matrices, and $\mathcal{O}$ represents higher order terms. As mentioned before, due to the development of projection law in Section 3.7, the elements of $\hat{W}$ and $\hat{V}$ can all be upper and lower bounded by known positive constants. Hence, $\tilde{W}^T\sigma'\left(\hat{V}^T\Xi\right)\tilde{V}^T\Xi$ and $W^T\mathcal{O}\left(\tilde{V}^T\Xi\right)^2$ can also be bounded by known positive constants, and therefore,

$$D(t) - \hat{D}(t) \leq \tilde{W}^T\sigma\left(\hat{V}^T\Xi\right) + \hat{W}^T\sigma'\left(\hat{V}^T\Xi\right)\tilde{V}^T\Xi + \bar{\epsilon}, \tag{3–30}$$

where $\bar{\epsilon} \in \mathbb{R}$ is a positive bounding constant.

3.5 Stability Analysis

To facilitate the stability analysis, let $y \in \mathbb{R}^{m(n+2)+1}$ be defined as

$$y \triangleq \begin{bmatrix} z^T & \sqrt{Q} \end{bmatrix}^T, \tag{3–31}$$

where $Q \in \mathbb{R}$ denotes an LK functional defined as

$$Q \triangleq \lambda_Q \int_0^1 e^{\omega_2 p}\|u(p,t)\|^2 dp, \tag{3–32}$$

where $\lambda_Q, \omega_2 \in \mathbb{R}$ are known, positive constants. Let $\mathscr{D}$ be an open and connected set, and $\mathcal{S}_{\mathscr{D}} \subset \mathscr{D}$ is defined as

$$\mathcal{S}_{\mathscr{D}} \triangleq \left\{ y \in \mathbb{R}^{3n+1} |\, \|y\| < \inf\left\{\rho^{-1}\left([\sqrt{\frac{\lambda_1\epsilon_1}{2}}, \infty)\right)\right\}\right\}, \tag{3–33}$$

where ϵ_1 and $\lambda_1 \in \mathbb{R}$ are known, positive constants.

Theorem 3.1. *Given the open-loop error system in (3–16), the controller in (3–17) ensures UUB tracking in the sense that*

$$\|e_1\| \leq \mathit{\Gamma}_0 \exp\left(-\mathit{\Gamma}_1 t\right) + \mathit{\Gamma}_2, \tag{3–34}$$

where $\mathit{\Gamma}_0$,$\mathit{\Gamma}_1$ and $\mathit{\Gamma}_2$ are known positive constants, provided that $y(\eta) \in \mathcal{S}_{\mathscr{D}}, \forall \eta \in [t_0, t_0 + \bar{D}]$.

Proof. Let $V_L : \mathscr{D} \times [t_0,\ \infty) \to \mathbb{R}$ be a continuously differentiable, positive-definite function defined as

$$V_L \triangleq \frac{1}{2}\sum_{i=1}^{n} e_i^T e_i + \frac{1}{2}r^T r + \frac{\omega_1}{2}e_u^T e_u + Q + \frac{1}{2}tr\left(\tilde{W}^T \Delta_1^{-1}\tilde{W}\right) + \frac{1}{2}tr\left(\tilde{V}^T \Delta_2^{-1}\tilde{V}\right), \tag{3–35}$$

where $\Phi_1 \left\|y\right\|^2 + c_L \leq V_L \leq \Phi_2 \left\|y\right\|^2 + c_U,\ c_L, c_U \in \mathbb{R}^+$ are known bounding constants. Taking the time derivative of (3–35) and using (3–11)-(3–14) and (3–24), yields

$$\begin{aligned}\dot{V}_L = {} & r^T\left(\tilde{N} + N_d - e_n\right) - e_n^T\beta e_u + e_n^T r + r^T\left(-U\left(t - D(t)\right)\right) - \lambda_Q\omega_2\int_0^1 \delta e^{\omega_2 p}\|u\|^2 dp \\ & + r^T\left(\frac{-\delta_1\beta r}{k} + \delta_0\beta u(0,t) + \delta_p\beta e_u\right) - \sum_{i=1}^{n-1} e_i^T e_i - e_n^T\alpha e_n + e_{n-1}^T e_n \\ & + \omega_1 e_u^T\left(\frac{\delta_1 r}{k} - \delta_0 u(0,t) - \delta_p e_u\right) + \lambda_Q\delta_1 e^{\omega_2}\|u(1,t)\|^2 - \lambda_Q\delta_0||u(0,t)||^2 \\ & - \lambda_Q\delta_p\int_0^1 e^{\omega_2 p}\|u\|^2 dp + tr\left(\tilde{W}^T\Delta_1^{-1}\dot{\tilde{W}}\right) + tr\left(\tilde{V}^T\Delta_2^{-1}\dot{\tilde{V}}\right).\end{aligned} \tag{3–36}$$

By using (3–17), (3–20), (3–23), and the facts that $\dot{\tilde{W}} = -\dot{\hat{W}}$ and $\dot{\tilde{V}} = -\dot{\hat{V}}$, an upper bound on $\dot{V}_L$ can be obtained as

$$\begin{aligned}\dot{V}_L \leq {} & \|r\|\rho\left(\|z\|\right)\|z\| + \|r\|\Theta + \left(\bar{\delta}_0\beta + 1\right)\|r^T u(0,t)\| - \sum_{i=1}^{n-1} e_i^T e_i - \alpha\|e_n\|^2 \\ & - \frac{\underline{\delta_1}\beta\|r\|^2}{k} + |r^T\left(u(0,t) - U\left(t - D(t)\right)\right)| + |\bar{\delta}_p|\beta\|r^T e_u\| + \|e_{n-1}^T e_n\| \\ & + \beta\|e_n^T e_u\| + \omega_1|\bar{\delta}_p|\|e_u\|^2 + \omega_1\bar{\delta}_1\|\frac{e_u^T r}{k}\| + \omega_1\bar{\delta}_0\|e_u^T u(0,t)\| \\ & - \lambda_Q\omega_2\int_0^1 \delta e^{\omega_2 p}\|u\|^2 dp + \lambda_Q|\bar{\delta}_p|\int_0^1 e^{\omega_2 p}\|u\|^2 dp - tr\left(\tilde{W}^T\Delta_1^{-1}\dot{\hat{W}}\right) \\ & + \frac{\lambda_Q\bar{\delta}_1 e^{\omega_2}\|r\|^2}{k^2} - \lambda_Q\underline{\delta_0}||u(0,t)||^2 - tr\left(\tilde{V}^T\Delta_2^{-1}\dot{\hat{V}}\right).\end{aligned} \tag{3–37}$$

To facilitate the subsequent analysis, note that

$$\delta\left(p,t\right) = \frac{1 + p\left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)}{\phi^{-1}(t) - t} \geq \min\left\{\delta_0, \delta_1\right\}. \tag{3–38}$$

Using the Cauchy-Schwarz inequality,

$$\|e_u\|^2 \leq \int_0^1 \|u^2(p,t)\| dp \int_0^1 1 dp,$$
$$\|e_u\|^2 \leq \int_0^1 \|u(p,t)\|^2 dp. \tag{3–39}$$

Using the Mean Value Theorem and the expression in (3–30), the following inequalities can be developed:

$$\begin{aligned} |r^T(u(0,t) - U(t-\tau(t)))| &\leq |r^T\left(D(t) - \hat{D}(t)\right)\dot{u}(0,t)| \\ &\leq M|r^T\tilde{W}^T\sigma\left(\hat{V}^T\Xi\right)| + M|r^T\hat{W}^T\sigma'\left(\hat{V}^T\Xi\right)\tilde{V}^T\Xi| \\ &+ M|r^T\bar{\epsilon}|. \end{aligned} \tag{3–40}$$

The inequalities in (3–38)-(3–40), can be used to upper bound (3–37) as

$$\begin{aligned} \dot{V}_L \leq{}& \frac{1}{\epsilon_1}\rho^2(\|z\|)\|z\|^2 + \frac{1}{\epsilon_2}\left(\Theta^2 + M\bar{\epsilon}^2\right) - tr\left(\tilde{W}^T\Delta_1^{-1}\dot{\hat{W}}\right) - \left(\alpha - \frac{1}{2} - \frac{\beta}{2k}\right)||e_n||^2 \\ &+M|r^T\left[\tilde{W}^T\sigma\left(\hat{V}^T\Xi\right) + \hat{W}^T\sigma'\left(\hat{V}^T\Xi\right)\tilde{V}^T\Xi\right]| - tr\left(\tilde{V}^T\Delta_2^{-1}\dot{\hat{V}}\right) \\ &-\sum_{i=1}^{n-2} e_i^Te_i - \frac{1}{2}\|e_{n-1}\|^2 - \left(\frac{\delta_1\beta}{k} - \frac{|\bar{\delta_p}|\beta\alpha^2\epsilon}{2k} - \frac{\omega_1\bar{\delta_1}}{2k^3}\right)\|r\|^2 \\ &+\left(\frac{\lambda_Q\bar{\delta_1}e^{\omega_2}}{k^2} + \frac{(\epsilon_1+\epsilon_2)}{4} + \frac{M\epsilon_2}{4}\right)\|r\|^2 + \left(\frac{\epsilon}{2k} + \frac{\bar{\delta_0}\epsilon}{2k}\right)\|r\|^2 \\ &-\left(\lambda_Q\underline{\delta_0} - \frac{\omega_1\bar{\delta_0}k}{2\epsilon}\right)\|u(0,t)\|^2 + \left(\frac{k\bar{\delta_0}\beta^2\epsilon}{2} + \frac{k}{2\epsilon}\right)\|u(0,t)\|^2 \\ &-\left(\frac{\lambda_Q\omega_2\min\{\delta_0,\delta_1\}}{2} - \lambda_Q|\bar{\delta_p}|\right)Q - \left(\frac{\lambda_Q\omega_2\min\{\delta_0,\delta_1\}}{2} - \frac{|\bar{\delta_p}|\beta k}{2\epsilon\alpha^2} - \omega_1|\bar{\delta_p}|\right)\|e_u\|^2 \\ &+\left(\frac{\omega_1\bar{\delta_1}k}{2} + \frac{\omega_1\bar{\delta_0}\epsilon}{2k} + \frac{k\beta}{2}\right)||e_u||^2. \end{aligned} \tag{3–41}$$

Using the fact that $a^Tb = trace(ba^T)$, $\dot{\hat{W}}(t)$ and $\dot{\hat{V}}(t)$ are designed to cancel cross terms as

$$\dot{\hat{W}} = proj\left(\Delta_1 M\sigma\left(\hat{V}^T\Xi\right)r^T, \hat{W}\right), \tag{3–42}$$

$$\dot{\hat{V}} = \Delta_2 M\Xi r^T\hat{W}^T\sigma'\left(\hat{V}^T\Xi\right). \tag{3–43}$$

The function $proj(.,.)$ in (3–42) denotes a Lipschitz continuous (discussed in Section 3.7) projection operator, which ensures $\underline{D} + \epsilon_l \leq \hat{D}(t) \leq \bar{D} + \epsilon_u$, where $\epsilon_l, \epsilon_u \in \mathbb{R}$ are subsequently defined positive constants (see Section 3.8). Since $\|y\| \geq \|z\|$, the following upper bound can be obtained

$$\dot{V}_L \leq -\left(\frac{\lambda_1}{2} - \frac{1}{\epsilon_1}\rho^2\left(\|y\|\right)\right)\|z\|^2 + \frac{1}{\epsilon_2}\left(\Theta^2 + M\bar{\epsilon}^2\right) - \frac{\lambda_1}{2}||z||^2 - \lambda_{Q_{res}}\|u(0,t)\|^2 - \left(\frac{\omega_2 \min\left\{\delta_0, \delta_1\right\}}{2} - |\bar{\delta}_p|\right)Q, \tag{3–44}$$

where $\lambda_1, \mu, \zeta, \lambda_{Q_{res}} \in \mathbb{R}$ are defined as

$$\lambda_1 \triangleq \min\left\{\left(\alpha - \frac{1}{2} - \frac{\beta}{2k}\right), \frac{1}{2}, \zeta, \mu\right\}, \tag{3–45}$$

$$\mu \triangleq \frac{\delta_1\beta}{k} - \frac{|\bar{\delta_p}|\beta\alpha^2\epsilon}{2k} - \frac{\omega_1\bar{\delta_1}}{2k^3} - \frac{M\epsilon_2}{4} - \frac{\lambda_Q\bar{\delta}_1 e^{\omega_2}}{k^2} - \frac{(\epsilon_1+\epsilon_2)}{4} - \frac{\epsilon}{2k} - \frac{\bar{\delta}_0\epsilon}{2k}, \tag{3–46}$$

$$\zeta \triangleq \frac{\lambda_Q\omega_2\min\left\{\delta_0, \delta_1\right\}}{2} - \frac{|\bar{\delta_p}|\beta k}{2\epsilon\alpha^2} - \omega_1|\bar{\delta_p}| - \frac{\omega_1\bar{\delta}_1 k}{2} - \frac{\omega_1\bar{\delta}_0\epsilon}{2k} - \frac{k\beta}{2}, \tag{3–47}$$

$$\lambda_{Q_{res}} \triangleq \lambda_Q\underline{\delta_0} - k\left(\frac{\bar{\delta}_0\omega_1 + 1}{2\epsilon} + \frac{\bar{\delta}_0\beta^2\epsilon}{2}\right). \tag{3–48}$$

Provided $y(\eta) \in \mathcal{S}_{\mathscr{D}}, \forall\eta \in [t-\bar{D}, t]$, the expression in (3–44) reduces to

$$\dot{V}_L \leq -\lambda_2\left\|y\right\|^2 + \varphi, \tag{3–49}$$

where $\varphi, \lambda_2 \in \mathbb{R}$ are defined as

$$\varphi \triangleq \frac{1}{\epsilon_2}\left(\Theta^2 + M\bar{\epsilon}^2\right), \tag{3–50}$$

$$\lambda_2 \triangleq \min\left\{\frac{\lambda_1}{2}, \frac{\omega_2\min\left\{\delta_0, \delta_1\right\}}{2} - |\bar{\delta}_p|\right\}. \tag{3–51}$$

An upper bound can be obtained for (3–49) as

$$\dot{V}_L \leq -\frac{\lambda_2}{\Phi_2}V_L + \frac{c_U}{\Phi_2} + \varphi. \tag{3–52}$$

The solution of the inequality in (3–52) can be obtained as

$$V_L(t) \leq V_L\left(0\right)\exp\left(-\frac{\lambda_2}{\Phi_2}t\right) + \frac{\Phi_2\varphi + c_U}{\lambda_2}\left(1 - \exp\left(-\frac{\lambda_2}{\Phi_2}t\right)\right). \tag{3–53}$$

Using (3–35) and (3–53), the inequality in (3–34) can be obtained. Since $e_1^{(i)}(t)$, $r(t)$, $e_u(t) \in \mathcal{L}_\infty$, (3–17) can be used to conclude that $u(t) \in \mathcal{L}_\infty$. $\square$

3.6 Simulation Results

A numerical simulation was performed to examine the performance of the controller in (3–17) along with the NN weights update laws described in (3–42) and (3–43). For the simulation the following second order system was used

$$\begin{bmatrix} U_1(t-D(t)) \\ U_2(t-D(t)) \end{bmatrix} = \begin{bmatrix} p_1 + 2p_3c_2 & p_2 + p_3c_2 \\ p_2 + p_3c_2 & p_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}$$
$$+ \begin{bmatrix} -p_3s_2\dot{x}_2 & -p_3s_2\left(\dot{x}_1 + \dot{x}_2\right) \\ p_3s_2\dot{x}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} f_{d1} & 0 \\ 0 & f_{d2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}. \tag{3–54}$$

In (3–54), d_1, d_2 represent added disturbances defined as $d_1 = 0.2\sin(0.5t)$ and $d_2 = 0.1\sin(0.25t)$. Additionally, $p_1 = 3.473$ kg $\cdot$ m^2, $p_2 = 0.196$ kg $\cdot$ m^2, $p_3 = 0.242$ kg $\cdot$ m^2, $p_4 = 0.238$ kg $\cdot$ m^2, $p_5 = 0.146$ kg $\cdot$ m^2, $f_{d1} = 5.3$ Nm $\cdot$ sec, $f_{d2} = 1,1$ Nm $\cdot$ sec, and s_2, c_2 denote $\sin(x_2)$, and $\cos(x_2)$, respectively.

The initial conditions for the system were selected as $x_1(0), x_2(0) = 0$. The desired trajectories were selected as

$$x_{d1}(t) = (30\sin(1.5t) + 20)\left(1 - e^{-0.01t^3}\right),$$
$$x_{d2}(t) = -(20\sin(t/2) + 10)\left(1 - e^{-0.01t^3}\right).$$

The dynamics described in (3–54), is simulated for time varying delay magnitude of $D(t) = 0.04\tanh(\frac{t}{30}) + 0.08$, where $0.08 \leq D(t) \leq 0.12$. A neural network of two hidden layers with 5 neurons in each layer is used for the delay estimation. First hidden layer has a sigmoid activation function and the second hidden layer has a linear activation function . Update laws developed in (3–42) and (3–43) are used to update neural network weights in each training iteration. Figure 3-1 shows the tracking error variation and the control force variation for the 2-link robot dynamics. Using the auxiliary tracking error data, available in discrete time from the simulation, NN weights are trained using 90 percent available training data, and tested on remaining 10 percent available simulated data. In order to enforce learning, k-fold method is used while training the designed network, for delay magnitude $D(t)$ estimation, with $k = 10$. Figure shows the percentage variation of delay mismatch, with training iteration, which indicates a successful training while estimating unknown delay magnitude using the update laws in (3–42) and (3–43).

3.7 Projection Law

Based on Assumption , unknown time-varying delay $D(t)$ belong to the compact convex set $\Omega := \{D(t) : \underline{D} \leq D(t) \leq \bar{D}\}$, where $\underline{D}$, $\bar{D} \in \mathbb{R}$ are known positive constants. The standard Lipschitz continuous projection operator (e.g., [89, 90]), introduced in (3–42) is given by

$$\dot{\hat{W}} = proj\left(\upsilon, \hat{W}\right)$$

$$\dot{\hat{W}} = \begin{cases} \upsilon, & \text{if } p_{j_{low}}\left(\hat{D}\right) \geq 0 \text{ or } p_{j_{high}}\left(\hat{D}\right) \leq 0 \\ \upsilon, & \text{if } p_{j_{low}}\left(\hat{D}\right) \leq 0 \text{ and } \nabla p_{j_{low}}\left(\hat{D}\right)^T \upsilon \geq 0 \\ \upsilon, & \text{if } p_{j_{high}}\left(\hat{D}\right) \geq 0 \text{ and } \nabla p_{j_{high}}\left(\hat{D}\right)^T \upsilon \leq 0 \\ \left(I - \underline{\varsigma}\right)\upsilon, \text{if } p_{j_{low}}\left(\hat{D}\right) \leq 0 \text{ and } \nabla p_{j_{low}}\left(\hat{D}\right)^T \upsilon < 0 \\ \left(I - \bar{\varsigma}\right)\upsilon, \text{if } p_{j_{high}}\left(\hat{D}\right) \geq 0 \text{ and } \nabla p_{j_{high}}\left(\hat{D}\right)^T \upsilon > 0 \end{cases}$$

where $\underline{\varsigma} \triangleq \frac{p_{j_{low}}(\hat{D})\nabla p_{j_{low}}(\hat{D})\nabla p_{j_{low}}(\hat{D})^T}{\nabla p_{j_{low}}\left(\hat{D}\right)^T \nabla p_{j_{low}}\left(\hat{D}\right)} \in \mathbb{R}$, $\bar{\varsigma} \triangleq \frac{p_{j_{high}}(\hat{D})\nabla p_{j_{high}}(\hat{D})\nabla p_{j_{high}}(\hat{D})^T}{\nabla p_{j_{high}}\left(\hat{D}\right)^T \nabla p_{j_{high}}\left(\hat{D}\right)} \in \mathbb{R}$, $\upsilon \triangleq \Delta_1 M\sigma\left(\hat{V}^T\Xi\right) r^T \in \mathbb{R}$, $p_{j_{low}}\left(\hat{D}\right) \triangleq \frac{\hat{D}^T\hat{D} - \underline{D}^2}{\epsilon_l^2 + 2\epsilon_l \underline{D}} \in \mathbb{R}$, $p_{j_{high}}\left(\hat{D}\right) \triangleq \frac{\hat{D}^T\hat{D} - \bar{D}^2}{\epsilon_u^2 + 2\epsilon_u \bar{D}} \in \mathbb{R}$, $\epsilon_l, \epsilon_u \in \mathbb{R}$ are positive constants, and ∇ is the gradient operator. Given $\hat{D}(0) \in \Omega$, the projection operator mentioned in (3–42) has the properties, $\underline{D} + \epsilon_l \leq \hat{D}(t) \leq \bar{D} + \epsilon_u$ and $proj\left(\upsilon, \hat{W}\right)$ is Lipschitz continuous. Lipschitz continuity of $\dot{\hat{W}}$, along with the continuous update law of $\dot{\hat{V}}$ in (3–43), guarantee the existence of $\dot{\hat{D}}(t)$, which gives a continuous estimate of unknown delay $\hat{D}(t)$ at all time.

3.8 Control Gain Selection

Control gains, such as λ_1 in (3–45), μ in (3–46), ζ in (3–47), $\lambda_{Q_{res}}$ in (3–48) and λ_2 in (3–51), introduced in the stability analysis (Section 3.5) are required to be positive constants. Based on the designed bounds of time-delay estimate $\hat{\tau}$ in this section, and subsequently derived bounds of δ_0, δ_1 and δ_p (i.e., $\underline{\delta_0}$, $\bar{\delta}_0$, $\underline{\delta_1}$, $\bar{\delta}_1$, $|\bar{\delta}_p|$), this appendix develops sufficient gain conditions to ensure λ_1, μ, ζ, $\lambda_{Q_{res}}$ and $\lambda_2 > 0$. Using the definitions of λ_1 in (3–45) and λ_2 in (3–51), sufficient lower bounds for α and ω_2 can be obtained as

$$\alpha > \frac{\beta + k}{2k}, \tag{3–55}$$

$$\omega_2 > \frac{2|\bar{\delta}_p|}{\min\{\delta_0, \delta_1\}}. \tag{3–56}$$

Using the definition of $\lambda_{Q_{res}}$ in (3–48), it is clear that for an arbitrarily large k and β and arbitrarily small ω_1 and ϵ, that λ_Q can be selected sufficiently large as

$$\lambda_Q > \frac{k}{\underline{\delta_0}}\left(\frac{\bar{\delta}_0\omega_1 + 1}{2\epsilon} + \frac{\bar{\delta}_0\beta^2\epsilon}{2}\right), \tag{3–57}$$

to ensure that $\lambda_{Q_{res}} > 0$. Also from the definition of ζ in (3–47), λ_Q also needs to satisfy the following inequality

$$\lambda_Q > \frac{2}{\omega_2 \min\{\delta_0, \delta_1\}}\left(\frac{|\bar{\delta}_p|\beta k}{2\epsilon\alpha^2} + \omega_1|\bar{\delta}_p| + \frac{\omega_1\bar{\delta}_1 k}{2} + \frac{\omega_1\bar{\delta}_0\epsilon}{2k} + \frac{k\beta}{2}\right) \tag{3–58}$$

to ensure $\zeta > 0$. By selecting α on the order of k^2, β on the order of k^3, ϵ on the order of $\frac{1}{k^4}$, and ω_1 on the order of $\frac{1}{k^5}$, the lower bound in (3–57) can be proven to be larger than (3–58). For example, if $\alpha = k^2$, $\beta = k^3$, $\epsilon = \frac{1}{k^4}$ and $\omega_1 = \frac{1}{k^5}$ then

$$\frac{1}{\underline{\delta_0}}\left(\frac{k^5}{2} + \frac{\bar{\delta}_0 k^3}{2} + \frac{\bar{\delta}_0}{2}\right) > \frac{2}{\omega_2 \min\{\delta_0, \delta_1\}}\left(\frac{|\bar{\delta}_p|k^4}{2} + \frac{|\bar{\delta}_p|}{k^5} + \frac{\bar{\delta}_1}{2k^4} + \frac{\bar{\delta}_0}{2k^{10}} + \frac{k^4}{2}\right), \tag{3–59}$$

for large values of $k > 1$ and ω_2.

From (3–46), λ_Q is multiplied by a negative term in the definition of μ. To develop a sufficient condition to ensure $\mu > 0$, the lower bound for λ_Q in (3–57) is substituted into

(3–46) to develop the following sufficient inequality:

$$\beta\left(\frac{\underline{\delta_1}}{k}-\frac{|\bar{\delta}_p|\epsilon\alpha^2}{2k}-\frac{\bar{\delta}_0\alpha\epsilon\bar{\delta}_1e^{\omega_2}}{2\underline{\delta_0}}\right)>\frac{(\epsilon_1+\epsilon_2)}{4}+\frac{\epsilon}{2k}+\frac{\bar{\delta}_0\epsilon}{2k}$$
$$+\left(\frac{\omega_1\bar{\delta}_0+1}{2\epsilon\underline{\delta_0}}\right)\frac{\bar{\delta}_1e^{\omega_2}}{k}+\frac{\omega_1\bar{\delta}_1}{2k^3}+\frac{M\epsilon_2}{4}. \tag{3–60}$$

Based on (3–60), a sufficient condition for the upper bound on ϵ can be established as

$$\epsilon<\frac{2\underline{\delta_1}\underline{\delta_0}}{|\bar{\delta}_p|\underline{\delta_0}\alpha^2+\bar{\delta}_0\alpha k\bar{\delta}_1e^{\omega_2}}, \tag{3–61}$$

to ensure the parenthetical terms on the left side of (3–60) are positive. Based on (3–60) and (3–61) a sufficient lower bound for β can be established for an arbitrarily large k, ϵ_2, ω_2 and arbitrarily small ω_1 and ϵ, to ensure $\mu>0$ as

$$\beta>\frac{\frac{(\epsilon_1+\epsilon_2)}{4}+\frac{\epsilon}{2k}+\frac{\bar{\delta}_0\epsilon}{2k}+\left(\frac{\omega_1\bar{\delta}_0+1}{2\epsilon\underline{\delta_0}}\right)\frac{\bar{\delta}_1e^{\omega_2}}{k}+\frac{\omega_1\bar{\delta}_1}{2k^3}+\frac{M\epsilon_2}{4}}{\frac{\underline{\delta_1}}{k}-\frac{|\bar{\delta}_p|\epsilon\alpha^2}{2k}-\frac{\bar{\delta}_0\alpha\epsilon\bar{\delta}_1e^{\omega_2}}{2\underline{\delta_0}}}. \tag{3–62}$$

To obtain the sufficient condition in (3–59) and to satisfy the sufficient conditions in (3–57), (3–61), and (3–62), ϵ_2, α, λ_Q and β are selected sufficiently large and ϵ and ω_1 are selected sufficiently small. For example, selecting $\alpha=k^2$, $\beta=k^3$ and $\epsilon=\frac{1}{k^4}$, as previously, so that

$$k^6\left(\frac{\omega_1\bar{\delta}_0+1}{2\underline{\delta_0}}\bar{\delta}_1e^{\omega_2}+\frac{\bar{\delta}_0}{2}\right)+k^5\left(\frac{\bar{\delta}_0\bar{\delta}_1e^{\omega_2}}{2\underline{\delta_0}}+\frac{1}{2}+\frac{(\epsilon_1+\epsilon_2)}{4}+\frac{M\epsilon_2}{4}-\underline{\delta_1}\right)$$
$$+\frac{k^3|\bar{\delta}_p|}{2}+\frac{\omega_1\bar{\delta}_1}{2}>0,$$

which clearly holds for any $k>1$, then the sufficient condition in (3–62) is satisfied.

3.9 Conclusion

For a class of uncertain nonlinear systems subject to unknown time-varying input delay, a tracking controller is designed where the control input varies with both time and a spatial variable. The designed controller features gains to compensate for the delay and the delay derivative independently. Due to the separation of the delay term outside the control input, a NN-based estimation scheme is used to estimate the unknown input delay magnitude. A nonlinear mapping is used to transform the non-compact time interval to a compact set to facilitate the use of a NN. A Lyapunov-Krasovskii functional is used in the Lyapunov-based stability analysis to prove uniform ultimate boundedness of the error signals.

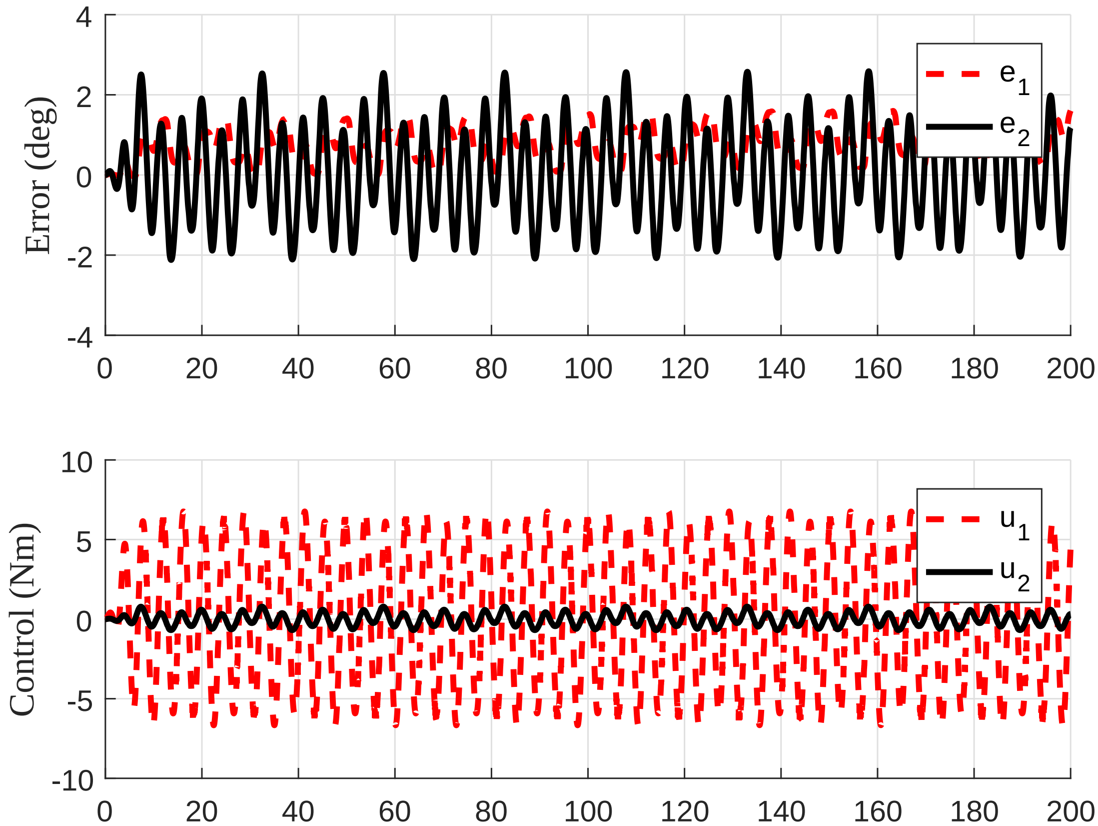

Figure 3-1. Variation of errors and control forces.

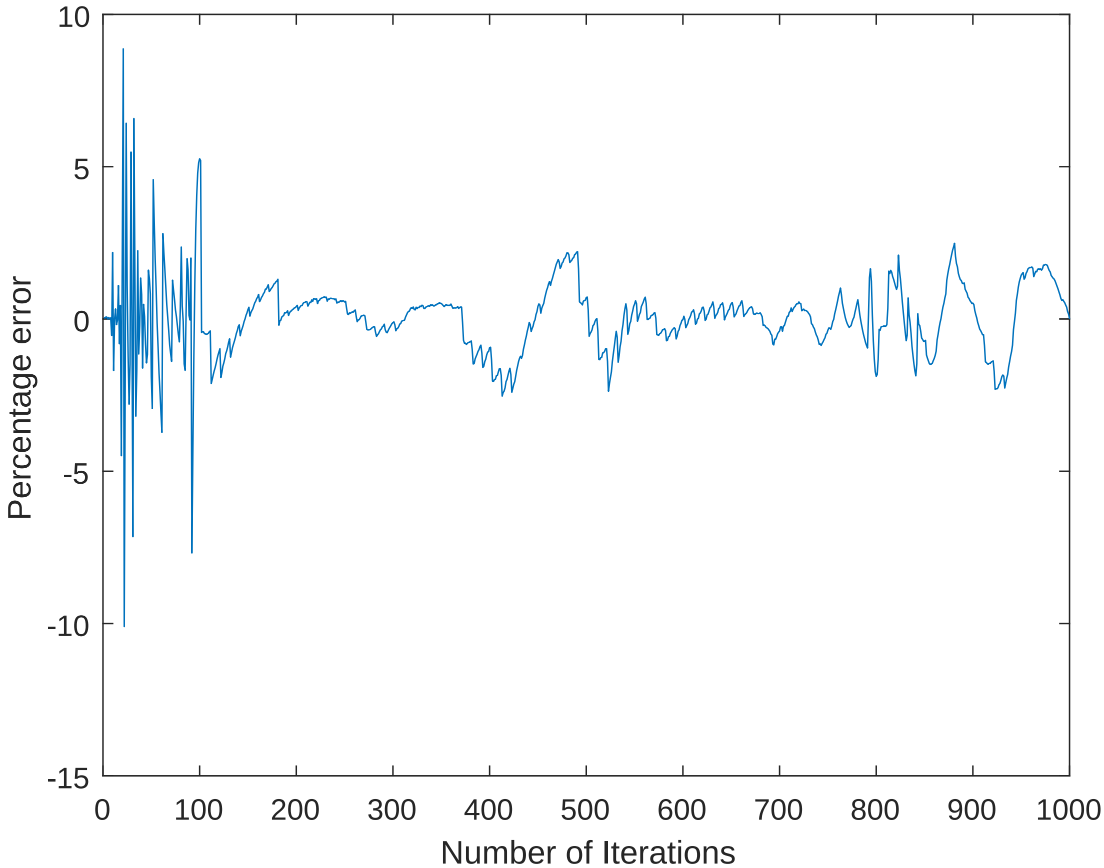

Figure 3-2. Delay mismatch variation with training iteration.

CHAPTER 4
CONTROL OF AN UNCERTAIN NONLINEAR SYSTEM WITH AN UNKNOWN TIME-VARYING INPUT DELAY USING AN ACCELERATED GRADIENT DESCENT BASED DELAY ESTIMATE

The developed robust PDE-based design approach doesn't restrict the delay rate magnitude. The delay derivative gain, as well as the conventional delay magnitude gain, can be independently adjusted. The amalgam of the predictor-based transformation, the robust control error system, and the delay estimate allows for new control development and stability analysis methods that can be applied to uncertain nonlinear systems with unknown time-varying delays. These contributions are based on Nesterov's AGD based strategy [91] to successfully estimate the delay magnitude at all time. Two different observer based methods are developed, where one requires the knowledge of highest order state derivative, and the other uses Nesterov's AGD based approach. A constrained optimization problem is formulated to estimate the delay. Subsequently, an augmented Lagrangian based unconstrained optimization in the dual space is formulated, which is solved using Nesterov's AGD based technique. Lyapunov-Krasovskii functionals are used in the Lyapunov-based analysis to prove the tracking errors exponentially converge to a steady-state residual that is a function of system uncertainty (i.e., uniform ultimately bounded (UUB) tracking). Simulation results demonstrate the controller performance for a second order nonlinear system and shows an estimation of delay and delay rate magnitudes.

4.1 Dynamic Model

Consider a class of $(n+1)^{th}$ order nonlinear systems model developed in Section 3.1. Along with the dynamic model, Assumptions 3.1, and Assumptions 3.2 are considered for this chapter. In addition, following assumptions are specific for this chapter.

Assumption 4.1. *The nonlinear exogenous disturbance term and its first derivative (i.e., d, $\dot{d}$) exist and are bounded by known positive constants, (cf. [88]).*

Assumption 4.2. *The unknown first and second delay derivatives $\dot{D}(t), \ddot{D}(t) \in \mathbb{R}$ are upper bounded by known positive constants $\bar{\dot{D}}$, $\bar{\ddot{D}}$ respectively, as $\dot{D}(t) \leq \bar{\dot{D}}, \ddot{D}(t) \leq \bar{\ddot{D}}, \forall t$.*

Assumption 4.3. *The estimate $\hat{D}(t)$ is sufficiently accurate such that $\tilde{D}(t) \triangleq D(t) - \hat{D}(t)$, can be upper bounded as $|\tilde{D}(t)| \leq \bar{\tilde{D}} \quad \forall t \in \mathbb{R}$, where $\bar{\tilde{D}} \in \mathbb{R}$ is a known positive constant, (cf. [88]).*

Based on the delay estimation in Section 4.5 and Assumptions 3.1-4.2, $\underline{\delta_0} \leq \delta_0 \leq \bar{\delta}_0$, $\underline{\delta_1} \leq \delta_1 \leq \bar{\delta}_1$ and $|\delta_p| \leq |\bar{\delta}_p|$, where $\underline{\delta_0}, \bar{\delta}_0, \underline{\delta_1}, \bar{\delta}_1$ and $|\bar{\delta}_p|$ are known positive constants.

4.2 Control Objective

The objective is to develop a controller which ensures that the state $x_1(t)$ of the input-delayed system in (3–2) tracks $x_d(t)$, despite uncertainties and additive disturbances in the dynamics. To quantify the control objective, a tracking error, denoted by $e_1(t) \in \mathbb{R}^m$, is defined in (3–10). Measurable auxiliary tracking errors, $e_i(t) \in \mathbb{R}^m, i = 2, 3, \ldots, n$, are defined in (3–11). A general expression for $e_i(t), \quad i = 2, 3, \ldots, n$ can be expressed as in (3–12). Another measurable auxiliary tracking error signal $r(t) \in \mathbb{R}^m$, is defined in (3–13). $e_u : [0, \infty) \to \mathbb{R}^m$ is designed as in (3–14) to obtain a delay dependent control term to negate the effect of the delayed input in (3–1).

4.3 Development of Error Signals

The open-loop error system for $r(t)$ can be obtained by taking the time derivative of $r(t)$, in (3–13), and can be expressed as (3–16). Based on the subsequent stability analysis, the delay-free control input is designed as

$$U(t) \triangleq u(1, t) \triangleq kr, \tag{4–1}$$

where $k \in \mathbb{R}$ is a constant, positive, adjustable control gain. Using the definition of $\tilde{N}, N_d \in \mathbb{R}^m$ in (3–19) and (3–22), and using the definition of $z \in \mathbb{R}^{(n+2)m}$ in (3–21), the closed-loop error system for r can be obtained as

$$\dot{r} = \tilde{N} + N_d - e_n - \delta_1 k\beta r + \delta_0 \beta u(0, t) - U(t - D(t)) + \delta_p \beta e_u. \tag{4–2}$$

Using the Mean Value Theorem, and Assumption 3.2, the expression in (3–19) can be upper bounded as

$$\left\|\tilde{N}\right\| \leq \rho\left(\|z\|\right)\|z\|, \tag{4–3}$$

where $\rho : \mathbb{R} \to \mathbb{R}$ is a known positive definite, non-decreasing, radially unbounded function. Using Assumptions 4.1 and 3.2, N_d can be upper bounded as

$$\sup_{t\in[0,\infty)} \|N_d\| \leq \Theta, \tag{4–4}$$

where $\Theta \in \mathbb{R}$ is a known positive constant.

4.4 Stability Analysis

To facilitate the subsequent stability analysis, let $y(t) \in \mathbb{R}^{(n+2)m+1}$ be defined as

$$y \triangleq \begin{bmatrix} z^T & \sqrt{Q} \end{bmatrix}^T, \tag{4–5}$$

where $Q(t) \in \mathbb{R}$ denotes an LK functional defined as

$$Q \triangleq \lambda_Q \int_0^1 e^{\omega_2 p}\|u(p,t)\|^2 dp, \tag{4–6}$$

where $\lambda_Q, \omega_2 \in \mathbb{R}$ are known, positive constants. Let $\mathscr{D}$ be an open and connected set, and $\mathcal{S}_{\mathscr{D}} \subset \mathscr{D}$ is defined as $\mathcal{S}_{\mathscr{D}} \triangleq \left\{y \in \mathbb{R}^{(n+2)m+1} |\, \|y\| < \sqrt{\frac{\min\{1,\omega_1\}}{\max\{2,\omega_1\}}}\text{inf}\left\{\rho^{-1}\left([\sqrt{\sigma},\infty)\right)\right\}\right\}$, where $\omega_1, \sigma \in \mathbb{R}$ are known, positive constants.

Theorem 4.1. *Given the dynamics in (3–2), the controller in (4–1) ensures UUB tracking in the sense that*

$$\|e_1\| \leq \mathit{\Gamma}_0 \exp\left(-\mathit{\Gamma}_1 t\right) + \mathit{\Gamma}_2, \tag{4–7}$$

where $\mathit{\Gamma}_0 \triangleq \sqrt{|2V\left(0\right) - \frac{2\Phi_2\varphi}{\Delta}|}$, $\mathit{\Gamma}_1 \triangleq -\frac{\Delta}{\Phi_2}$ *and* $\mathit{\Gamma}_2 \triangleq \sqrt{\frac{2\Phi_2\varphi}{\Delta}}$, *provided that* $y\left(\eta\right) \in \mathcal{S}_{\mathscr{D}}, \forall\eta \in [t_0, t_0 + \bar{D}]$, *where* $\Phi_1, \Phi_2 \in \mathbb{R}$, $\Phi_2 \triangleq \max\left\{2,\omega_1\right\}, \Phi_1 \triangleq \min\left\{1,\omega_1\right\}$ *and* $\varphi, \Delta \in \mathbb{R}$ *are subsequently developed control gains.*

Proof. Let $V : \mathscr{D} \times [t_0,\ \infty) \to \mathbb{R}$ be a continuously differentiable, positive-definite functional defined as

$$V \triangleq \frac{1}{2}\sum_{i=1}^{n} e_i^T e_i + \frac{1}{2} r^T r + \frac{\omega_1}{2} e_u^T e_u + Q, \tag{4–8}$$

where $\Phi_1 \left\Vert y \right\Vert^2 \leq V \leq \Phi_2 \left\Vert y \right\Vert^2$. The time derivative of (4–8) can be obtained after applying Leibniz integral rule to obtain the time derivative of (4–6), and utilizing (3–11)-(3–14) and (4–2), as

$$\begin{aligned}\dot{V} = {} & r^T \left(\tilde{N} + N_d - e_n\right) + r^T \left(-U\left(t - D(t)\right)\right) + \omega_1 e_u^T \left(\delta_1 k r - \delta_0 u(0,t) - \delta_p e_u\right) \\ & + r^T \left(-\delta_1 \beta k r + \delta_0 \beta u(0,t) + \delta_p \beta e_u\right) - \sum_{i=1}^{n-1} e_i^T e_i - e_n^T \alpha e_n + e_{n-1}^T e_n \\ & - e_n^T \beta e_u + e_n^T r + \lambda_Q \delta_1 e^{\omega_2} \|u(1,t)\|^2 - \lambda_Q \delta_0 ||u(0,t)||^2 - \lambda_Q \delta_p \int_0^1 e^{\omega_2 p} \|u\|^2 dp \\ & - \left(\frac{\bar{\delta}_0 \bar{\beta} + 1 + \omega_1 \bar{\delta}_0}{2\epsilon \left(\underline{\delta_0} - 2\bar{\delta}_1 e^{\omega_2}\right)}\right) \delta_0 ||u(0,t)||^2 - \lambda_Q \omega_2 \int_0^1 \delta e^{\omega_2 p} \|u\|^2 dp. \end{aligned} \tag{4–9}$$

By using (4–1), (4–3), (4–4), and canceling common terms in (4–9), an upper bound can be obtained as

$$\begin{aligned}\dot{V} \leq {} & \rho\left(\|z\|\right) \|r\| \|z\| + \|r\| \Theta - \underline{\delta_1 \beta} k \|r\|^2 + \left(\bar{\delta}_0 \bar{\beta} + 1\right) \|r^T u(0,t)\| \\ & + \|e_{n-1}^T e_n\| + \bar{\beta} \|e_n^T e_u\| + |r^T \left(u(0,t) - U\left(t - D(t)\right)\right)| + \bar{\beta} |\bar{\delta_p}| \|r^T e_u\| \\ & + \omega_1 \bar{\delta}_1 k \|e_u^T r\| + \omega_1 \bar{\delta}_0 \|e_u^T u(0,t)\| + \omega_1 |\bar{\delta_p}| \|e_u\|^2 + \lambda_Q \delta_1 e^{\omega_2} \|u(1,t)\|^2 \\ & - \sum_{i=1}^{n-1} e_i^T e_i - \underline{\alpha} \|e_n\|^2 - \left[\lambda_Q - \left(\frac{\bar{\delta}_0 \bar{\beta} + 1 + \omega_1 \bar{\delta}_0}{2\epsilon \left(\underline{\delta_0} - 2\bar{\delta}_1 e^{\omega_2}\right)}\right) \underline{\delta_0}\right] ||u(0,t)||^2 \\ & - \lambda_Q \omega_2 \int_0^1 \delta e^{\omega_2 p} \|u\|^2 dp + \lambda_Q |\bar{\delta_p}| \int_0^1 e^{\omega_2 p} \|u\|^2 dp. \end{aligned} \tag{4–10}$$

By using Assumption 3.1, Young's Inequality, the Cauchy-Schwartz Inequality, the fact that $\delta(p,t) \geq min\left\{\delta_0, \delta_1\right\}$ (derived from the definition of $\delta(p,t)$ in (3–5)), the fact that $\|e_u\|^2 = \left(\|\int_0^1 u\left(p,t\right) dp\|\right)^2 \leq \int_0^1 ||u^2(p,t)||dp. \int_0^1 1.dp \leq \int_0^1 ||u^2(p,t)||dp$, the following inequalities can be developed

$$u(1,t) = c\dot{u}(0,t)\bar{D} + u(0,t) \le cM\bar{D} + u(0,t), \tag{4–11}$$

$$\|u(0,t) - U\left(t - D(t)\right)\|^2 \le \left(\left(1-c\right)\bar{D} + \bar{\dot{D}}\right)^2 M^2, \tag{4–12}$$

where M [1] is a positive constant. Inequalities in (4–11)-(4–12) can be used to upper bound (4–10) as

$$\begin{aligned}
\dot{V} \le \; & \frac{1}{2k}\rho^2\left(\|z\|\right)\|z\|^2 + \frac{1}{2k}\Theta^2 + \left(\frac{\left(\bar{\delta}_0\bar{\beta} + \omega_1\bar{\delta}_0\right)c^2M^2\bar{D}^2}{\epsilon\left(\underline{\delta_0} - 2\bar{\delta}_1 e^{\omega_2}\right)}\right)\bar{\delta}_1 e^{\omega_2} \\
& + \left(\frac{c^2M^2\bar{D}^2}{\epsilon\left(\underline{\delta_0} - 2\bar{\delta}_1 e^{\omega_2}\right)}\right)\bar{\delta}_1 e^{\omega_2} + \frac{1}{2\epsilon}\left(\left(1-c\right)\bar{D} + \bar{\dot{D}}\right)^2 M^2 - \sum_{i=1}^{n-2} e_i^T e_i - \frac{1}{2}\|e_{n-1}\|^2 \\
& - \left(\underline{\alpha} - \frac{1}{2} - \frac{\bar{\beta}}{2}\right)||e_n||^2 - \left(\frac{\underline{\delta_1}\underline{\beta}k}{2} - \epsilon - k - \frac{\epsilon\bar{\delta}_0\bar{\beta}}{2}\right)\|r\|^2 \\
& - \left(\frac{\underline{\delta_1}\underline{\beta}k}{4} - \frac{\epsilon|\bar{\delta}_p|\bar{\beta}}{2}\right)\|r\|^2 - \left(\frac{\underline{\delta_1}\underline{\beta}k}{4} - \frac{\epsilon\omega_1\bar{\delta}_1 k}{2}\right)\|r\|^2 \\
& - \left(\frac{\lambda_Q\omega_2\min\left\{\underline{\delta_0},\underline{\delta_1}\right\}}{2} - \lambda_Q|\bar{\delta}_p|\right)Q - \left[\lambda_Q - \left(\frac{\bar{\delta}_0\bar{\beta} + 1 + \omega_1\bar{\delta}_0}{2\epsilon\left(\underline{\delta_0} - 2\bar{\delta}_1 e^{\omega_2}\right)}\right)\underline{\delta_0}\right]||u(0,t)||^2 \\
& - \left(\frac{\lambda_Q\omega_2\min\left\{\underline{\delta_0},\underline{\delta_1}\right\}}{2} - \frac{\bar{\beta}|\bar{\delta}_p|}{2\epsilon} - \frac{\bar{\beta}}{2}\right)||e_u||^2 - \left(1 + \frac{\omega_1\bar{\delta}_1 k}{2\epsilon} + \frac{\epsilon\omega_1\bar{\delta}_0}{2} + \omega_1|\bar{\delta}_p|\right)\|e_u\|^2.
\end{aligned} \tag{4–13}$$

Since $\|y\| \ge \|z\|$, (4–13) can be simplified to obtain

$$\begin{aligned}
\dot{V} \le & -\left(\frac{\sigma}{2} - \frac{1}{2k}\rho^2\left(\|y\|\right)\right)\|z\|^2 - \frac{\sigma}{2}||z||^2 - \left(\frac{\lambda_Q\omega_2\min\left\{\underline{\delta_0},\underline{\delta_1}\right\}}{2} - \lambda_Q|\bar{\delta}_p|\right)Q \\
& -\lambda_{Q_{res}}||u(0,t)||^2 + \frac{1}{2\epsilon}\left(\left(1-c\right)\bar{D} + \bar{\dot{D}}\right)^2 M^2 + \left(\frac{\left(\bar{\delta}_0\bar{\beta} + \omega_1\bar{\delta}_0\right)c^2M^2\bar{D}^2}{\epsilon\left(\underline{\delta_0} - 2\bar{\delta}_1 e^{\omega_2}\right)}\right)\bar{\delta}_1 e^{\omega_2} \\
& + \left(\frac{c^2M^2\bar{D}^2}{\epsilon\left(\underline{\delta_0} - 2\bar{\delta}_1 e^{\omega_2}\right)}\right)\bar{\delta}_1 e^{\omega_2} + \frac{1}{2k}\Theta^2,
\end{aligned} \tag{4–14}$$

[1] Similar to [88], the subsequent analysis does not assume that the inequality $\dot{u} < M$ holds for all time. The subsequent analysis only exploits the fact that provided $\|z(\eta)\| < \gamma$, $\forall\eta \in [t_0, t]$, then $\dot{u} < M$.

where $\sigma, \kappa, \zeta, \lambda_{Q_{res}} \in \mathbb{R}$ are defined as

$$\sigma \triangleq \min\left\{\left(\underline{\alpha} - \frac{1}{2} - \frac{\bar{\beta}}{2}\right), \frac{1}{2}, \zeta, \kappa\right\}, \tag{4–15}$$

$$\kappa \triangleq \frac{\lambda_Q \omega_2 \min\left\{\underline{\delta_0}, \underline{\delta_1}\right\}}{2} - \frac{\bar{\beta}|\bar{\delta_p}|}{2\epsilon} - \frac{\bar{\beta}}{2} - 1 - \frac{\omega_1 \bar{\delta_1} k}{2\epsilon} - \frac{\epsilon \omega_1 \bar{\delta_0}}{2} - \omega_1 |\bar{\delta_p}|, \tag{4–16}$$

$$\lambda_{Q_{res}} \triangleq \lambda_Q - \left(\frac{\bar{\delta_0}\bar{\beta} + 1 + \omega_1 \bar{\delta_0}}{2\epsilon\left(\underline{\delta_0} - 2\bar{\delta_1} e^{\omega_2}\right)}\right)\underline{\delta_0}, \tag{4–17}$$

$$\zeta \triangleq \left(\underline{\delta_1}\underline{\beta} - 1\right)k - \epsilon - \frac{\epsilon\left(\bar{\delta_0}\bar{\beta} + |\bar{\delta_p}|\bar{\beta}\right)}{2} - \frac{\epsilon \omega_1 \bar{\delta_1} k}{2}. \tag{4–18}$$

Provided $y(\eta) \in \mathscr{D}, \forall \eta \in [t - \bar{D}, t]$ the expression in (4–14) reduces to

$$\dot{V} \leq -\lambda \left\|y\right\|^2 + \varphi, \tag{4–19}$$

where $\lambda, \varphi \in \mathbb{R}$ are defined as

$$\lambda \triangleq \min\left\{\frac{\sigma}{2}, \frac{\lambda_Q \omega_2 \min\left\{\underline{\delta_0}, \underline{\delta_1}\right\}}{2} - \lambda_Q |\bar{\delta_p}|\right\}, \tag{4–20}$$

$$\varphi \triangleq \frac{1}{2k}\Theta^2 + \frac{1}{2\epsilon}\left(\left(1 - c\right)\bar{D} + \bar{\tilde{D}}\right)^2 M^2 + \left(\frac{\left(\bar{\delta_0}\bar{\beta} + 1 + \omega_1 \bar{\delta_0}\right)c^2 M^2 \bar{D}^2}{\epsilon\left(\underline{\delta_0} - 2\bar{\delta_1} e^{\omega_2}\right)}\right)\bar{\delta_1} e^{\omega_2}. \tag{4–21}$$

An upper bound can be obtained for (4–19) as

$$\dot{V} \leq -\frac{\Delta}{\Phi_2} V + \varphi. \tag{4–22}$$

The solution of the differential equation in (4–22) can be obtained as

$$\dot{V} \leq V\left(0\right)\exp\left(-\frac{\Delta}{\Phi_2}t\right) + \frac{\Phi_2 \varphi}{\Delta}\left(1 - \exp\left(-\frac{\Delta}{\Phi_2}t\right)\right). \tag{4–23}$$

Using (4–8) and (4–23), following upper bounds can be obtained for $e_1^{(i)}, i = 0, 1, \ldots, n, \quad r,$ and e_u as

$$\|e_1^{(i)}\| \leq \sqrt{2V\left(0\right)\exp\left(-\frac{\Delta}{\Phi_2}t\right) + \frac{2\Phi_2 \varphi}{\Delta}\left(1 - \exp\left(-\frac{\Delta}{\Phi_2}t\right)\right)}, \tag{4–24}$$

$$\|r\| \leq \sqrt{2V\left(0\right)\exp\left(-\frac{\Delta}{\Phi_2}t\right) + \frac{2\Phi_2 \varphi}{\Delta}\left(1 - \exp\left(-\frac{\Delta}{\Phi_2}t\right)\right)}, \tag{4–25}$$

$$\|e_u\| \leq \sqrt{\frac{2V\left(0\right)}{\omega_1}\exp\left(-\frac{\Delta}{\Phi_2}t\right)+\frac{2\Phi_2\varphi}{\Delta\omega_1}\left(1-\exp\left(-\frac{\Delta}{\Phi_2}t\right)\right)}. \tag{4–26}$$

Since $e_1^{(i)}$, r, $e_u \in \mathcal{L}_\infty$, from (4–1), $u \in \mathcal{L}_\infty$. Based on inequalities developed in (4–24)-(4–26),

$$\|e_1^{(i)}\|, \|r\| \leq C_{e_1}, \forall i = 0, 1, \ldots n \tag{4–27}$$

$$\|e_u\| \leq C_{e_u}. \tag{4–28}$$

where $\sqrt{2V\left(0\right)\exp\left(-\frac{\Delta}{\Phi_2}t\right)+\frac{2\Phi_2\varphi}{\Delta}\left(1-\exp\left(-\frac{\Delta}{\Phi_2}t\right)\right)} \leq C_{e_1} \in \mathbb{R}^+, \forall t$ and $C_{e_u} \triangleq \frac{C_{e_1}}{\sqrt{\omega_1}} \in \mathbb{R}^+$. □

4.5 Accelerated Gradient Descent based Estimation of Delay

From the relationship in (3–4) the time and spatial variation of the control input can be related. Evaluating (3–4) at $p = 1$, yields

$$u_t\left(1,t\right) = \delta_1 u_p\left(1,t\right). \tag{4–29}$$

The left side of (4–29) equals to the time derivative of the delay free control input, which after using (4–1), is $u_t\left(1,t\right) = k\dot{r}$. To determine the expression for $u_p\left(1,t\right)$, the linear variation of $u(p,t)$ over x is used (as defined in (3–4)), which states that at a fixed time instant, $u(p,t)$ varies linearly in p. Using the Mean Value Theorem, $u_p\left(1,t\right)$ can be expressed as

$$u_p\left(1,t\right) = \left(\frac{u\left(1,t\right)-u\left(0,t\right)}{1}\right) = u\left(1,t\right)-u\left(0,t\right). \tag{4–30}$$

Using (4–1), (4–29), and (4–30), and the time derivative of (3–13), the following expression can be obtained:

$$k\left(\ddot{e}_n+\alpha\dot{e}_n-\beta\dot{e}_u\right)-\delta_1\left(u(1,t)-u(0,t)\right)=0. \tag{4–31}$$

Substituting for (3–15) yields

$$k\left(\ddot{e}_n+\alpha\dot{e}_n\right)-k\delta_1\beta u\left(1,t\right)+k\delta_0\beta u\left(0,t\right)+k\delta_p\beta e_u-\delta_1\left(u\left(1,t\right)-u\left(0,t\right)\right)=0. \tag{4–32}$$

After substituting for δ_0, δ_1 and δ_p, the expression in (4–32) can be written as

$$\left(k\ddot{e}_n + k\alpha\dot{e}_n\right) c\hat{D}(\phi^{-1}(t)) \left(1 - c\dot{\hat{D}}(\phi^{-1}(t))\right) - \left(\dot{e}_n + \alpha e_n - \beta e_u\right)\left(k^2\beta + k\right)$$
$$+k\beta\left(1 - c\dot{\hat{D}}(\phi^{-1}(t))\right) u\left(0, t\right) + c\dot{\hat{D}}(\phi^{-1}(t))k\beta e_u + u\left(0, t\right) = 0. \tag{4–33}$$

Given initial conditions for $e_u(t)$ and $\hat{D}(\phi^{-1}(t))$, (4–33) can be solved for $\hat{D}\left(\phi^{-1}(t)\right)$.

Remark 4.1. To estimate $\hat{D}(t)$, $\ddot{e}_n(t)$ needs to be measurable, i.e., the highest order state derivative needs to be measurable.

Remark 4.1 motivates the necessity of removing the requirement of highest order state derivative measurement by designing an accelerated gradient descent based delay estimation scheme. From the relationship in (3–4) the time and spatial variation of the control input can be related. Evaluating (3–4) at $p = 1$ and at $p = 0$, yields

$$u_t\left(1, t\right) = \delta_1 u_p\left(1, t\right), \tag{4–34}$$

$$u_t\left(0, t\right) = \delta_0 u_p\left(0, t\right). \tag{4–35}$$

After dividing (4–34) by (4–35), and using the relations in (3–8) and (3–9), following relation has been developed

$$\frac{u_t\left(1, t\right)}{u_t\left(0, t\right)} = \frac{1}{1 - c\dot{\hat{D}}} \frac{u_p\left(1, t\right)}{u_p\left(0, t\right)}. \tag{4–36}$$

After taking a partial with respect to p of the relation in (3–3), the following relationship is developed

$$\frac{\partial}{\partial p} u(p, t) = \frac{\partial}{\partial p} U\left(\phi(t + p\left(\phi^{-1}(t) - t\right)\right),$$

$$u_p\left(p, t\right) = \dot{U}\left(\phi(t + p\left(\phi^{-1}(t) - t\right)\right)\dot{\phi}\left(t + p\left(\phi^{-1}(t) - t\right)\right) \times \left(\phi^{-1}(t) - t\right),$$

$$u_p\left(1, t\right) = t\dot{U}\left(t\right)\left(\phi^{-1}(t) - t\right), \tag{4–37}$$

$$u_p\left(0, t\right) = \left(1 - c\dot{\hat{D}}(t)\right)\dot{U}\left(t - c\hat{D}(t)\right)\left(\phi^{-1}(t) - t\right). \tag{4–38}$$

Before proceeding further with the estimation, the following upper bound on $\dot{r}(t)$ is derived using the relationship in (4–2) and substituting the inequalities from (4–3), (4–4), (4–27) and

(4–28) as

$$\dot{r} = \tilde{N} + N_d - e_n - \delta_1 k\beta r + \delta_0\beta u(0,t) - U\left(t - D(t)\right) + \delta_p\beta e_u,$$

$$\|\dot{r}\| \leq \sqrt{\sigma k} + \Theta + C_{e_n} + \bar{\delta}_1\bar{k}\bar{\beta}C_r + \bar{\delta}_0\bar{\beta}\bar{k}C_r + \bar{k}C_r + |\bar{\delta}_p|\bar{\beta}C_{e_u}, \tag{4–39}$$

$$\|\dot{r}\| \leq C_{\dot{r}}, \tag{4–40}$$

where $C_{\dot{r}} \triangleq \sqrt{\sigma k} + \Theta + C_{e_1} + \bar{\delta}_1\bar{k}\bar{\beta}C_{e_1} + +\bar{\delta}_0\bar{\beta}\bar{k}C_{e_1} + \bar{k}C_{e_1} + |\bar{\delta}_p|\bar{\beta}C_{e_u}$.

Remark 4.2. $|\ddot{r}(\kappa)| \leq \epsilon_{\ddot{r}}$, where [2] $\kappa \in [t, t - \bar{D}]$, $\forall t$ and $\epsilon_{\ddot{r}} > 0$. Also, $\epsilon_{\ddot{r}}$ is small enough such that $\epsilon_{\ddot{r}} << C_{\dot{r}}$, and $c\bar{D}\frac{\epsilon_{\ddot{r}}}{C_{\dot{r}}} \leq \delta_\epsilon$, where $\delta_\epsilon > 0$.

Using Remark 4.2, and using Taylor's Remainder Theorem, the following upper bound is developed for $\frac{u_p(1,t)}{u_p(0,t)}$

$$\begin{aligned}\frac{u_p(1,t)}{u_p(0,t)} &= \frac{\dot{r}(t)}{\dot{r}\left(t - c\hat{D}(t)\right)} \leq \frac{\dot{r}(t)}{\left(\dot{r}(t) - c\hat{D}(t)\ddot{r}(\kappa)\right)}, \\ &\leq \frac{1}{\left(1 - c\bar{D}\frac{\epsilon_{\ddot{r}}}{C_{\dot{r}}}\right)} \leq \left(1 + \frac{\epsilon_{\ddot{r}}}{C_{\dot{r}}}\right) \leq 1 + \delta_\epsilon,\end{aligned} \tag{4–41}$$

where $\kappa \in [t - c\hat{D}(t), t]$. Substituting (4–41), the expression in (4–36) can be written as

$$u_t\left(1,t\right) \leq \frac{1+\delta_\epsilon}{1 - c\dot{\hat{D}}} u_t\left(0,t\right). \tag{4–42}$$

After integrating (4–42) and using (4–1) and the fact that $u(0,t) = kr\left(t - \hat{D}\right)$,

$$\begin{aligned}\int_{t_0}^{t} d\left(u\left(1,x\right)\right) &\leq \left(1+\delta_\epsilon\right)\int_{t_0}^{t}\frac{1}{1-\dot{\hat{D}}}d\left(u\left(0,x\right)\right), \\ \left(r\left(t\right) - r\left(t_0\right)\right) &\leq \left(1+\delta_\epsilon\right)\left[\left.\frac{r(\tau - \hat{D}(\tau))}{1 - \dot{\hat{D}}(\tau)}\right|_{t_0}^{t} - \int_{t_0}^{t} r\left(\tau - \hat{D}(\tau)\right)\frac{\ddot{\hat{D}}}{\left(1-\dot{\hat{D}}\right)^2}d\tau\right].\end{aligned} \tag{4–43}$$

[2] The subsequent analysis does not assume that the inequality $|\ddot{r}(\kappa)| \leq \epsilon_{\ddot{r}}$ holds for all time. The subsequent analysis only exploits the fact that provided $\kappa \in [t, t - \bar{D}]$, then $|\ddot{r}(\kappa)| \leq \epsilon_{\ddot{r}}$.

Based on the development in Section 3.2, (3–13) provides a continuous $r(t)$ that is composed of measurable signals (i.e., $e_n(t)$, $\dot{e}_n(t)$) and $e_u(t)$, which can be computed from (3–15) given an initial condition of the delay and its derivative (i.e., $\hat{D}(t_0)$, $\dot{\hat{D}}(t_0)$) and $r(t_0)$. Considering the upper bound of $r(t)$ developed in (4–43), a discrete estimate of $r(t)$, denoted by $\hat{r}(\tau)$ is given by[3]

$$\hat{r}(\tau) = \hat{r}(t_0) + (1+\delta_\epsilon)\left[\frac{\left(r(\tau - \hat{D}(\tau)\right)}{1-\dot{\hat{D}}(\tau)} - \frac{\left(r(t_0 - \hat{D}(t_0)\right)}{1-\dot{\hat{D}}(t_0)} \right.$$
$$\left. - \int_0^\tau r\left(p - \hat{D}(p)\right)\frac{\ddot{\hat{D}}(p)}{\left(1-\dot{\hat{D}}(p)\right)^2}dp\right]. \tag{4–44}$$

The subsequent designed adaptive estimate for $\hat{D}(t)$ is motivated by the desire to minimize the mismatch between the auxiliary error signal $r(\tau)$ and the discrete estimate $\hat{r}(\tau)$. To quantify this objective, an objective function is defined as

$$E = \int_{t_0}^{\bar{t}} \left(\hat{r}(\tau) - r(\tau)\right)^2 d\tau, \tag{4–45}$$

where $\bar{t} \in \mathbb{R}^+$ is the time of interest while discretizing for delay estimation. From (4–45), the following gradient is derived

$$\frac{\partial E}{\partial \hat{D}} = 2\left(r(\tau) - \hat{r}(\tau)\right)(1+\delta_\epsilon)\left(-\frac{\dot{r}(\tau - \hat{D}(\tau))}{1-\dot{\hat{D}}(\tau)} - 2\frac{d}{d\tau}\int_0^\tau r(p-\hat{D}(p))\frac{\ddot{\hat{D}}(p)}{(1-\dot{\hat{D}}(p))^3}dp\right.$$
$$\left. + \int_0^\tau \dot{r}(p-\hat{D}(p))\frac{\ddot{\hat{D}}(p)}{(1-\dot{\hat{D}}(p))^2}dp - \frac{d}{d\tau}\frac{r(\tau-\hat{D}(\tau))}{(1-\dot{\hat{D}}(\tau))^2} + \frac{d^2}{d\tau^2}\int_0^\tau \frac{r(p-\hat{D}(p))}{(1-\dot{\hat{D}}(p))^2}dp\right). \tag{4–46}$$

Note that, in the above equation, "$\frac{\partial}{\partial \dot{\hat{D}}}$" and "$\frac{\partial}{\partial \ddot{\hat{D}}}$" are interchanged with "$\int$" by using the Leibniz integral rule, since the partial derivative (with respect to "$\frac{\partial}{\partial \dot{\hat{D}}}$" and "$\frac{\partial}{\partial \ddot{\hat{D}}}$") of the integrand is continuous and is pointwise bounded. Nesterov's AGD technique is used to update $\hat{D}$. Since the update of $\hat{D}$ at each iteration is $\frac{\partial \hat{D}}{\partial t} = \frac{\partial E}{\partial \hat{D}}$ (see [92]), the expression in

[3] The inequality sign in (4–43) has been replaced by an equality sign for discrete estimate of $r(t)$, by considering the worst case scenario.

([4–46](#)) can be used to update $\hat{D}$. Before using the Nesterov's AGD algorithm to update $\hat{D}$, development is provided to indicate that given a bound on the error, E, there will always be a smooth $\hat{D}$, i.e., the Nesterov's AGD algorithm converges. In other words, given an arbitrary ϵ, such that $\int_{t_0}^{\bar{t}} \left(\hat{r}(\tau) - r(\tau)\right)^2 d\tau < \epsilon$, $\frac{\partial \hat{D}}{\partial t} \to 0$.

Theorem 4.2. *For all $\epsilon > 0$, if at some $t = T$, $E = \int_{t_0}^{\bar{t}} \left(\hat{r}(\tau) - r(\tau)\right)^2 d\tau < \epsilon$, there exists a $\delta > 0$ such that, at $t = T$, $\int_{t_0}^{\bar{t}} \left(\frac{\partial \hat{D}(\tau)}{\partial t}\right)^2 d\tau < \delta$ for some T, i.e., $E \to 0$ implies $\frac{\partial \hat{D}}{\partial t} \to 0$ in L^2 sense.*

Proof. Observe in (4–46) that, $\frac{\dot{r}(\tau - \hat{D}(\tau))}{1 - \dot{\hat{D}}(\tau)} < C_1$, $\frac{r(\tau - \hat{D}(\tau))}{(1 - \dot{\hat{D}}(\tau))^2} < C_2$, $\dot{r}(\tau - \hat{D}(\tau))\frac{\ddot{\hat{D}}(\tau)}{(1 - \dot{\hat{D}}(\tau))^2} < C_3$, $r(\tau - \hat{D}(\tau))\frac{\ddot{\hat{D}}(\tau)}{(1 - \dot{\hat{D}}(\tau))^3} < C_4$ and $\frac{r(\tau - \hat{D}(\tau))}{(1 - \dot{\hat{D}}(\tau))^2} < C_5$, for some constants $C_i > 0$, where $C_1 \triangleq \frac{C_{\dot{r}}}{1 - \dot{\bar{D}}}$, $C_2 \triangleq \frac{C_{e_1}}{\left(1 - \dot{\bar{D}}\right)^2}$, $C_3 \triangleq \frac{C_{\dot{r}} \ddot{\bar{D}}}{\left(1 - \dot{\bar{D}}\right)}$, $C_4 \triangleq \frac{C_{e_1} \ddot{\bar{D}}}{\left(1 - \dot{\bar{D}}\right)^3}$ and $C_5 \triangleq \frac{C_{e_1}}{\left(1 - \dot{\bar{D}}\right)^2}$. At $t = T$,

$$\int_{t_0}^{\bar{t}} \left(\frac{\partial \hat{D}(\tau)}{\partial t}\right)^2 \leq (1 + \delta_\epsilon)\, 4(C_1 + C_3 - 2C_4)^2(\bar{t} - t_0),$$

$$\int_{t_0}^{\bar{t}} \left(\hat{r}(\tau) - r(\tau)\right)^2 d\tau < 4\epsilon\,(1 + \delta_\epsilon)\,(C_1 + C_3 - 2C_4)^2(\bar{t} - t_0).$$

Now choose, $\delta = 4\epsilon\,(1 + \delta_\epsilon)\,(C_1 + C_3 - 2C_4)^2(\bar{t} - t_0)$ to conclude the proof. □

The subsequent development uses $\hat{D}$. The analytic expression of the gradient derived in (4–46) is used. Note that, this approach involves derivative and Riemann integration. Central difference is used to approximate the derivative. Since the Riemann integral for the integrands in (4–46) is assumed to exist, the integral with upper is approximated with Darboux sum with uniform interval length of dt (dt is selected based on the discretization of the time axis of the data). To approximate $\hat{D}$, to minimize the error E is minimized, the following objective function is solved

$$\mathrm{argmin}_{\hat{D}} E + \lambda_1 \|\dot{\hat{D}}\|_2 + \lambda_2 \|\ddot{\hat{D}}\|_2 \tag{4–47}$$

subject to (based on Assumption 3.1),

$$\underline{\mathsf{D}} \leq \hat{D} \leq \bar{D}.$$

Also, λ_1, λ_2 are the regularizers to ensure that the $\hat{D}$ is twice differentiable. The regularizer λ_1 is adjusted to enforce the inequality on $\dot{\hat{D}}$, as $\dot{\hat{D}} \leq \bar{\dot{D}}$ (based on Assumption 4.2). Similarly, λ_2 is adjusted to enforce the inequality on $\ddot{\hat{D}}$, as $\ddot{\hat{D}} \leq \bar{\ddot{D}}$ (based on Assumption 4.2). For T discrete time steps, then the equivalent Karush-Kuhn-Tucker (KKT) conditions (see [93]) are given below

$$\mathcal{C}(\hat{D}) \triangleq \frac{\partial E}{\partial \hat{D}} + \lambda_1 \frac{\partial \|\dot{\hat{D}}\|}{\partial \hat{D}} + \lambda_2 \frac{\partial \|\ddot{\hat{D}}\|}{\partial \hat{D}} + \sum_{i=1}^{T} \mu_i \frac{\partial}{\partial \hat{D}} (\hat{D}(i) - \bar{D}) + \sum_{j=1}^{T} \nu_j \frac{\partial}{\partial \hat{D}} (-\hat{D}(j) + \underline{\mathrm{D}}) = 0$$

$$\mu_i(\hat{D}(i) - \bar{D}) = 0, \forall i$$

$$\mu_i \geq 0, \forall i$$

$$\nu_i(-\hat{D}(i) + \underline{\mathrm{D}}) = 0, \forall i$$

$$\nu_i \geq 0, \forall i.$$

Nesterov's AGD is used to solve the optimization by formulating an augmented Lagrangian of the above constrained problem to get

$$\mathrm{argmin}_{\hat{D}} F \triangleq \frac{\gamma_1}{2} (\mathcal{C}(\hat{D}))^2 - \lambda_1^1 \mathcal{C}(\hat{D}) + \frac{\gamma_2}{2} \sum_{i=1}^{T} (\mu_i(\hat{D}(i) - b))^2$$
$$- \sum_{i=1}^{T} \lambda_2^i (\mu_i(\hat{D}(i) - b)) + \frac{\gamma_3}{2} \sum_{j=1}^{T} (\nu_j(-\hat{D}(j) + a))^2 - \sum_{j=1}^{T} \lambda_3^j (\nu_j(-\hat{D}(j) + a))$$

subject to,

$$\mu_i \geq 0, \forall i$$

$$\nu_i \geq 0. \forall i$$

Here, γ_1, γ_2, γ_3 are taken as user specified parameters, and in Algorithm 1 $\left\{\lambda_i^j\right\}$ are learned. Now, Nesterov's AGD will use to solve the objective function F in (4–47). The algorithm is given below.

4.6 Simulation Results

A numerical simulation was performed on the 2-link robot described in (3–54). In (3–54), $D(t)$ is the actual delay injected in the simulation, which is of the form $D(t)$ =

Algorithm 1 The algorithm to learn $\hat{D}$

Require: $r(\tau)$, for $x \in [t_0, \bar{t}]$ in dt incremental steps, λ_1, λ_2, γ_1, γ_2, $\eta > 0$, $a < b$, max_iter, $\beta^1 = 0$
Ensure: $\hat{D}$

while iter $\leq$ max_iter **do**
  $\beta^{\text{iter}+1} \leftarrow \frac{1+\sqrt{1+4(\beta^{\text{iter}})^2}}{2}$;
  $\alpha^{\text{iter}} \leftarrow \frac{1-\beta^{\text{iter}}}{\beta^{\text{iter}+1}}$;
  $\hat{D}^{\text{iter}+1}(i) = \bar{D}^{\text{iter}}(i) - \eta \frac{\partial F}{\partial \bar{D}^{\text{iter}}(i)}$, $\forall i$;
  $\bar{D}^{\text{iter}}(i) = (1-\alpha^{\text{iter}})\hat{D}^{\text{iter}+1}(i) + \alpha^{\text{iter}}\hat{D}^{\text{iter}}(i)$, $\forall i$;
  $\mu_i = \mu_i - \eta \frac{\partial F}{\partial \mu_i}$, $\forall i$;
  $\nu_i = \nu_i - \eta \frac{\partial F}{\partial \nu_i}$, $\forall i$;
  $\mu_i = \max\{0, \mu_i\}$, $\forall i$;
  $\nu_i = \max\{0, \nu_i\}$, $\forall i$;
  $\lambda_1^1 = \lambda_1^1 - \gamma_1 \mathcal{C}(\hat{D}^{\text{iter}})$;
  $\lambda_2^i = \lambda_2^i - \gamma_2(\mu_i(\hat{D}^{\text{iter}}(i) - b))$, $\forall i$;
  $\lambda_3^j = \lambda_3^j - \gamma_3(\nu_j(-\hat{D}^{\text{iter}}(j) + a))$, $\forall j$;
  iter $\leftarrow$ iter $+1$;
end while
$\hat{D} \leftarrow \hat{D}^{\text{iter}}$

$0.04\tanh(\frac{t}{30}) + 0.08$, where $0.08 \leq D(t) \leq 0.12$. Figure 4-1 shows the tracking error (i.e., e_1 and e_2) of two states, along with required control forces (i.e., $U(t)$). Based on the objective function introduced in (4–45), Nesterov's AGD algorithm in Algorithm 1, successfully estimates delay magnitude $\hat{D}(t)$ as shown in Figure 4-2. Figure 4-2 shows that the estimated delay is bounded between the known bounds of actual delay $D(t)$, i.e., $\bar{D}(t)$ and $\underline{D}(t)$, while Figure 4-3 shows the existence of $\dot{\hat{D}}(t)$ and $\ddot{\hat{D}}(t)$. To justify the performance of Nesterov's AGD based delay estimation strategy, Figure 4-4 shows the objective function E, defined in (4–45), decreases with number of iterations and stays constant. This result shows the utility of using Nesterov's AGD based strategy for estimating unknown time-varying delay magnitude present in the system. Also unlike existing literature of constant delay estimate, i.e., [88], the developed Nesterov's AGD based time-varying estimate gives less control force and less tracking error.

4.7 Control Gain Selection

Control gains, such as σ in (4–15), κ in (4–16), ζ in (4–18), λ in (4–20) and $\lambda_{Q_{res}}$ in (4–17), introduced in the stability analysis (Section 4.4) are required to be positive constants. Based on Nesterov's AGD estimation scheme developed in Section 4.5, the estimated delay is designed to be bounded between given bounds as mentioned in Assumption 3.1. Given the fact that $\underline{D} \leq \hat{D} \leq \bar{D}$, and subsequently derived bounds of δ_0, δ_1 and δ_p (i.e., $\underline{\delta_0}$, $\bar{\delta}_0$, $\underline{\delta_1}$, $\bar{\delta}_1$, $|\bar{\delta_p}|$), this section develops sufficient gain conditions to ensure σ, κ, ζ, $\lambda_{Q_{res}}$ and $\lambda > 0$. Using the definitions of σ in (4–15) sufficient lower bounds for $\underline{\alpha}$ can be obtained as

$$\underline{\alpha} > \frac{\bar{\beta}+1}{2}. \tag{4–48}$$

From the definition of λ in (4–20), ω_2 also needs to satisfy the following inequality

$$\omega_2 > \frac{2|\bar{\delta_p}|}{\min\left\{\underline{\delta_0}, \underline{\delta_1}\right\}}. \tag{4–49}$$

Now combining the definitions of κ in (4–16) and $\lambda_{Q_{res}}$ in (4–17), a lower bound over λ_Q can be obtained as

$$\lambda_Q > \frac{\left(\frac{\bar{\beta}|\bar{\delta_p}|}{\epsilon} + \bar{\beta} + 2 + \frac{\omega_1 \bar{\delta}_1 k}{\epsilon} + \epsilon\omega_1\bar{\delta}_0 + 2\omega_1|\bar{\delta_p}|\right)}{\omega_2 \min\left\{\underline{\delta_0}, \underline{\delta_1}\right\}} \triangleq \lambda_{Q_1}, \tag{4–50}$$

$$\lambda_Q > \left(\frac{\bar{\delta}_0\bar{\beta} + 1 + \omega_1\bar{\delta}_0}{2\epsilon\left(\underline{\delta_0} - 2\bar{\delta}_1 e^{\omega_2}\right)}\right)\underline{\delta_0} \triangleq \lambda_{Q_2}, \tag{4–51}$$

$$\lambda_Q > \max\left\{\lambda_{Q_1}, \lambda_{Q_2}\right\}. \tag{4–52}$$

To make $\zeta > 0$ from the definition of ζ in (4–18), the following lower bound of k can be obtained as

$$k > \frac{2\epsilon + \epsilon\left(\bar{\delta}_0\bar{\beta} + |\bar{\delta_p}|\bar{\beta}\right)}{2\left[\underline{\delta_1}\underline{\beta} - 1 - \frac{\epsilon\omega_1\bar{\delta}_1}{2}\right]}, \tag{4–53}$$

where ω_1 must satisfy

$$\omega_1 < \frac{2\left(\underline{\delta_1}\underline{\beta} - 1\right)}{\epsilon\bar{\delta}_1}, \tag{4–54}$$

to make the denominator positive which gives rise to the following inequality over β (both $\bar{\beta}$ and $\underline{\beta}$)

$$\bar{\beta} > \underline{\beta} > \frac{1}{\underline{\delta_1}}. \tag{4–55}$$

From the definition of φ in (4–21), ϵ is selected sufficiently small.

4.8 Conclusion

A robust controller is developed for a class of uncertain nonlinear systems with an additive disturbance subject to unknown time varying input delay without delay rate constraints. A filtered tracking error signal is designed to facilitate the control design and stability analysis. A novel Lyapunov-Krasovskii functional is used in the Lyapunov-based stability analysis to provide UUB of the tracking error. An observer based method is developed for unknown delay estimation, which uses highest order state derivative measurement, and can cause potential drawback in practical applications. In order to remove necessity of highest order state derivative measurement, Nesterov's AGD based estimation is used to provide a time-varying estimate of the delay. Simulation results show the performance of the controller along with the estimation of the delay and delay rates magnitude.

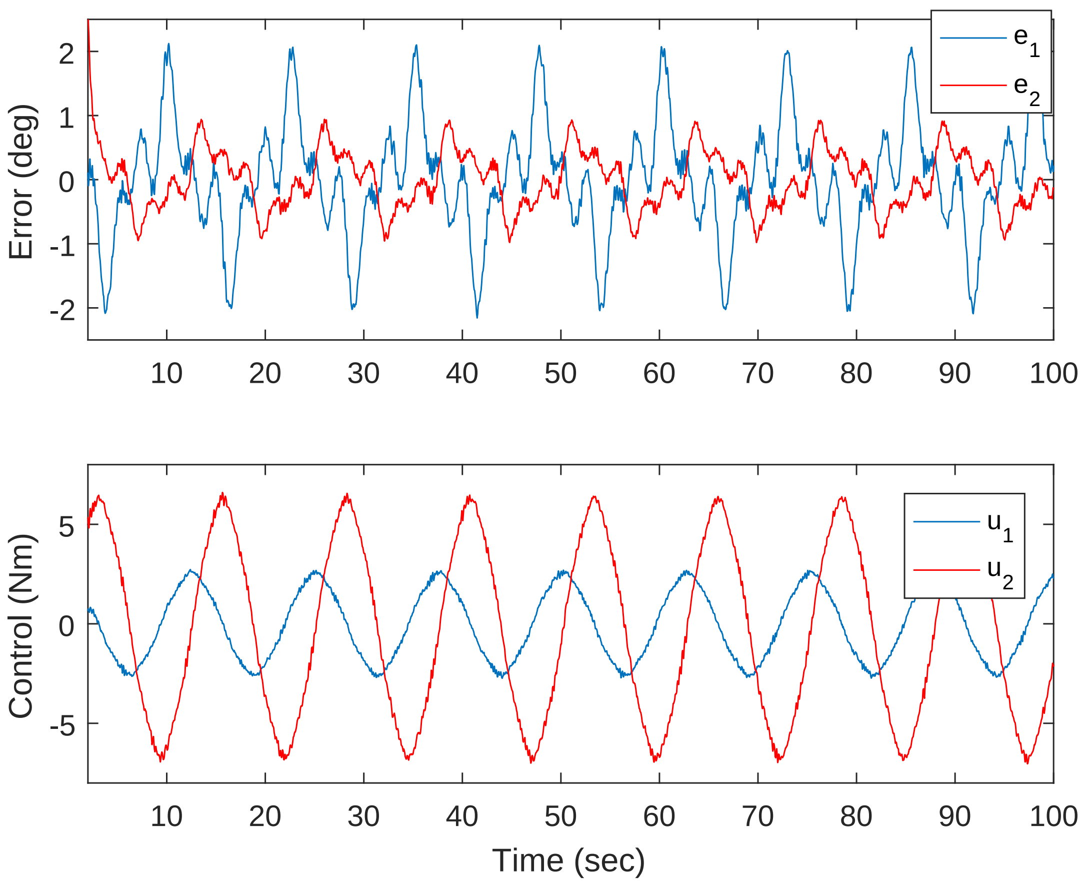

Figure 4-1. Variation of errors and control forces.

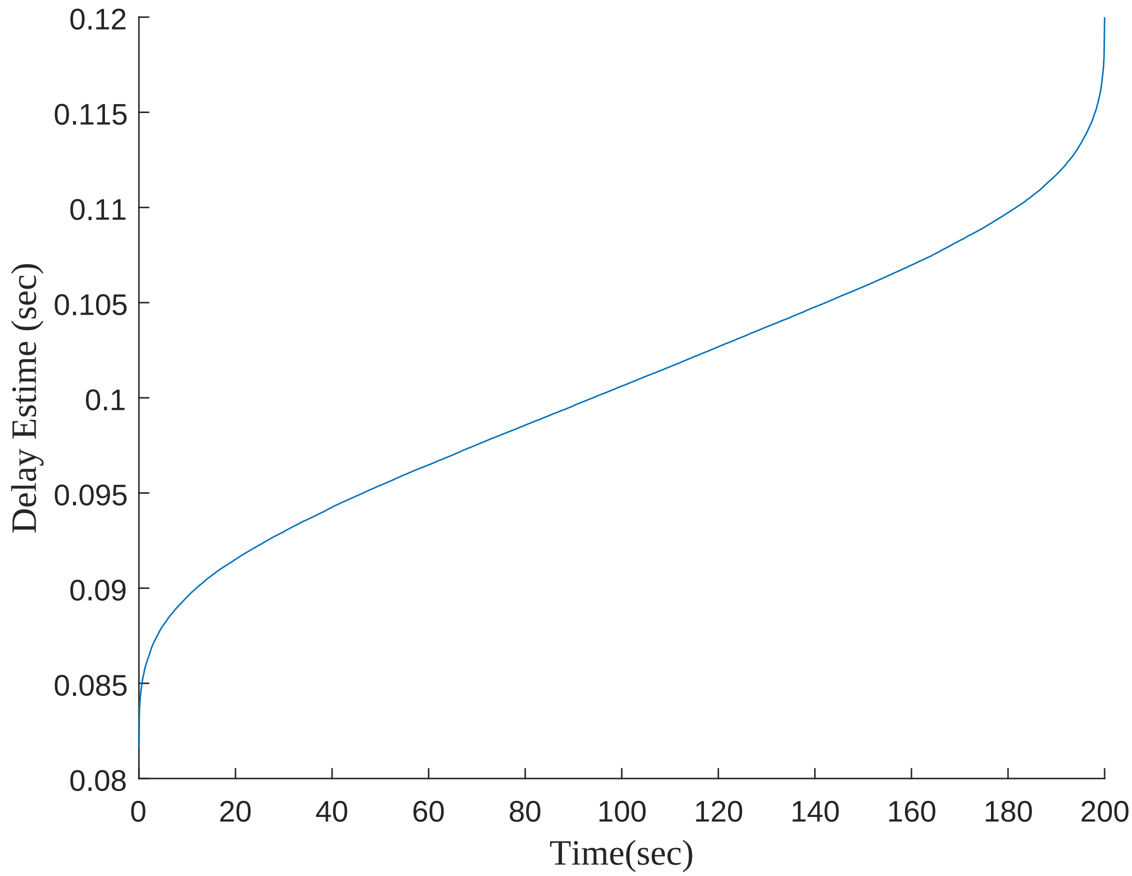

Figure 4-2. Delay estimate vs. time.

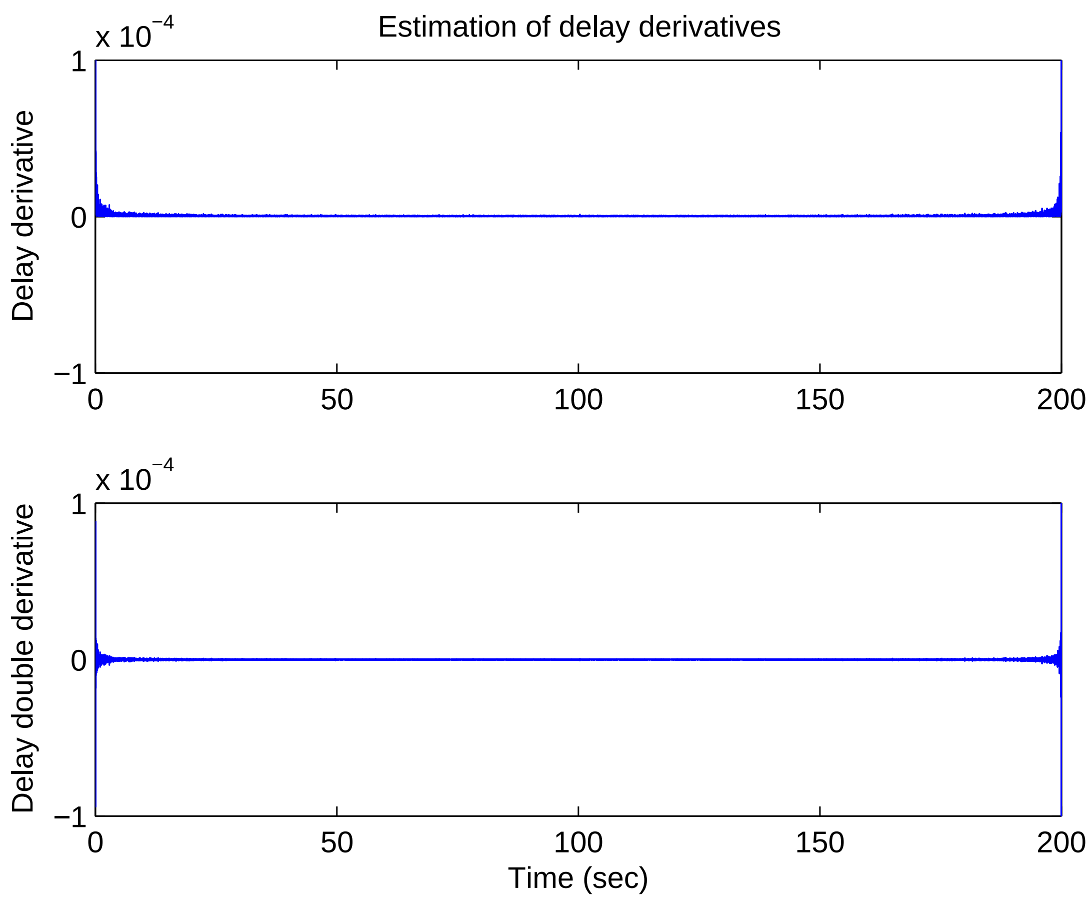

Figure 4-3. Delay derivatives vs. time.

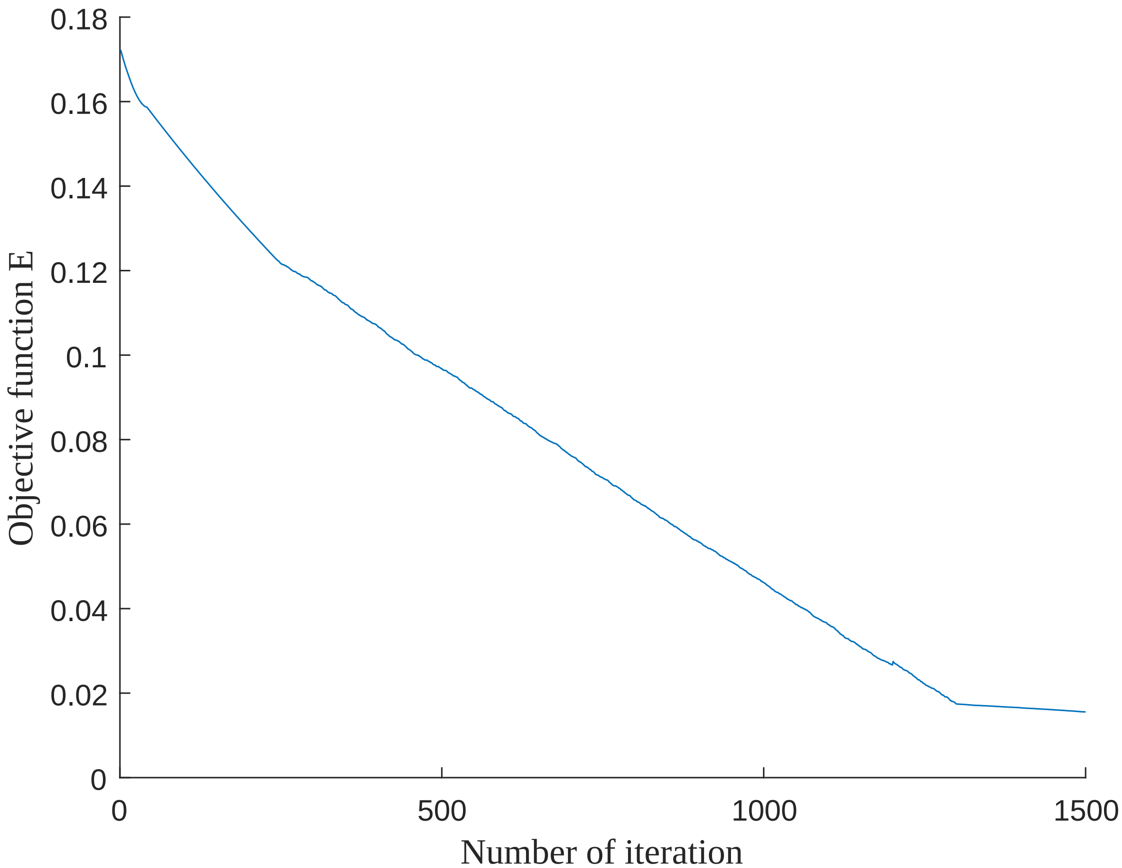

Figure 4-4. Objective function E in (4–45).

CHAPTER 5
BOUNDARY CONTROL OF STORE INDUCED OSCILLATIONS IN A FLEXIBLE AIRCRAFT WING WITH UNKNOWN TIME-VARYING INPUT DELAY

This chapter presents a robust controller for an elastic wing subjected to store induced oscillation with unknown time-varying input delay in boundary control feedback. 2D elastic aircraft wing is described by uncertain coupled nonlinear PDEs via regulation of the state variables as in ([1]) . An adaptive boundary controller added with a PDE based robust controller is designed to ensure the distributed states of the flexible wing are regulated exponentially to a residual ball of given radius. Unlike Krstic's work in [94], uncertain nonlinear PDE can not be transformed into an exponentially stable target system using Voltera Integral method. As a result, the controller is developed through a Lyapunov-based analysis. In the Lyapunov analysis, wing energy terms are used along with a novel Lyapunov Krasvoskii function, introduced in the PDE based delay work as in [44, 45, 95]. The developed controller uses the linear mapping approach inspired by predictor-based approaches such as [30] to map the time dependent control input to a modified control input that depends both on time and a spatial variable. Similar to predictor-based approaches, the modified input can be segregated into delayed and delay-free components. This segregation impacts the stability analysis in a way that allows for arbitrarily large delay rates, unlike existing results (cf., [3, 8, 31, 32]).

Another contribution of this result is that a neural network (NN) estimation scheme is introduced to estimate the unknown delay magnitude. Since the universal functional approximation theorem only holds for continuous functions whose domain is compact, a nonlinear mapping is introduced to map the non-compact time domain to a compact domain. Simulation results demonstrate the controller effectiveness to damp out the oscillation despite the presence of unknown time-varying input delay in boundary feedback.

5.1 2D Euler-Bernoulli Beam

Flexible aircraft wing can be modeled by a 2D cantilever Euler-Bernoulli beam model (as shown in Figure 5-1) of length $l \in \mathbb{R}$, chord length $c \in \mathbb{R}$, mass per unit span of $\rho \in \mathbb{R}$, moment of inertia per unit length of $I_w \in \mathbb{R}$, and bending and torsional stiffnesses of $EI \in \mathbb{R}$

and $GJ \in \mathbb{R}$, respectively, with a store of mass $m_s \in \mathbb{R}$ and moment of inertia $J_s \in \mathbb{R}$ attached at the free end of the beam.

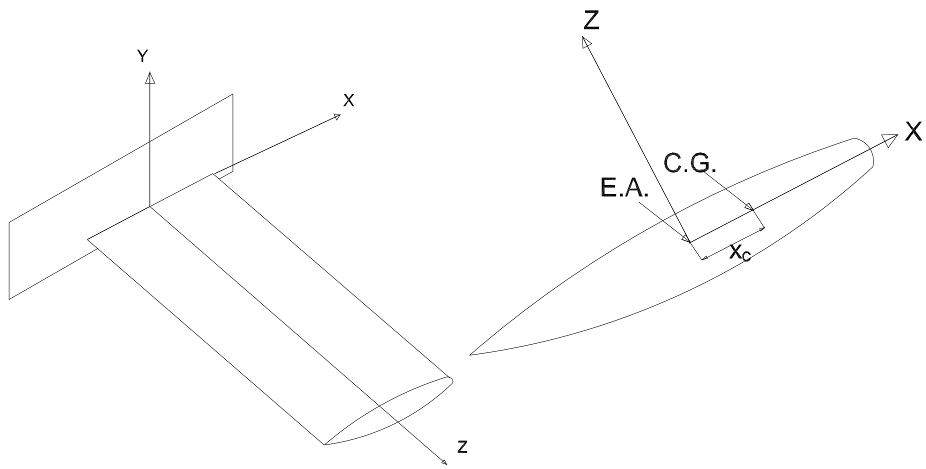

Figure 5-1. Schematic of the wing section, where E.A. denotes the elastic axis and C.G. denotes the center of gravity..

Similar to [1], Hamiltonian mechanics is used to develop bending (ω) and twisting (ϕ) dynamics of the cantilever beam as

$$L(t) = \rho\omega_{tt}(y,t) - \rho x_c \sin(\phi(y,t))\,\phi_t^2(y,t) + \rho x_c \cos(\phi(y,t))\,\phi_{tt}(y,t) + EI\omega_{yyyy}(y,t), \tag{5–1}$$

$$M(t) = \left(I_w + \rho x_c^2\right)\phi_{tt}(y,t) + \rho x_c \cos(\phi(y,t))\,\omega_{tt}(y,t) - GJ\phi_{yy}(y,t), \tag{5–2}$$

where $\omega : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ and $\phi : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ denote the bending and twisting displacements, respectively, $y \in [0,l]$ denotes spanwise location on the wing, $x_c \in \mathbb{R}$ represents the distance from the wing elastic axis to the wing center of gravity, $L : [0,\infty) \to \mathbb{R}$ and $M : [0,\infty) \to \mathbb{R}$ denote aerodynamic lift and moment per unit length, respectively. Throughout this paper, $(.)_t$ and $(.)_y$ denote partial derivatives of corresponding variable with respect to time and the spanwise position along a wing, respectively. In addition to the derived dynamics, boundary control conditions for the 2D cantilever beam are developed as

$$\omega(0,t) = \omega_y(0,t) = \omega_{yy}(l,t) = \phi(0,t) = 0, \tag{5–3}$$

$$L_{tip}(t - D(t)) = m_s\omega_{tt}(l,t) - m_s x_s \sin(\phi(l,t))\,\phi_t^2(l,t) - EI\omega_{yyy}(l,t)$$

$$+m_s x_s \cos\left(\phi\left(l,t\right)\right)\phi_{tt}\left(l,t\right), \tag{5–4}$$

$$M_{tip}(t-D(t)) = \left(m_s x_s^2 + J_s\right)\phi_{tt}\left(l,t\right) + GJ\phi_y\left(l,t\right) + m_s x_s \cos\left(\phi\left(l,t\right)\right)\omega_{tt}\left(l,t\right), \tag{5–5}$$

where L_{tip} : [0,∞) $\to \mathbb{R}$ and M_{tip} : [0,∞) $\to \mathbb{R}$ denote the aerodynamic lift and moment at the wing tip, $D : [0,\infty) \to \mathbb{R}$ denotes unknown time-varying input delay, associated with the time taken for the control forces to get applied on the system, and $x_s \in \mathbb{R}$ is the distance from the wing elastic axis to the store center of gravity. As in [44], a linear transformation is used to transform time-varying control input to a control input of two independent variables, i.e., p and t, where $t \in [0,\infty)$ and $p \in [0,1]$. This transformation produces control input as a two variable function (i.e., p and t), where evaluating at $p = 0$ and $p = 1$ gives delayed and delay-free control input, respectively. The linear transformation is of the form

$$u(p,t) \triangleq U\left(\psi(t + p\left(\psi^{-1}(t) - t\right)\right) \quad \psi(t) \le t, \forall t \ge 0, \tag{5–6}$$

where $u : [0,1] \times [0,\infty) \to \mathbb{R}^2$, $U \triangleq [L_{tip} \quad M_{tip}]^T \in \mathbb{R}^2$, $\psi : [0,\infty) \to \mathbb{R}$ is a known delay dependent invertible monotonous time function, defined as $\psi(t) \triangleq t - \hat{D}(t)$, where $\hat{D}(t) \in \mathbb{R}$ represents a known time-varying subsequently designed delay estimate, $\psi^{-1}(t)$ exists at all time. Transformation defined in (5–6), is used to express the delayed control input as $U(t - \hat{D}(t)) = u(0,t)$, and the delay-free control input as $U(t) = u(1,t)$. Similar as in [44], the spatial and time variation of $u(p,t)$, denoted by $u_p(p,t)$ and $u_t(p,t)$ respectively, can be related as

$$u_t(p,t) = \delta(p,t)u_p(p,t), \tag{5–7}$$

and the auxiliary function $\delta : [0,1] \times (0,\infty) \in \mathbb{R}$ is defined as

$$\delta(p,t) \triangleq \frac{1 + p\left(\frac{d\left(\psi^{-1}(t)\right)}{dt} - 1\right)}{\psi^{-1}(t) - t}, \tag{5–8}$$

where

$$\delta(0,t) \triangleq \delta_0 = \frac{1}{\hat{D}\left(\psi^{-1}(t)\right)}, \tag{5–9}$$

$$\delta(1,t) \triangleq \delta_1 = \frac{1}{\left(1-\dot{\hat{D}}(\psi^{-1}(t))\right)}\delta_0, \tag{5–10}$$

$$\frac{\partial}{\partial p}\delta(p,t) \triangleq \delta_p = \frac{\dot{\hat{D}}(\psi^{-1}(t))}{\left(1-\dot{\hat{D}}(\psi^{-1}(t))\right)}\delta_0. \tag{5–11}$$

In rest of the paper, $\bar{()}$ and $\underset{\sim}{()}$ indicate maximum and minimum value of the bracketed variable, respectively. First two assumptions are stated based on Remark 5.1 in [96].

Property 5.1. *Based on Remark 5.1 in [96], the potential energy of the system,* $E_P(t) \triangleq \frac{1}{2}\int_0^l \left(EI\omega_{yy}^2 + GJ\phi_y^2\right) dy$ *is assumed to be bounded* $\forall t \in [0,\infty)$*, and* $\frac{\partial^n \omega}{\partial y^n}$ *and* $\frac{\partial^m \phi}{\partial y^m}$ *are assumed to be bounded, uniformly in* y $\forall t \in [0,\infty)$ *for* $n = 2,3,4$ *and* $m = 1,2$*.*

Property 5.2. *Similarly, the kinetic energy of the system*

$$E_K(t) \triangleq \frac{1}{2}\int_0^l \left(\rho\omega_t^2 + 2\rho x_c \cos(\phi)\,\phi_t\omega_t\right) dy + \frac{1}{2}m_s\omega_t^2(l,t) + \frac{1}{2}J_s\phi_t^2(l,t)$$
$$+\frac{1}{2}\int_0^l \left(\left(I_w + \rho x_c^2\right)\phi_t^2\right) dy,$$

is assumed to be bounded $\forall t \in [0,\infty)$*, and* $\frac{\partial^q \omega}{\partial t^q}$ *and* $\frac{\partial^q \phi}{\partial t^q}$ *are assumed to be bounded, uniformly in* t $\forall y \in [0,l]$ *for* $q = 1,2,3$*.*

Assumption 5.1. *The subsequent control development is based on the assumption that* $\phi(l,.)$, $\omega_{tyyy}(l,.)$, $\phi_t(l,.)$, $\omega_{yyy}(l,.)$, $\phi_y(l,.)$, *and* $\phi_{ty}(l,.)$ *are measurable.*

Assumption 5.2. *The unknown time-varying input delay* $D(t) \in \mathbb{R}$ *is bounded by known positive constants,* $\bar{D}$ *and* $\underline{D}$ *respectively, as* $\underline{D} \leq D(t) \leq \bar{D}$*.*

Remark 5.1. In practice, time variation of both wing tip bending and twisting deflection can be measured by transducers. Spatial variation of bending deflection can be measured by strain gauges (as mentioned in [73]) or shear sensors (as discussed in [97]), based on the order of differentiation. Time variations of these sensor measurements can be obtained through numerical methods. Such measurements and numerical methods can introduce noise, and motivation exists for additional research to eliminate these higher-order measurements. Advances in fiber optic sensing (both Long Period Fiber Gratings and Fiber Bragg Grating) can also be used to measure the deformation of the wing. For example, fiber optic strain data from a ground load test of a full-scale aircraft wing can be used to measure the deflection of the wing and corrugated long-period fiber grating can be used to measure strain, bending and torsion of the wing as in.

5.2 Instability in Presence of Input Time Delay

An adaptive controller is developed in [1], to stabilize the store induced oscillation in an aircraft wing. In order to motivate the input delay problem, constant input delays of magnitude between $0-900$ ms, with an increment of 1 ms are injected in the system dynamics described in (5–1)-(5–2), to show impact of injected delay on the stability of the system. Simulation results validate the claim, that even a slight presence of input delay can destabilize a stable system. Figure 5-2, shows the designed controller in [1], is not sufficient to handle the presence of input delay in the system, in case of bending deflection of the wing. Similar result is noticed for twisting deflection, as shown in Figure 5-3. Figure 5-2 and 5-3 motivate the necessity of a controller that can compensate the presence of input delay in the system. Although, for this simulation demonstration, delay magnitude was known, it is needed to generalize the system by incorporating delay of unknown magnitude. Motivated by this necessity, in the following section, an adaptive controller is developed for the system described in (5–1)-(5–2), subjected to an unknown time-varying delay.

5.3 Control Development

Control objective is to ensure that in presence of time delay, both bending and twisting deflection go to zero throughout the whole wing span, as time progresses, i.e., $\omega(y,t)\to 0$ and $\phi(y,t)\to 0,\ \forall y\in[0,l]$ as $t\to\infty$. To facilitate subsequent stability analysis, an auxiliary error signal, $e:[0,\infty)\to\mathbb{R}^2$ and mass matrix, $\mathscr{M}:[0,\infty)\to\mathbb{R}^{2\times 2}$ defined as

$$e(t)\triangleq\begin{bmatrix}\omega_t(l,t)-\omega_{yyy}(l,t)\\ \phi_t(l,t)+\phi_y(l,t)\end{bmatrix}, \tag{5–12}$$

$$\mathscr{M} \triangleq \begin{bmatrix} m_s & m_s x_s \cos(\phi(l,t)) \\ m_s x_s \cos(\phi(l,t)) & m_s x_s^2 + J_s \end{bmatrix}. \tag{5–13}$$

In order to include boundary control equations (5–4)-(5–5) in the open-loop dynamics, another auxiliary error signal, $r : [0, \infty) \to \mathbb{R}^2$ defined as

$$r(t) \triangleq e(t) + \alpha e_u(t), \tag{5–14}$$

where $\alpha \in \mathbb{R}$ is known, diagonal, positive definite, constant gain. In (5–14), $e_u : [0, \infty) \to \mathbb{R}^2$ is an auxiliary error term, introduced to obtain a delay-free control expression for the input in the closed loop error system and can be expressed as

$$e_u(t) \triangleq \int_0^1 u(p,t)dp. \tag{5–15}$$

Before proceeding further with the error signal development, using Leibniz rule and definitions of δ_0, δ_1 and δ_p from (5–9)-(5–11), time derivative of $e_u(t)$ is calculated as

$$\dot{e}_u(t) = \delta_1 u(1,t) - \delta_0 u(0,t) - \delta_p e_u(t). \tag{5–16}$$

The open-loop dynamics for the error signal $r(t)$, can be expressed as

$$\dot{r}(t) = \begin{bmatrix} \omega_{tt}(l,t) \\ \phi_{tt}(l,t) \end{bmatrix} + \begin{bmatrix} -\omega_{tyyy}(l,t) \\ \phi_{ty}(l,t) \end{bmatrix} + \alpha \dot{e}_u(t). \tag{5–17}$$

Before proceeding further, (5–4) and (5–5) are rearranged to facilitate open-loop error signal development, and can be expressed as

$$\begin{aligned} \mathscr{M}_{inv}(t) U(t - D(t)) &= I_{2\times 2} \begin{bmatrix} \omega_{tt}(l,t) \\ \phi_{tt}(l,t) \end{bmatrix} \\ &+ \mathscr{M}_{inv}(t) \begin{bmatrix} -m_s x_s \sin(\phi(l,t)) \phi_t^2(l,t) - EI\omega_{yyy}(l,t) \\ GJ\phi_y(l,t) \end{bmatrix}, \end{aligned} \tag{5–18}$$

where $I_{2\times2}$ is 2×2 identity matrix and $\mathscr{M}_{inv} : [0,\infty) \to \mathbb{R}^{2\times2}$ can be expressed as

$$\mathscr{M}_{inv}(t) \triangleq \begin{bmatrix} \frac{m_s x_s^2 + J_s}{m_s^2 x_s^2 \sin^2(\phi(l,t)) + m_s J_s} & \frac{x_s \cos(\phi(l,t)}{m_s x_s^2 \sin^2(\phi(l,t)) + J_s} \\ \frac{x_s \cos(\phi(l,t)}{m_s x_s^2 \sin^2(\phi(l,t)) + J_s} & \frac{1}{m_s x_s^2 \sin^2(\phi(l,t)) + J_s} \end{bmatrix} = \frac{\mathcal{M}}{\bar{M}}, \tag{5–19}$$

where $\mathcal{M} : [0,\infty) \to \mathbb{R}^{2\times2}$ is defined as $\mathcal{M} \triangleq \begin{bmatrix} m_s x_s^2 + J_s & m_s x_s \cos(\phi(l,t) \\ m_s x_s \cos(\phi(l,t) & m_s \end{bmatrix}$ and $\bar{M} : [0,\infty) \to \mathbb{R}$ is defined as $\bar{M}(t) \triangleq m_s^2 x_s^2 \sin^2(\phi(l,t)) + m_s J_s$.

Proposition 5.1. *As $\mathscr{M}_{inv} \in \mathbb{R}^{2\times2}$, it is necessary and sufficient to prove that both $det(\mathscr{M}_{inv})$ and $trace(\mathscr{M}_{inv})$ are positive, in order to show that $\mathscr{M}_{inv}$ defined in (5–19) is positive definite $\forall t \in [0,\infty)$. $det(\mathscr{M}_{inv}) = \frac{1}{det(\mathscr{M})}$, from the definition of $\mathscr{M}$ in (5–13), $det(\mathscr{M}) = m_s^2 x_s^2 \sin^2(\phi(l,t)) + m_s J_s > 0$, which gives $det(\mathscr{M}_{inv}) = \frac{1}{m_s^2 x_s^2 \sin^2(\phi(l,t)) + m_s J_s} > 0$. Also, $trace(\mathscr{M}_{inv}) = \frac{m_s x_s^2 + J_s}{m_s^2 x_s^2 \sin^2(\phi(l,t)) + m_s J_s} + \frac{1}{m_s x_s^2 \sin^2(\phi(l,t)) + J_s} = \frac{m_s x_s^2 + J_s + m_s}{m_s^2 x_s^2 \sin^2(\phi(l,t)) + m_s J_s} > 0$. This means $\mathscr{M}_{inv}$ is positive definite $\forall t \in [0,\infty)$.*

Proposition 5.2. *A set of wing and store parameters satisfying these conditions are listed in [98]. Based on the sample data values in [98], $|m_s x_s^2| \ll |J_s|$, so $|\frac{m_s x_s^2 \sin^2(\phi(l,t))}{J_s}| < 1$. $\bar{M}(t)^{-1}$ can be expanded as a binomial series while $|\frac{m_s x_s^2 \sin^2(\phi(l,t))}{J_s}| < 1$.*

$$\begin{aligned} \bar{M}(t)^{-1} &= \left[m_s^2 x_s^2 \sin^2(\phi(l,t)) + m_s J_s\right]^{-1} = \frac{1}{m_s J_s}\left[1 + \frac{m_s x_s^2 \sin^2(\phi(l,t))}{J_s}\right]^{-1} \\ &\approx \frac{1}{m_s J_s}\left[1 - \frac{m_s x_s^2 \sin^2(\phi(l,t))}{J_s} + \frac{1}{2}\left(\frac{m_s x_s^2 \sin^2(\phi(l,t))}{J_s}\right)^2\right] \\ &= \left[\frac{1}{m_s J_s} - \frac{x_s^2 \sin^2(\phi(l,t))}{J_s^2} + \frac{1}{2}\left(\frac{m_s x_s^4 \sin^4(\phi(l,t))}{J_s^3}\right)\right]. \end{aligned}$$

After substituting, (5–18) and (5–16) in (5–17), yields

$$\dot{r}(t) = \frac{\mathcal{M}}{\bar{M}} U(t - D(t)) + \alpha\left[\delta_1 u(1,t) - \delta_0 u(0,t) - \delta_p e_u(t)\right] + Y(t)\theta, \tag{5–20}$$

where $Y : [0, \infty) \to \mathbb{R}^{2\times 27}$ is a regression matrix of known time-varying quantities and $\theta \in \mathbb{R}^{27}$ is a vector of unknown parameters, defined as

$$Y(t) \triangleq \begin{bmatrix} Y_1^1 & \dots & Y_6^1 & \dots & Y_{12}^1 & 0 & 0 & \dots & 0 \\ Y_1^2 & \dots & Y_6^2 & 0 & \dots & 0 & Y_{22}^2 & \dots & Y_{27}^2 \end{bmatrix}, \tag{5–21}$$

$$\theta \triangleq \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \dots & \theta_{27} \end{bmatrix}^T, \tag{5–22}$$

where $\theta_1 = J_s m_s x_s^2$, $\theta_2 = \frac{m_s^2 x_s^4}{J_s^2}$, $\theta_3 = \frac{m_s^3 x_s^6}{2J_s^3}$, $\theta_4 = J_s^2$, $\theta_5 = \frac{m_s x_s^2}{J_s}$, $\theta_6 = \frac{m_s^2 x_s^4}{2J_s^2}$, $\theta_7 = J_s m_s x_s^3$, $\theta_8 = \frac{m_s^2 x_s^5}{J_s^2}$, $\theta_9 = \frac{m_s^3 x_s^7}{2J_s^3}$, $\theta_{10} = J_s x_s^2 EI$, $\theta_{11} = \frac{x_s^4 m_s EI}{J_s^2}$, $\theta_{12} = \frac{m_s^2 x_s^6 EI}{2J_s^3}$, $\theta_{13} = x_s J_s^2$, $\theta_{14} = \frac{m_s x_s^3}{J_s}$, $\theta_{15} = \frac{m_s^2 x_s^5}{2J_s^2}$, $\theta_{16} = \frac{J_s^2 EI}{m_s}$, $\theta_{17} = \frac{x_s^2 EI}{J_s}$, $\theta_{18} = \frac{m_s x_s^4 EI}{2J_s^2}$, $\theta_{19} = J_s x_s GJ$, $\theta_{20} = \frac{x_s^3 GJ m_s}{J_s^2}$, $\theta_{21} = \frac{m_s^2 x_s^5 GJ}{2J_s^3}$, $\theta_{22} = J_s GJ$, $\theta_{23} = \frac{x_s^2 m_s GJ}{J_s^2}$, $\theta_{24} = \frac{m_s^2 x_s^4 GJ}{2J_s^3}$, $\theta_{25} = J_s x_s EI$, $\theta_{26} = \frac{x_s^3 m_s EI}{J_s^2}$, $\theta_{27} = \frac{m_s^2 x_s^5 EI}{2J_s^3}$ and $Y_1^1 = -\sin(\phi(l,t))\,\omega_{tyyy}(l,t)$, $Y_2^1 = \sin^4(\phi(l,t))\,\omega_{tyyy}(l,t)$, $Y_3^1 = -\sin^6(\phi(l,t))\,\omega_{tyyy}(l,t)$, $Y_4^1 = -\omega_{tyyy}(l,t)$, $Y_5^1 = \sin^2(\phi(l,t))\,\omega_{tyyy}(l,t)$, $Y_6^1 = -\sin^4(\phi(l,t))\,\omega_{tyyy}(l,t)$, $Y_7^1 = \sin(\phi(l,t))\,\phi_t^2(l,t)$, $Y_8^1 = -\sin^3(\phi(l,t))\,\phi_t^2(l,t)$, $Y_9^1 = \sin^5(\phi(l,t))\,\phi_t^2(l,t)$, $Y_{10}^1 = \omega_{yyy}(l,t)$, $Y_{11}^1 = -\sin^2(\phi(l,t))\,\omega_{yyy}(l,t)$, $Y_{12}^1 = \sin^4(\phi(l,t))\,\omega_{yyy}(l,t)$, $Y_{13}^1 = \sin(\phi(l,t))\,\phi_t^2(l,t)$, $Y_{14}^1 = -\sin^3(\phi(l,t))\,\phi_t^2(l,t)$, $Y_{15}^1 = \sin^5(\phi(l,t))\,\phi_t^2(l,t)$, $Y_{16}^1 = \omega_{yyy}(l,t)$, $Y_{17}^1 = -\sin^2(\phi(l,t))\,\omega_{yyy}(l,t)$, $Y_{18}^1 = \sin^4(\phi(l,t))\,\omega_{yyy}(l,t)$, $Y_{19}^1 = -\phi_y(l,t)\cos(\phi(l,t))$, $Y_{20}^1 = \sin^2(\phi(l,t))\,\phi_y(l,t)\cos(\phi(l,t))$, $Y_{21}^1 = -\sin^4(\phi(l,t))\,\phi_y(l,t)\cos(\phi(l,t))$, $Y_1^2 = \sin(\phi(l,t))\cos(\phi(l,t))\,\phi_t^2(l,t) + \sin^2(\phi(l,t))\,\phi_{ty}(l,t)$, $Y_2^2 = -\sin^4(\phi(l,t))\,\phi_{ty}(l,t) - \sin^3(\phi(l,t))\cos(\phi(l,t))\,\phi_t^2(l,t)$, $Y_3^2 = \sin^6(\phi(l,t))\,\phi_{ty}(l,t) + \sin^5(\phi(l,t))\cos(\phi(l,t))\,\phi_t^2(l,t)$, $Y_4^2 = \phi_{ty}(l,t)$, $Y_5^2 = -\sin^2(\phi(l,t))\,\phi_{ty}(l,t)$, $Y_6^2 = \sin^4(\phi(l,t))\,\phi_{ty}(l,t)$, $Y_{22}^2 = -\phi_y(l,t)$, $Y_{23}^2 = \sin^2(\phi(l,t))\,\phi_y(l,t)$, $Y_{24}^2 = -\sin^4(\phi(l,t))\,\phi_y(l,t)$, $Y_{25}^2 = \omega_{yyy}(l,t)\cos(\phi(l,t))$, $Y_{26}^2 = -\sin^2(\phi(l,t))\,\omega_{yyy}(l,t)\cos(\phi(l,t))$, $Y_{27}^2 = \sin^4(\phi(l,t))\,\omega_{yyy}(l,t)\cos(\phi(l,t))$.

Remark 5.2. Projection algorithm and adaptation law are used to estimate the unknown parameters in θ. Due to use of projection algorithm in the estimation of θ, $\mathscr{M}$ can be upper and lower bounded by $\bar{\mathscr{M}}$ and $\underline{\mathscr{M}}$ respectively. Similarly using the same argument, $\bar{M}(t)$ can be upper and lower bounded by $\bar{M}_{up}$ and $\bar{M}_{low}$ respectively.

Based on the subsequent stability analysis, delay-free control input is designed as

$$U(t) = u(1,t) \triangleq -\frac{K}{\alpha\delta_1}r(t) - \frac{1}{\alpha\delta_1}Y(t)\hat{\theta}(t), \tag{5–23}$$

where $K \in \mathbb{R}$ is a positive constant control gain, $\hat{\theta} : [0,\infty) \to \mathbb{R}^{27}$ is a time-varying estimate of unknown parameters as in θ. Gradient update law is used to update the estimate of unknown parameters, defined as

$$\dot{\hat{\theta}}(t) = \Gamma Y(t)^T r(t), \tag{5–24}$$

where $\Gamma \in \mathbb{R}^{27\times 27}$ is a positive definite control gain. The closed-loop error dynamics is developed by substituting delay-free control input (5–23) into open-loop error dynamics in (5–20), and can be expressed as

$$\dot{r}(t) = \frac{\mathcal{M}}{\bar{M}}U\left(t - D(t)\right) - Kr - \alpha\delta_0 u(0,t) - \alpha\delta_p e_u(t) + Y(t)\tilde{\theta}, \tag{5–25}$$

where $\tilde{\theta} : [0,\infty) \to$, defined as $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$, is difference between actual and estimation of unknown parameter values.

5.4 Neural Network Based Delay Estimation

A neural network (NN) based function approximator is used to estimate the unknown delay magnitude. The universal function approximation theorem only holds over a compact domain. Therefore, to approximate the unknown delay function, a nonlinear mapping is defined to map the non-compact domain to a compact spatial domain. Let $f_L : t \to \xi$ be defined as

$$f_L \triangleq \frac{\kappa t}{1+\kappa t}, \quad t \in [0,\infty), \quad \xi \in [0,1], \tag{5–26}$$

where $\kappa \in \mathbb{R}^+$ is a user defined saturation coefficient. Using (5–26), $D(t)$ can be mapped into the domain ξ as

$$D(t) = D\left(f_L^{-1}(\xi)\right) \triangleq D_{f_L}(\xi). \tag{5–27}$$

The universal functional approximation theorem can be used to represent $D_{f_L}(\xi)$ by a three-layer NN as

$$D_{f_L}(\xi) \triangleq W^T\sigma\left(V^T\Xi\right) + \epsilon, \tag{5–28}$$

where $W \in \mathbb{R}^{(L+1)\times 1}$ and $V \in \mathbb{R}^{3\times L}$ are the bounded constant ideal weights for the first-to-second and second-to-third layers, respectively, L is the number of neurons in the hidden layer, $\sigma \in \mathbb{R}^{(L+1)}$ is an activation function, ϵ is the functional reconstruction error, and $\Xi = [1 \quad \xi]^T$ denotes the input to the NN. Based on (5–27), the NN estimation for $\hat{D}(t)$ is given by

$$\hat{D}(t) = \hat{W}^T\sigma\left(\hat{V}^T\Xi\right), \tag{5–29}$$

where $\hat{W}$ and $\hat{V}$ are estimates of the ideal weights. Using (5–28) and (5–29), the mismatch between $D(t)$ and $\hat{D}(t)$ can be obtained using a Taylor's series approximation, which after some algebraic manipulation, can be expressed as

$$\begin{aligned} D(t) - \hat{D}(t) &= W^T\sigma\left(V^T\Xi\right) - \hat{W}^T\sigma\left(\hat{V}^T\Xi\right) + \epsilon, \\ &= \tilde{W}^T\sigma\left(\hat{V}^T\Xi\right) + \hat{W}^T\sigma'\left(\hat{V}^T\Xi\right)\tilde{V}^T\Xi + W^T\mathcal{O}\left(\tilde{V}^T\Xi\right)^2 + \epsilon \\ &\quad +\tilde{W}\sigma'\left(\hat{V}^T\Xi\right)\tilde{V}^T\Xi, \end{aligned} \tag{5–30}$$

where $\tilde{W} = W - \hat{W} \in \mathbb{R}^{(L+1)\times 1}$ and $\tilde{V} = V - \hat{V} \in \mathbb{R}^{2\times L}$, are the estimate mismatch for the ideal weight matrices, and $\mathcal{O}$ represents higher order terms. In the subsequent development a continuously differential projection algorithm as shown in Section 3.7, is used to design the adaptive update laws for $\hat{W}$ and $\hat{V}$. As a result, the elements of $\hat{W}$ and $\hat{V}$ can all be upper and lower bounded by known positive constants. Hence, $\tilde{W}^T\sigma'\left(\hat{V}^T\Xi\right)\tilde{V}^T\Xi$ and $W^T\mathcal{O}\left(\tilde{V}^T\Xi\right)^2$ can also be bounded by known positive constants, and therefore,

$$D(t) - \hat{D}(t) \leq \tilde{W}^T\sigma\left(\hat{V}^T\Xi\right) + \hat{W}^T\sigma'\left(\hat{V}^T\Xi\right)\tilde{V}^T\Xi + \bar{\epsilon}, \tag{5–31}$$

where $\bar{\epsilon} \in \mathbb{R}$ is a positive bounding constant.

5.5 Lyapunov-based Stability Analysis

To facilitate the subsequent stability analysis, energy terms $E_T : [0,\infty) \to \mathbb{R}$, $E_C : [0,\infty) \to \mathbb{R}$, $E_B : [0,\infty) \to \mathbb{R}$ are defined as

$$E_T(t) \triangleq \frac{1}{2}\int_0^l \left(\rho\omega_t^2 + 2\rho x_c \cos(\phi)\,\phi_t\omega_t + EI\omega_{yy}^2\right) dy + \frac{1}{2}\int_0^l \left(\left(I_w + \rho x_c^2\right)\phi_t^2 + GJ\phi_y^2\right) dy, \tag{5–32}$$

$$E_C(t) \triangleq \beta \int_0^l \rho\omega_y y\left(\omega_t + x_c\cos(\phi)\,\phi_t\right) dy + \beta\int_0^l \phi_y y\left(I_w + \rho x_c^2\right)\phi_t dy$$
$$+\beta\int_0^l \phi_y y\rho x_c\cos(\phi)\,\omega_t dy, \tag{5–33}$$

$$E_B(t) \triangleq \int_0^l \left(\omega_t^2 + \omega_{yy}^2 + \phi_t^2 + \phi_y^2\right) dy, \tag{5–34}$$

where $\beta \in \mathbb{R}$ is a positive control gain. Young's Inequality is used to upper and lower bound $E_T(t)$, $E_C(t)$ and can be expressed as

$$E_T(t) \leq \frac{1}{2}\max\Big\{\left(\rho + \rho\left|x_c\right|\right)\left(I_w + \rho x_c^2 + \rho\left|x_c\right|\right), EI, GJ\Big\}E_B(t), \tag{5–35}$$

$$E_T(t) \geq \frac{1}{2}\min\Big\{\left(\rho - \rho\left|x_c\right|\right), \left(I_w + \rho x_c^2 - \rho\left|x_c\right|\right), EI, GJ\Big\}E_B(t), \tag{5–36}$$

$$E_C(t) \leq \beta l\max\Big\{\left(\rho + \rho\left|x_c\right|\right), l^2\left(\rho + \rho\left|x_c\right|\right), \left(I_w + \rho x_c^2 + \rho\left|x_c\right|\right)\Big\}E_b(t), \tag{5–37}$$

$$E_C(t) \geq \beta l\max\Big\{\left(\rho + \rho\left|x_c\right|\right), l^2(\rho + \rho\left|x_c\right|), \left(I_w + \rho x_c^2 + \rho\left|x_c\right|\right)\Big\}E_b(t). \tag{5–38}$$

Remark 5.3. Provided that $|x_c| < 1$ and $I_w > \rho x_c^2 - \rho\left|x_c\right|$, E_T will be non-negative. The conditions $|x_c| < 1$ and $I_w > \rho x_c^2 - \rho|x_c|$ are engineering design considerations that ensure the store is mounted sufficiently close to the wing center of mass [98].

From (5–35) and (5–38), if β is selected as $\beta < \frac{\psi_1}{2l\psi_2}$,where

$$\psi_1 \triangleq \min\left\{\left(\rho - \rho\left|x_c\right|\right), \left(I_w + \rho x_c^2 - \rho\left|x_c\right|\right), EI, GJ\right\},$$
$$\psi_2 \triangleq \max\left\{\left(\rho + \rho\left|x_c\right|\right), l^2\left(\rho + \rho\left|x_c\right|\right), \left(I_w + \rho x_c^2 + \rho\left|x_c\right|\right)\right\}$$

then

$$\zeta_1 E_B(t) \leq E_T(t) + E_C(t) \leq \zeta_2 E_B(t) \tag{5–39}$$

where the positive constants ζ_1 and ζ_2 are defined as

$$\zeta_1 \triangleq \frac{1}{2}\psi_1 - \beta l\psi_2, \quad \zeta_2 \triangleq \frac{1}{2}\psi_2 + \beta l\psi_2.$$

Before proceeding further with the analysis, an LK functional $Q : [0,\infty) \to \mathbb{R}$ defined as

$$Q(t) \triangleq \lambda_Q \int_0^1 e^{\omega_2 p} u^T(p,t) u(p,t) dp, \tag{5–40}$$

where $\omega_2, \lambda_Q \in \mathbb{R}$ are known, positive constants.

Theorem 5.1. *Given the open-loop error system in (5–20), the controller in (5–23) along with the adaptive law in (5–24), ensures the system states ω, ϕ are UUB $\forall y \in [0,l]$ as $t \to \infty$ provided the following sufficient gain conditions are satisfied:*

$$\beta l < K_\beta, \tag{5–41}$$

$$\beta\rho - \beta\rho x_c - \bar{L}_w > 0, \tag{5–42}$$

$$\frac{3\bar{E}I}{2} - \frac{\bar{L}_w l^3}{2} > 0, \tag{5–43}$$

$$\beta\left(I_w + \rho x_c^2\right) - \beta\rho x_c - \bar{M}_w > 0, \tag{5–44}$$

$$\beta\bar{G}J - \beta\bar{M}_w l^3 - \beta\bar{M}_w l - \beta\bar{L}_w l^3 - \left(\bar{M}_w + \bar{L}_w\right) l^2 > 0, \tag{5–45}$$

$$\beta\bar{E}Il + \bar{E}I - \beta\rho - \beta\rho x_c l > 0, \tag{5–46}$$

$$\bar{G}J - \beta l\left(I_w + \rho x_c^2\right) - \beta\rho x_c l > 0, \tag{5–47}$$

and all the gain conditions are satisfied in Section 5.7.

Remark 5.4. The sufficient gain conditions in (5–41-5–47) can be satisfied by a combination of gain selections and engineering design considerations. Selection of the wing aerodynamic properties can be done to satisfy aircraft performance criteria (e.g., minimum takeoff distance, maximum range, etc.). The structural properties of the wing can then be selected to satisfy the sufficient conditions. Increasing the stiffness and mass of the wing or mounting the store closer to the wing center of mass will satisfy the sufficient conditions. A set of wing and store parameters satisfying these conditions are listed in [98].

Proof. Let $V_L : [0, \infty) \to \mathbb{R}^+$, continuously differentiable function defined as

$$V_L(t) \triangleq E_T(t) + E_C(t) + \frac{1}{2}r(t)^T r(t) + \frac{\omega_1}{2}e_u(t)^T e_u(t) + Q(t) + \frac{\tilde{\theta}(t)^T \Gamma^{-1} \tilde{\theta}(t)}{2}$$
$$+\frac{1}{2}tr\left(\tilde{W}(t)^T \Delta_1^{-1} \tilde{W}(t)\right) + \frac{1}{2}tr\left(\tilde{V}(t)^T \Delta_2^{-1} \tilde{V}(t)\right). \tag{5–48}$$

From the definition of the error signals and Q, following inequality is developed and can be expressed as

$$\varphi_L \|y\|^2 + c_L \leq \left\{ \frac{1}{2}tr\left(\tilde{W}(t)^T \Delta_1^{-1} \tilde{W}(t)\right) + \frac{1}{2}tr\left(\tilde{V}(t)^T \Delta_2^{-1} \tilde{V}(t)\right) + \frac{1}{2}r(t)^T r(t) \right.$$
$$\left. + \frac{\omega_1}{2}e_u(t)^T e_u(t) + Q(t) \right\} \leq \varphi_U \|y\|^2 + c_U, \tag{5–49}$$

where $y : [0, \infty) \to \mathbb{R}^7$, defined as $y(t) \triangleq [e(t)^T \quad r(t)^T \quad e_u(t)^T \quad Q]^T$ and $c_L, c_U \in \mathbb{R}^+$ are known bounding constants. Using inequalities in ((5–39)), and the relation that V_L can be bounded as

$$V_L(t) \geq \zeta_1 E_B(t) + \frac{\lambda_{min}\left[\Gamma^{-1}\right]}{2}\left\|\tilde{\theta}(t)\right\|^2 + \varphi_L \|y\|^2 + c_L, \tag{5–50}$$
$$V_L(t) \leq \zeta_2 E_B(t) + \frac{\lambda_{max}\left[\Gamma^{-1}\right]}{2}\left\|\tilde{\theta}(t)\right\|^2 + \varphi_U \|y\|^2 + c_U, \tag{5–51}$$

where $\lambda_{min}()$ and $\lambda_{max}()$ denote the minimum and maximum eigenvalues of respective item. Taking the time derivative of (5–48) and using (5–16), (5–25) and the update law in (5–24), yields

$$\dot{V}_L = r(t)^T \left(\frac{\mathcal{M}}{\bar{M}}U\left(t - D(t)\right) + Y(t)\tilde{\theta}\right) + \dot{E}_T(t) + \dot{E}_C(t) - \tilde{\theta}(t)^T \Gamma^{-1} \dot{\tilde{\theta}}(t)$$
$$+r(t)^T\left(-Kr(t) - \alpha\delta_0 u(0,t) - \alpha\delta_p e_u(t)\right) + \lambda_Q \delta_1 e^{\omega_2} u^T(1,t)u(1,t)$$
$$+\omega_1 e_u(t)^T\left(\delta_1 u(1,t) - \delta_0 u(0,t) - \delta_p e_u(t)\right) - \lambda_Q \omega_2 \int_0^1 \delta e^{\omega_2 p} u^T(p,t)u(p,t)dp$$
$$+\frac{\left[\frac{\bar{\mathcal{M}}}{\bar{M}_{low}} - \alpha\underline{\delta_0}\right] + \omega_1 \bar{\delta}_p}{2\epsilon_2\left(\underline{\delta_0} - \bar{\delta}_1 e^{\omega_2}\left(1 + \lambda_M\right)\right)}\delta_0 u^T(0,t)u(0,t) - \lambda_Q \delta_0 u^T(0,t)u(0,t)$$
$$+\lambda_Q \delta_p \int_0^1 e^{\omega_2 p} u^T(p,t)u(p,t)dp + tr\left(\tilde{W}(t)^T \Delta_1^{-1} \dot{\tilde{W}}(t)\right) + tr\left(\tilde{V}(t)^T \Delta_2^{-1} \dot{\tilde{V}}(t)\right). \tag{5–52}$$

By using (5–24), Young's inequality, and the facts that $\dot{\tilde{W}} = -\dot{\hat{W}}$and $\dot{\tilde{V}} = -\dot{\hat{V}}$, an upper bound on $\dot{V}_L$ can be obtained as

$$\dot{V}_L \leq r(t)^T \left(\frac{\bar{\mathscr{M}}}{\bar{M}} U\left(t - D(t)\right)\right) + \frac{\epsilon_1 \alpha \bar{\delta}_p}{2} \|r(t)\|^2 + \dot{E}_T(t) + \dot{E}_C(t)$$
$$-K\|r(t)\|^2 + r(t)^T\left(-\alpha\delta_0 u(0,t)\right) + \frac{\alpha\bar{\delta}_p}{2\epsilon_1}\|e_u(t)\|^2 + \lambda_Q\bar{\delta}_1 e^{\omega_2}\|u(1,t)\|^2$$
$$+\omega_1 e_u(t)^T\left(\delta_1 u(1,t) - \delta_0 u(0,t)\right) + \omega_1|\bar{\delta_p}|\|e_u(t)\|^2 - \lambda_Q\omega_2\int_0^1 \delta e^{\omega_2 p} u^T(p,t)u(p,t)dp$$
$$-\left(\lambda_Q - \left[\frac{\left[\frac{\bar{\mathscr{M}}}{\bar{M}_{low}} - \alpha\underline{\delta_0}\right] + \omega_1\bar{\delta}_p}{2\epsilon_2\left(\underline{\delta_0} - \bar{\delta}_1 e^{\omega_2}\left(1 + \lambda_M\right)\right)}\right]\right)\underline{\delta_0}\|u(0,t)\|^2$$
$$+\lambda_Q\bar{\delta}_p\int_0^1 e^{\omega_2 p}u^T(p,t)u(p,t)dp - tr\left(\tilde{W}(t)^T\Delta_1^{-1}\dot{\hat{W}}(t)\right) - tr\left(\tilde{V}(t)^T\Delta_2^{-1}\dot{\hat{V}}(t)\right). \quad (5–53)$$

In (5–53), $\dot{E}_T$ is determined by differentiating (5–32) with respect to time to obtain

$$\dot{E}_T(t) = \int_0^l \omega_t\left(\rho\omega_{tt} + \rho x_c\cos\left(\phi\right)\phi_{tt} - \rho x_c\sin\left(\phi\right)\phi_t^2\right)dy$$
$$+\int_0^l\left(EI\omega_{yy}\omega_{tyy} + GJ\phi_y\phi_{ty}\right)dy + \int_0^l\left(\left(I_w + \rho x_c^2\right)\phi_{tt} + \rho x_c\cos\left(\phi\right)\omega_{tt}\right)\phi_t dy. \quad (5–54)$$

Substituting (5–1) and (5–2) into the first two integrals of (5–54) yields

$$\dot{E}_T(t) = \int_0^l\left(L\omega_t + M\phi_t\right)dy - \int_0^l EI\omega_t\omega_{yyyy}dy + \int_0^l EI\omega_{yy}\omega_{tyy}dy + \int_0^l GJ\phi_t\phi_{yy}dy$$
$$+\int_0^l GJ\phi_y\phi_{ty}dy. \quad (5–55)$$

Integrating the third and fifth integrals in (5–55) by parts and applying the boundary conditions of the PDE system gives

$$\int_0^l EI\omega_{yy}\omega_{tyy}dy = -EI\omega_{yyy}(l,t)\omega_t(l,t) + \int_0^l EI\omega_t\omega_{yyyy}dy, \quad (5–56)$$
$$\int_0^l GJ\phi_y\phi_{ty}dy = GJ\phi_y(l,t)\phi_t(l,t) - \int_0^l GJ\phi_t\phi_{yy}dy. \quad (5–57)$$

Using the expressions in (5–56) and (5–57) and using the error signal definition in (5–12), (5–55) can be rewritten as

$$\dot{E}_T(t) = \int_0^l (L\omega_t + M\phi_t)\, dy + e(t)^T \begin{bmatrix} \frac{EI}{2} & 0 \\ 0 & \frac{k_\beta GJ}{2} \end{bmatrix} e(t) - \frac{EI}{2}\left(\omega_t^2(l,t) + \omega_{yyy}^2(l,t)\right)$$
$$-\tfrac{k_\beta GJ}{2}\left(\phi_y^2(l,t) + \phi_t^2(l,t)\right). \qquad (5\text{–}58)$$

After integrating and using Young's Inequality and Lemma A.12 from [96], $\dot{E}_C$ can be upper bounded as

$$\begin{aligned}\dot{E}_C(t) \leq & -(1-x_c)\frac{\beta\rho}{2}\int_0^l \omega_t^2 dy + \frac{\beta EIl}{2}e_1^2 + \frac{1}{2}\beta l\left(I_w + \rho x_c^2\right)\phi_t^2\left(l,t\right) - \frac{\beta EIl}{2}\omega_y^2\left(l,t\right) \\ & -\left(\frac{3EI}{2} - \frac{Ll^3}{2}\right)\beta\int_0^l \omega_{yy}^2 dy - \left(I_w + \rho x_c^2 - \rho x_c\right)\frac{\beta}{2}\int_0^l \phi_t^2 dy + \frac{1}{2}\beta GJl\phi_y^2\left(l,t\right) \\ & -\left(GJ - Ml^3 - Ml - Ll^3\right)\frac{\beta}{2}\int_0^l \phi_y^2 dy - \frac{\beta EIl}{2}\omega_{yyy}^2\left(l,t\right) \\ & +\frac{1}{2}\beta\rho l\omega_t^2\left(l,t\right) + \beta\rho x_c l\phi_t\left(l,t\right)\omega_t\left(l,t\right), \end{aligned} \qquad (5\text{–}59)$$

where e_1 denotes the first element of the vector e, (i.e., $e_1(t) \triangleq \omega_t\left(l,t\right) - \omega_{yyy}\left(l,t\right)$). Before proceeding further, note that

$$\begin{aligned}\delta\left(p,t\right) &= \frac{1 + p\left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)}{\phi^{-1}(t) - t},\ p \in [0,1] \\ &= \delta_0 + p\left(\delta_1\right) - p\delta_0, \\ &= \delta_0 + \left(\delta_1 - \delta_0\right)p, \\ &\geq \min\left\{\delta_0, \delta_1\right\}. \end{aligned} \qquad (5\text{–}60)$$

Using the Cauchy-Schwarz inequality,

$$\begin{aligned}\|e_u\|^2 &\leq \int_0^1 \|u^T\left(p,t\right)u(p,t)\|dp. \int_0^1 1dp, \\ \|e_u\|^2 &\leq \int_0^1 \|u^2\left(p,t\right)\|dp. \end{aligned} \qquad (5\text{–}61)$$

Using Mean Value Theorem, following inequalities have been developed

$$
\begin{aligned}
U(t) &= \dot{U}(c)\hat{D}(t) + U\left(t - \hat{D}(t)\right), \\
u(1,t) &\leq \lambda_M \bar{\hat{D}} + u(0,t). && \text{(5–62)} \\
u^T(1,t)u(1,t) &\leq u^T(0,t)u(0,t) + 2\lambda_M \bar{\hat{D}} u(0,t) + \lambda_M^2 \bar{\hat{D}}^2, \\
\|u(1,t)\|^2 &\leq (1+\lambda_M)\|u(0,t)\|^2 + \left(\lambda_M + \lambda_M^2\right)\bar{\hat{D}}^2. && \text{(5–63)}
\end{aligned}
$$

Using Mean Value Theorem and the expression in (5–31), the following equality can be developed as

$$
\begin{aligned}
\frac{\bar{\mathscr{M}}}{\bar{M}_{low}} \left| r^T \left(U\left(t - \hat{D}(t)\right) - U\left(t - D(t)\right)\right) \right| &\leq \frac{\bar{\mathscr{M}}\lambda_M}{\bar{M}_{low}} |r^T \bar{\epsilon}| \\
&\leq \frac{\bar{\mathscr{M}}\lambda_M}{\bar{M}_{low}} \left| r^T \tilde{W}^T \sigma\left(\hat{V}^T \Xi\right) \right| \\
&+ \frac{\bar{\mathscr{M}}\lambda_M}{\bar{M}_{low}} \left| r^T \hat{W}^T \sigma'\left(\hat{V}^T \Xi\right) \tilde{V}^T \Xi \right| \\
&+ \frac{\bar{\mathscr{M}}\lambda_M}{\bar{M}_{low}} \left| r^T \left(D(t) - \hat{D}(t)\right) \right|. && \text{(5–64)}
\end{aligned}
$$

The expressions in (5–58), (5–59), (5–62), (5–63) and Remark 5.2, can be used to upper bound (5–53) as

$$
\begin{aligned}
\dot{V}_L \leq{}& \frac{\bar{\mathscr{M}}}{\bar{M}_{low}} r(t)^T \left(U\left(t - D(t)\right) - U\left(t - \hat{D}(t)\right)\right) + \frac{\alpha\bar{\delta}_p}{2\epsilon_1}\|e_u(t)\|^2 + \frac{\epsilon_1 \alpha \bar{\delta}_p}{2}\|r(t)\|^2 \\
& -K\|r(t)\|^2 + \left[\frac{\bar{\mathscr{M}}}{\bar{M}_{low}} - \alpha\underline{\delta_0}\right] r(t)^T u(0,t) - \omega_1 |\bar{\delta_p}| \|e_u(t)\|^2 \\
& +\omega_1 e_u(t)^T \left[\delta_1 \lambda_M \bar{\hat{D}} + \delta_p u(0,t)\right] + \int_0^l \left(L\omega_t + M\phi_t\right) dy + e(t)^T \begin{bmatrix} \frac{EI}{2} & 0 \\ 0 & \frac{k_\beta GJ}{2} \end{bmatrix} e(t) \\
& -\frac{EI}{2}\left(\omega_t^2(l,t) + \omega_{yyy}^2(l,t)\right) - \tfrac{k_\beta GJ}{2}\left(\phi_y^2(l,t) + \phi_t^2(l,t)\right) \\
& -(1-x_c)\frac{\beta\rho}{2}\int_0^l \omega_t^2 dy + \frac{\beta EIl}{2} e_1^2 - \lambda_Q \underline{\delta_0}\|u(0,t)\|^2 \\
& -\left(\frac{3EI}{2} - \frac{Ll^3}{2}\right)\beta \int_0^l \omega_{yy}^2 dy - \left(I_w + \rho x_c^2 - \rho x_c\right)\frac{\beta}{2}\int_0^l \phi_t^2 dy \\
& +\frac{1}{2}\beta l\left(I_w + \rho x_c^2\right)\phi_t^2(l,t) - \frac{\beta EIl}{2}\omega_y^2(l,t) - \left(GJ - Ml^3 - Ml - Ll^3\right)\frac{\beta}{2}\int_0^l \phi_y^2 dy
\end{aligned}
$$

$$-\frac{\beta EIl}{2}\omega_{yyy}^{2}\left(l,t\right)+\frac{1}{2}\beta GJl\phi_{y}^{2}\left(l,t\right)+\frac{1}{2}\beta\rho l\omega_{t}^{2}\left(l,t\right)+\beta\rho x_{c}l\phi_{t}\left(l,t\right)\omega_{t}\left(l,t\right)$$
$$+\lambda_{Q}\bar{\delta}_{1}e^{\omega_{2}}\left[\left(1+\lambda_{M}\right)\|u(0,t)\|^{2}+\left(\lambda_{M}+\lambda_{M}^{2}\right)\bar{\hat{D}}^{2}\right]$$
$$+\frac{\left[\frac{\bar{\mathscr{M}}}{\bar{M}_{low}}-\alpha\underline{\delta_{0}}\right]+\omega_{1}\bar{\delta}_{p}}{2\epsilon_{2}\left(\underline{\delta_{0}}-\bar{\delta}_{1}e^{\omega_{2}}\left(1+\lambda_{M}\right)\right)}\underline{\delta_{0}}\|u(0,t)\|^{2}-\lambda_{Q}\omega_{2}\int_{0}^{1}\delta e^{\omega_{2}p}u^{T}(p,t)u(p,t)dp$$
$$+\lambda_{Q}\bar{\delta}_{p}\int_{0}^{1}e^{\omega_{2}p}u^{T}(p,t)u(p,t)dp-tr\left(\tilde{W}(t)^{T}\Delta_{1}^{-1}\dot{\hat{W}}(t)\right)-tr\left(\tilde{V}(t)^{T}\Delta_{2}^{-1}\dot{\hat{V}}(t)\right). \tag{5–65}$$

Young's inequality is used to upper bound (5–65) as

$$\dot{V}_{L}\leq\frac{\bar{\mathscr{M}}}{\bar{M}_{low}}r^{T}\left(U\left(t-D(t)\right)-U\left(t-\hat{D}(t)\right)\right)+\frac{\epsilon_{2}\omega_{1}\bar{\bar{\delta}}_{p}}{2}\|e_{u}(t)\|^{2}-\lambda_{Q}\underline{\delta_{0}}\|u(0,t)\|^{2}$$
$$-\left(\frac{K}{2}-\left[\frac{\bar{\mathscr{M}}}{\bar{M}_{low}}-\alpha\underline{\delta_{0}}\right]\frac{\epsilon_{2}}{2}-\frac{\epsilon_{1}\alpha\bar{\delta}_{p}}{2}\right)\|r(t)\|^{2}-\left(K-\omega_{1}|\bar{\delta_{p}}|-\frac{\alpha\bar{\delta}_{p}}{2\epsilon_{1}}-\frac{\omega_{1}\bar{\delta}_{1}\lambda_{M}}{2}\right)\|e_{u}(t)\|^{2}$$
$$+\int_{0}^{l}\left(L\omega_{t}+M\phi_{t}\right)dy-\left(1-x_{c}\right)\frac{\beta\rho}{2}\int_{0}^{l}\omega_{t}^{2}dy$$
$$-\frac{EI}{2}\left(\omega_{t}^{2}(l,t)+\omega_{yyy}^{2}(l,t)\right)-\tfrac{k_{\beta}GJ}{2}\left(\phi_{y}^{2}(l,t)+\phi_{t}^{2}(l,t)\right)$$
$$-\left(\frac{3EI}{2}-\frac{Ll^{3}}{2}\right)\beta\int_{0}^{l}\omega_{yy}^{2}dy-\left(I_{w}+\rho x_{c}^{2}-\rho x_{c}\right)\frac{\beta}{2}\int_{0}^{l}\phi_{t}^{2}dy$$
$$+\frac{1}{2}\beta l\left(I_{w}+\rho x_{c}^{2}\right)\phi_{t}^{2}\left(l,t\right)-\frac{\beta EIl}{2}\omega_{y}^{2}\left(l,t\right)+\frac{1}{2}\beta\rho l\omega_{t}^{2}\left(l,t\right)+\beta\rho x_{c}l\phi_{t}\left(l,t\right)\omega_{t}\left(l,t\right)$$
$$-\left(GJ-Ml^{3}-Ml-Ll^{3}\right)\frac{\beta}{2}\int_{0}^{l}\phi_{y}^{2}dy-\frac{\beta EIl}{2}\omega_{yyy}^{2}\left(l,t\right)+\frac{1}{2}\beta GJl\phi_{y}^{2}\left(l,t\right)$$
$$-\left(K-\frac{1}{2}\max\left\{EI+\beta EIl,k_{\beta}GJ\right\}\right)\|e(t)\|^{2}-\lambda_{Q}\omega_{2}\int_{0}^{1}\delta e^{\omega_{2}p}u^{T}(p,t)u(p,t)dp$$
$$+\lambda_{Q}\bar{\delta}_{1}e^{\omega_{2}}\left(\lambda_{M}+\lambda_{M}^{2}\right)\bar{\hat{D}}^{2}+\frac{\omega_{1}\bar{\delta}_{1}\lambda_{M}\bar{\hat{D}}^{2}}{2}+\lambda_{Q}\bar{\delta}_{p}\int_{0}^{1}e^{\omega_{2}p}u^{T}(p,t)u(p,t)dp$$
$$+\frac{\left[\frac{\bar{\mathscr{M}}}{\bar{M}_{low}}-\alpha\underline{\delta_{0}}\right]+\omega_{1}\bar{\delta}_{p}}{2\epsilon_{2}\left(\underline{\delta_{0}}-\bar{\delta}_{1}e^{\omega_{2}}\left(1+\lambda_{M}\right)\right)}\underline{\delta_{0}}\|u(0,t)\|^{2}$$
$$-tr\left(\tilde{W}(t)^{T}\Delta_{1}^{-1}\dot{\hat{W}}(t)\right)-tr\left(\tilde{V}(t)^{T}\Delta_{2}^{-1}\dot{\hat{V}}(t)\right). \tag{5–66}$$

Using the fact that $a^{T}b=trace(ba^{T})$, $\dot{\hat{W}}(t)$ and $\dot{\hat{V}}(t)$ are designed to cancel cross terms as

$$\dot{\hat{W}}=proj\left(\Delta_{1}\frac{\bar{\mathscr{M}}\lambda_{M}}{\bar{M}_{low}}\sigma\left(\hat{V}^{T}\Xi\right)r^{T}\right), \tag{5–67}$$
$$\dot{\hat{V}}=\Delta_{2}\frac{\bar{\mathscr{M}}\lambda_{M}}{\bar{M}_{low}}\Xi r^{T}\hat{W}^{T}\sigma'\left(\hat{V}^{T}\Xi\right). \tag{5–68}$$

The function $proj(.,.)$ in ((5–67)) denotes a projection operator, that is Lipschitz continuous (as discussed in Section (3.7)), which ensures $\underline{D} + _l \leq \hat{D}(t) \leq \bar{D} + _u$, where $_l, _u \in \mathbb{R}$ are subsequently defined positive constants (see Section 5.7). Using the inequalities in (5–60), (5–66) can be upper bounded as

$$\begin{aligned}
\dot{V}_L \leq & -\lambda_r \|r(t)\|^2 - \lambda_{e_u} \|e_u(t)\|^2 - \lambda_e \|e(t)\|^2 - \lambda_{Q_{res}} \underline{\delta_0} \|u(0,t)\|^2 \\
& + \int_0^l (L\omega_t + M\phi_t)\, dy - (1 - x_c) \frac{\beta\rho}{2} \int_0^l \omega_t^2 dy \\
& - \frac{EI}{2} \left(\omega_t^2(l,t) + \omega_{yyy}^2(l,t)\right) - \tfrac{k_\beta GJ}{2} \left(\phi_y^2(l,t) + \phi_t^2(l,t)\right) \\
& - \left(\frac{3EI}{2} - \frac{Ll^3}{2}\right) \beta \int_0^l \omega_{yy}^2 dy - \lambda_{Q1} Q + \lambda_{res} - \left(I_w + \rho x_c^2 - \rho x_c\right) \frac{\beta}{2} \int_0^l \phi_t^2 dy \\
& + \frac{1}{2} \beta l \left(I_w + \rho x_c^2\right) \phi_t^2 (l,t) - \frac{\beta EIl}{2} \omega_y^2 (l,t) - \left(GJ - Ml^3 - Ml - Ll^3\right) \frac{\beta}{2} \int_0^l \phi_y^2 dy \\
& - \frac{\beta EIl}{2} \omega_{yyy}^2 (l,t) + \frac{1}{2} \beta GJl \phi_y^2 (l,t) + \frac{1}{2} \beta \rho l \omega_t^2 (l,t) + \beta \rho x_c l \phi_t (l,t)\, \omega_t (l,t)\,,
\end{aligned} \tag{5–69}$$

where $\lambda_r, \lambda_e, \lambda_{e_u}, \lambda_{Q1}, \lambda_{Q_{res}}, \lambda_{res} \in \mathbb{R}$ are defined as

$$\lambda_r \triangleq \frac{K}{2} - \left[\frac{\bar{\mathscr{M}}}{\bar{M}_{low}} - \alpha \underline{\delta_0}\right] \frac{\epsilon_2}{2} - \frac{\epsilon_1 \alpha \bar{\delta}_p}{2}, \tag{5–70}$$

$$\lambda_e \triangleq K - \frac{1}{2} \max \left\{EI + \beta EIl, k_\beta GJ\right\}, \tag{5–71}$$

$$\lambda_{e_u} \triangleq K - \omega_1 |\bar{\delta}_p| - \frac{\alpha \bar{\delta}_p}{2\epsilon_1} - \frac{\omega_1 \bar{\delta}_1 \lambda_M}{2} - \frac{\epsilon_2 \omega_1 \bar{\delta}_p}{2}, \tag{5–72}$$

$$\lambda_{Q1} \triangleq \lambda_Q \left(\omega_2 \min\left(\delta_0, \delta_1\right) - |\bar{\delta}_p|\right), \tag{5–73}$$

$$\lambda_{Q_{res}} \triangleq \lambda_Q - \left[\frac{\left[\frac{\bar{\mathscr{M}}}{\bar{M}_{low}} - \alpha \underline{\delta_0}\right] + \omega_1 \bar{\delta}_p}{2\epsilon_2 \left(\underline{\delta_0} - \bar{\delta}_1 e^{\omega_2} \left(1 + \lambda_M\right)\right)}\right], \tag{5–74}$$

$$\lambda_{res} = \lambda_Q \bar{\delta}_1 e^{\omega_2} \left(\lambda_M + \lambda_M^2\right) \bar{\tilde{D}}^2 + \frac{\omega_1 \bar{\delta}_1 \lambda_M \bar{\tilde{D}}^2}{2}. \tag{5–75}$$

Using the definition of $z \triangleq [r^T \quad e^T \quad e_u^T]^T$ and $y \triangleq [z^T \quad \sqrt{Q}]^T$, and provided all the gain conditions are satisfied sufficiently (see Remark 5.4 and Section (5.7)), following upper bound of $\dot{V}_L$ is developed and can be expressed as

$$\dot{V}_L \leq -\lambda_1 \|y\|^2 - \lambda_2 E_B + \lambda_{res} \triangleq -g(t) + \lambda_{res}, \tag{5–76}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are defined as

$$\lambda_1 \triangleq \min\left\{\lambda_r, \lambda_e, \lambda_{e_u}, \lambda_Q\right\}, \tag{5–77}$$

$$\begin{aligned}\lambda_2 \triangleq \frac{1}{2}\min\Big\{&\beta\rho - \beta\rho x_c - L, 3EI - Ll^3, \beta\left(I_w + \rho x_c^2\right) - \beta\rho x_c - M,\\ &\beta\left(GJ - Ml^3 - Ml - Ll^3\right) - (M+L)\, l^2\Big\}.\end{aligned} \tag{5–78}$$

From (5–48) and (5–76), $V_L \in \mathcal{L}_\infty$; hence, $E_B \in \mathcal{L}_\infty$, $e, r, e_u \in \mathcal{L}_\infty$, and $\tilde{\theta} \in \mathcal{L}_\infty$. Since $E_B \in \mathcal{L}_\infty$, it can be concluded that $\int_0^l \omega_{yy}^2 dy \in \mathcal{L}_\infty$ and $\int_0^l \phi_y^2 dy \in \mathcal{L}_\infty$; hence, the elastic potential energy in the wing $E_P \in \mathcal{L}_\infty$ and by Property 5.1, $\omega_{yyy}\left(l, \cdot\right) \in \mathcal{L}_\infty$ and $\phi_y\left(l, \cdot\right) \in \mathcal{L}_\infty$. Since $e \in \mathcal{L}_\infty$, $\omega_{yyy}\left(l, \cdot\right) \in \mathcal{L}_\infty$, and $\phi_y\left(l, \cdot\right) \in \mathcal{L}_\infty$, (5–12) can be used to show $\omega_t\left(l, \cdot\right) \in \mathcal{L}_\infty$ and $\phi_t\left(l, \cdot\right) \in \mathcal{L}_\infty$. Since $\omega_t\left(l, \cdot\right) \in \mathcal{L}_\infty$, $\varphi_t\left(l, \cdot\right) \in \mathcal{L}_\infty$, and $E_B \in \mathcal{L}_\infty$, the kinetic energy of the system $E_K \in \mathcal{L}_\infty$ and by Property 5.2, $\frac{\partial^q \omega}{\partial t^q}$ and $\frac{\partial^q \phi}{\partial t^q}$ are bounded, uniformly in t $\forall y \in [0, l]$ for $q = 1, 2, 3$. Equations (5–4) and (5–5) and the fact that $e_u \in L_\infty$ can be used to show that the boundary control input, $U(t) \in \mathcal{L}_\infty$. Differentiating g from (5–76) with respect to time yields

$$\dot{g}(t) = \lambda_2 \dot{E}_B(t) + 2\lambda_1 \|y(t)\|^T \|\dot{y}(t)\|, \tag{5–79}$$

where

$$\dot{E}_B(t) = 2\int_0^l \left(\omega_t\omega_{tt} + \omega_{yy}\omega_{tyy} + \phi_t\phi_{tt} + \phi_{ty}\phi_y\right) dy. \tag{5–80}$$

After integrating by parts the second and fourth terms in (5–80), $\dot{E}_b$ can be expressed as

$$\begin{aligned}\dot{E}_B(t) = 2\int_0^l \left(\omega_t\left(\omega_{tt} + \omega_{yyyy}\right) + \phi_t\left(\phi_{tt} - \phi_{yy}\right)\right) dy - 2\omega_t\left(l, t\right)\omega_{yyy}\left(l, t\right)\\ +2\phi_t\left(l, t\right)\phi_y\left(l, t\right).\end{aligned} \tag{5–81}$$

Since all system signals are bounded, (5–81) can be used to conclude that $\dot{E}_B \in \mathcal{L}_\infty$. Equations (5–25) and (5–79) can be used to show that $\dot{g} \in \mathcal{L}_\infty$. Given that $V_L(t)$ is a non-negative function in time and $\dot{V}_L(t) \leq -g(t) + \lambda_{res}$, where $g(t)$ is a non-negative function and

$\dot{g}(t) \in L_\infty$, Lemma A.6 in [96] and Lemma 4.3 in [99] can be used to show that $E_B\left(t\right), y\left(t\right)$ are UUB. Using (5–34) and Lemma A.12 in [96] the following inequalities can be developed

$$E_B(t) \geq \int_0^l \omega_{yy}^2 dy \geq \frac{1}{l^3}\omega^2 \geq 0, \tag{5–82}$$

$$E_B(t) \geq \int_0^l \phi_y^2 dy \geq \frac{1}{l}\phi^2 \geq 0. \tag{5–83}$$

Since $y(t)$ is UUB as time approaches, that is all the auxiliary error terms (i.e., $e(t)$, $r(t)$ and $e_u(t)$), it can be concluded from (5–82) and (5–83) that ω, ϕ are UUB as $t \to \infty$ $\forall y \in [0, l]$. □

5.6 Numerical Simulation

A numerical simulation is presented to illustrate the performance of the developed controller. To approximate the simultaneous nonlinear system of PDEs that describe the bending and twisting of aircraft wing with a finite number of ODEs, a Galerkin-based method is used. The twisting and bending deflections of the wing are represented as a weighted sum of basis functions as given by

$$\begin{aligned} \varphi(y,t) &= a_0(t)h_0(y) + \sum_{i=1}^{n} a_i(t)h_i(y), \\ \omega(y,t) &= b_0(t)g_0(y) + \sum_{i=1}^{p} b_i(t)g_i(y), \end{aligned} \tag{5–84}$$

where $n = 5, p = 4$, denote the number of basis functions used in the approximations of the wing twisting and bending deflection, respectively. Equation (5–84) is a standard trail solution for Galerkin's weighted residual method. Selecting the trial solution in this way ensures that the solution satisfies the PDEs, by using principle of orthogonality between the basis functions and any arbitrary function. A set of linearly independent functions $\{h_i(y)\}_{i=0}^{n}$ and $\{g_i(y)\}_{i=0}^{p}$ is used satisfying the following boundary conditions.

$$h_0\left(0\right) = h_i\left(0\right) = 0, \quad h_{y_0}\left(l\right) = 1, h_{y_i}\left(l\right) = 0,$$

$$g_0(0) = g_i(0) = 0, \quad g_{y_0}(0) = g_{y_i}(0) = 0,$$

$$g_{yy_0}(l) = g_{yy_i}(l) = 0, \quad g_{yyy_0}(l) = 1, g_{yyy_i}(l) = 0.$$

First the approximation of the twisting and bending deflection given in (5–84) is substituted in the system of PDEs in (5–1) and (5–2), and then Taylor's approximation up to two terms is used to approximate sine and cosine terms, and the resulting equations can be written as a set of coupled nonlinear ODEs

$$G_1\ddot{b} + \dot{a}^2\left(G_{21}a + G_{22}a^3\right) + \ddot{a}\left(G_{31} + G_{32}a^2\right) + G_4 b + G_5 a = 0, \tag{5–85}$$

$$H_1\ddot{a} + H_{21}\ddot{b} + H_{22}\ddot{b}a^2 + H_3 a + H_4 a = 0. \tag{5–86}$$

In (5–85) and (5–86) $b(t) \triangleq \begin{bmatrix} b_0(t) & b_1(t) & \dots & b_p(t) \end{bmatrix}^T$, $a(t) \triangleq \begin{bmatrix} a_0(t) & a_1(t) & \dots & a_n(t) \end{bmatrix}^T$, $G_1 \triangleq \rho \int_0^l g(y)g^T(y)dy$, $G_{21} \triangleq -\rho x_c \int_0^l g(y)\left(h(y)h(y)^2\right)^T dy$, $G_{22} \triangleq \frac{\rho x_c}{3!} \int_0^l g(y)\left(h(y)^2 h(y)^3\right)^T dy$, $G_{31} \triangleq \rho x_c \int_0^l g(y)h^T(y)dy$, $G_{32} \triangleq -\frac{\rho x_c}{2!} \int_0^l g(y)\left(h(y)h(y)^2\right)^T dy$, $G_4 \triangleq EI \int_0^l g(y)g^T_{yyyy}(y)dy$, $G_5 \triangleq -\bar{L}_w \int_0^l g(y)h^T(y)dy$, $H_1 \triangleq \left(I_w + \rho x_c^2\right) \int_0^l h(y)h^T(y)dy$, $H_{21} \triangleq \rho x_c \int_0^l h(y)g^T(y)dy$, $H_{22} \triangleq -\frac{\rho x_c}{2!} \int_0^l h(y)\left(g(y)h(y)^2\right)^T dy$, $H_3 \triangleq -GJ \int h(y)h^T_{yy}(y)dy$, $H_4 \triangleq -\bar{M}_w \int_0^l h(y)h^T(y)dy$.

The coupled nonlinear ODEs are simulated with the following initial conditions: $\omega(y,0) = 0$ m and $\varphi(y,0) = \frac{y^2}{2l^2}$ rad. The performance of the controller designed in (5–23) along with the update laws in (5–24), in the simulation, demonstrated. As indicated in Figures 5-2 and 5-3, the coupled elastic system become unstable in the presence of time delay. Thus, the control objective is to regulate the twisting and bending deflection in presence of unknown time-varying input delay in the system. In order to estimate the time-varying delay and compensate for that, NN based update laws have been used for this simulation as in (5–67-5–68). Figures 5-4 and 5-5 show that the designed controller sufficiently mitigates the delay induced bending and twisting deflections respectively, along the length of the beam as time progresses. Figures 5-6 and 5-7 illustrate the time variation of the applied control force and moment, respectively.

5.7 Control Gain Selection

Control gains, such as λ_r in (5–70), λ_e in (5–71), λ_{e_u} in (5–72), λ_{Q_1} in (5–73), $\lambda_{Q_{res}}$ in (5–74), and λ_1 in (5–77), introduced in the stability analysis (Section 5.5) require to be positive constants. Based on the designed bounds of time-delay estimate ($\hat{D}$), and subsequently derived bounds of δ_0, δ_1 and δ_p (i.e., $\underline{\delta_0}$, $\bar{\delta}_0$, $\underline{\delta_1}$, $\bar{\delta}_1$, $|\bar{\delta}_p|$), this section develops sufficient gain conditions to ensure λ_r, λ_e, λ_{e_u}, λ_{Q_1} and $\lambda_{Q_{res}} > 0$. Using the definitions of λ_r in (5–70) and λ_e in (5–71), sufficient lower bounds for K can be obtained as

$$K > \left[\frac{\bar{\mathscr{M}}}{\bar{M}_{low}} - \alpha\underline{\delta_0}\right]\epsilon_2 + \epsilon_1\alpha\bar{\delta}_p, \tag{5–87}$$

$$K > \frac{1}{2}\max\left\{EI + \beta EIl, k_\beta GJ\right\}, \tag{5–88}$$

$$K > \max\left\{\left[\frac{\bar{\mathscr{M}}}{\bar{M}_{low}} - \alpha\underline{\delta_0}\right]\epsilon_2 + \epsilon_1\alpha\bar{\delta}_p, \frac{1}{2}\max\left\{EI + \beta EIl, k_\beta GJ\right\}\right\}. \tag{5–89}$$

Using the definition of λ_{e_u} in (5–72), following upper bounds of ω_1 can be obtained as

$$\omega_1 < \frac{K - \frac{\alpha\bar{\delta}_p}{2\epsilon_1}}{\left(|\bar{\delta}_p| + \frac{\bar{\delta}_1\lambda_M}{2} + \frac{\epsilon_2\bar{\delta}_p}{2}\right)}. \tag{5–90}$$

In order to ensure that the numerator of inequality in (5–90), stays positive, following upper bound of α can be obtained as

$$\alpha < \frac{2K\epsilon_1}{\bar{\delta}_p}. \tag{5–91}$$

Also from the definition of λ_{Q1} in (5–73), ω_2 needs to satisfy the following inequality

$$\omega_2 > \frac{|\bar{\delta}_p|}{\min\left(\delta_0, \delta_1\right)}. \tag{5–92}$$

In order to satisfy $\lambda_{Q_{res}}$ in (5–74), λ_Q has to be selected sufficiently large to ensure

$$\lambda_Q > \left[\frac{\left[\frac{\bar{\mathscr{M}}}{\bar{M}_{low}} - \alpha\underline{\delta_0}\right] + \omega_1\bar{\delta}_p}{2\epsilon_2\left(\underline{\delta_0} - \bar{\delta}_1 e^{\omega_2}\left(1 + \lambda_M\right)\right)}\right]. \tag{5–93}$$

In (5–93), ω_1 has to be selected sufficiently to satisfy

$$\alpha < \frac{\bar{\mathscr{M}}}{\underline{\delta_0} \bar{M}_{low}}. \tag{5–94}$$

From (5–77), it is quite clear that by selecting K, ω_1, α, λ_Q, and ω_2 sufficiently as in (5–89)-(5–94), λ_1 can be made positive. Based on Remark 5.4, $\lambda_2 > 0$ has been satisfied.

5.8 Conclusion

This paper presents a novel approach of developing a boundary control strategy added with delay compensation, for mitigating store induced oscillations in a flexible aircraft wing, subjected to unknown time-varying input delay. The designed controller guarantees to provide a UUB type stability as shown in stability analysis, unlike regular boundary controller in [1], which ensures asymptotic stability without delay presence in the system. The main contributions of this work is two fold. First, the designed controller is the first of this kind which ensures the stability of a coupled PDE based elastic system in presence of unknown time-varying input delay. Second, NN based update laws have been developed to model the unknown delay in the system, which uses a nonlinear mapping to transform the time domain to a compact domain, in order to utilize the universal function approximation theorem. A potential drawback to the developed method is the need for measurements of high-order spatial derivatives of the distributed states (e.g., $\omega_{yyy}(l,t)$), as shown in (5–21). Future efforts are focused on developing PDE-based output feedback boundary control strategies that would eliminate the need for high-order spatial derivative measurements. Finally, numerical simulation demonstrates the performance of the designed controller along with the adaptive update laws.

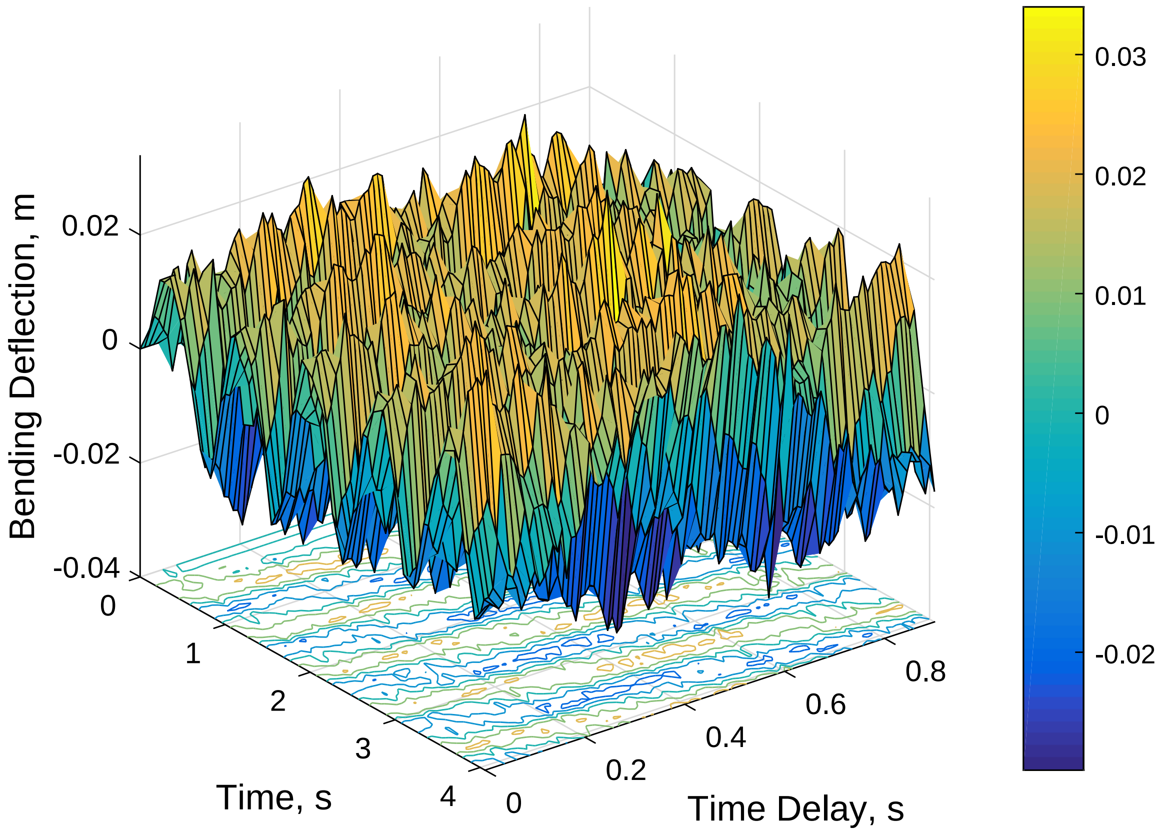

Figure 5-2. Closed loop bending deflection with constant input delays using the controller in [1].

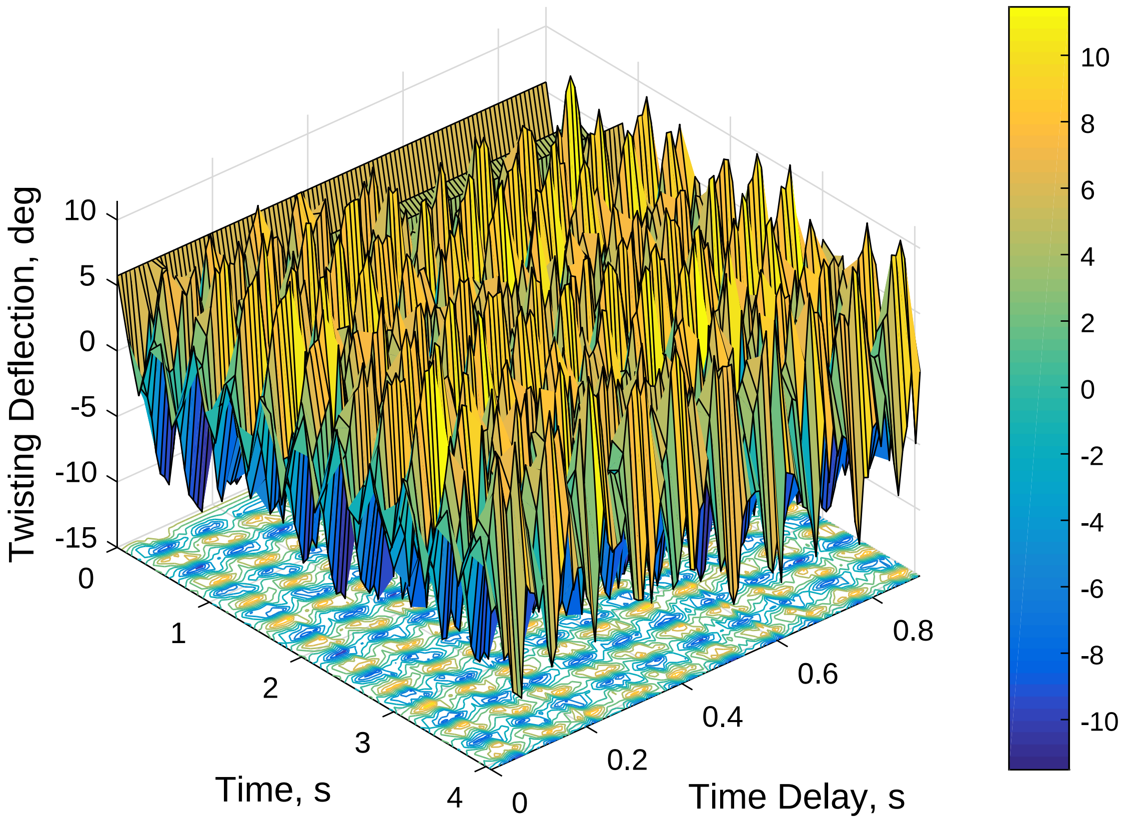

Figure 5-3. Closed loop twisting deflection with constant input delays using the controller in [1].

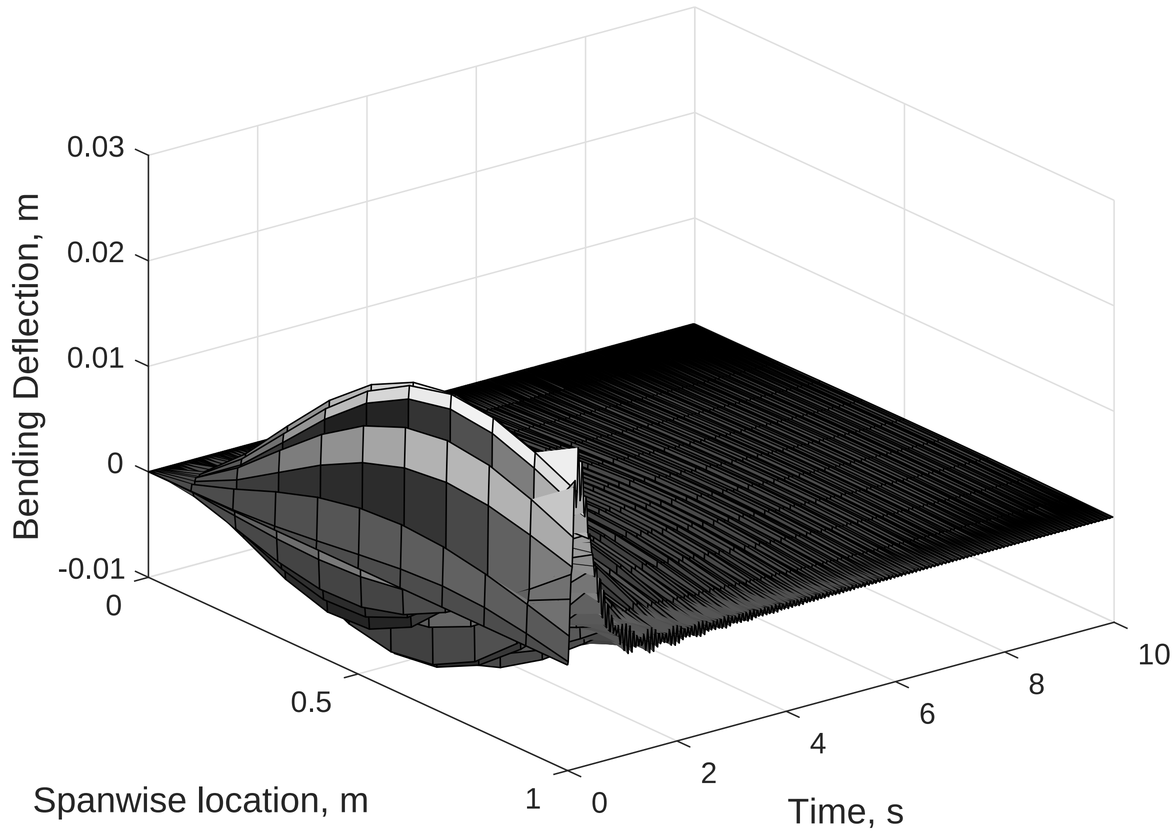

Figure 5-4. Bending deflection after applying the deigned controller in (5–23).

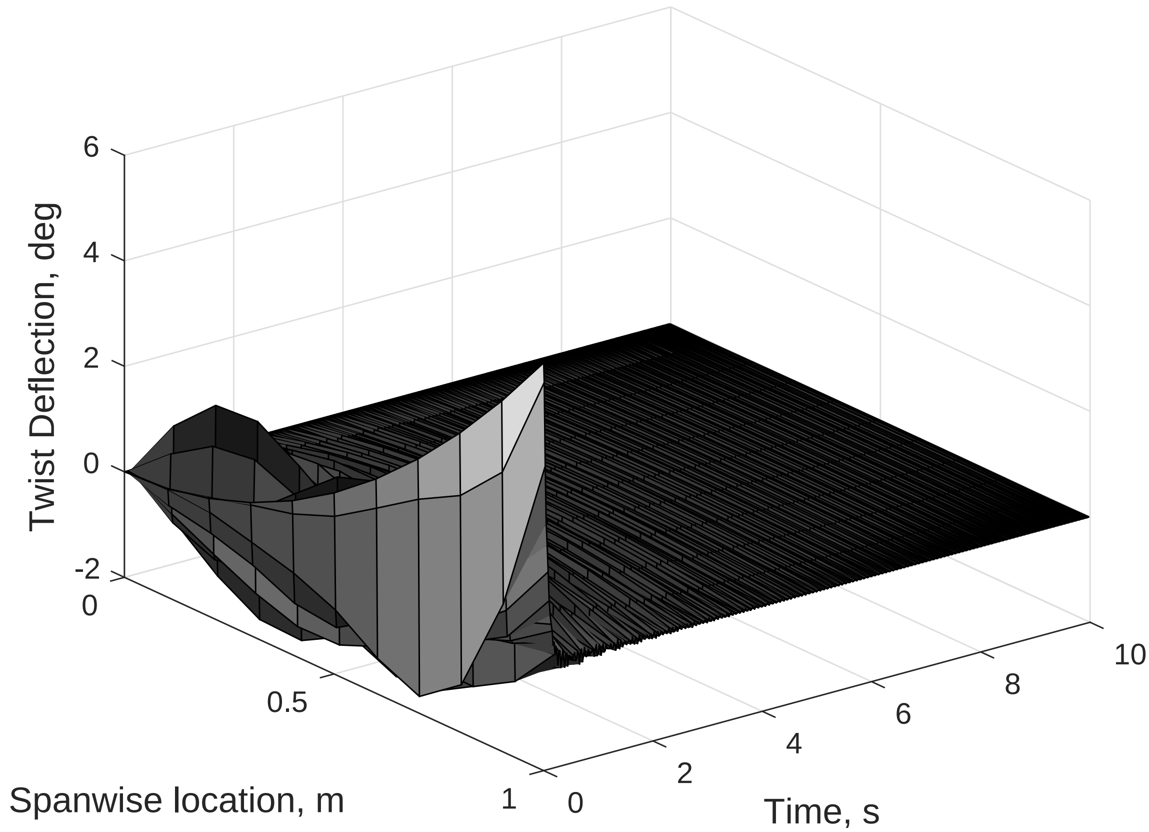

Figure 5-5. Twisting deflection after applying the deigned controller in (5–23).

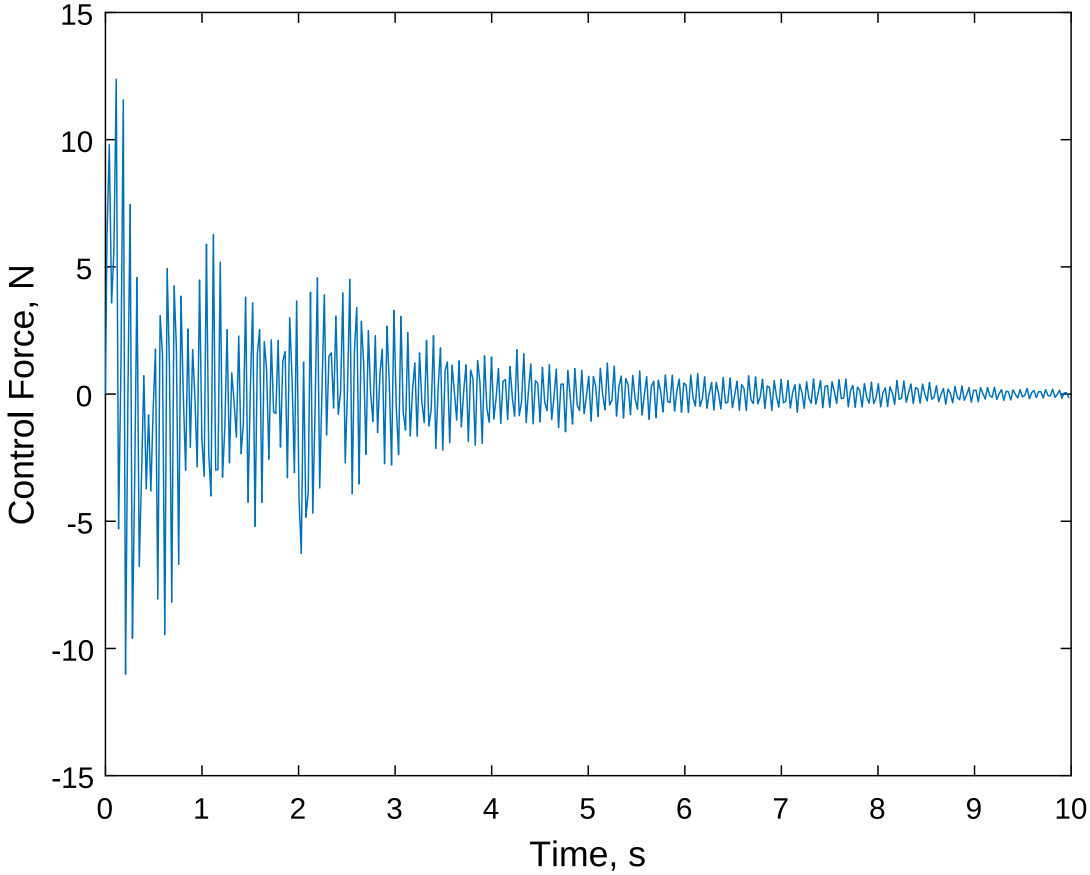

Figure 5-6. Control force variation vs. time.

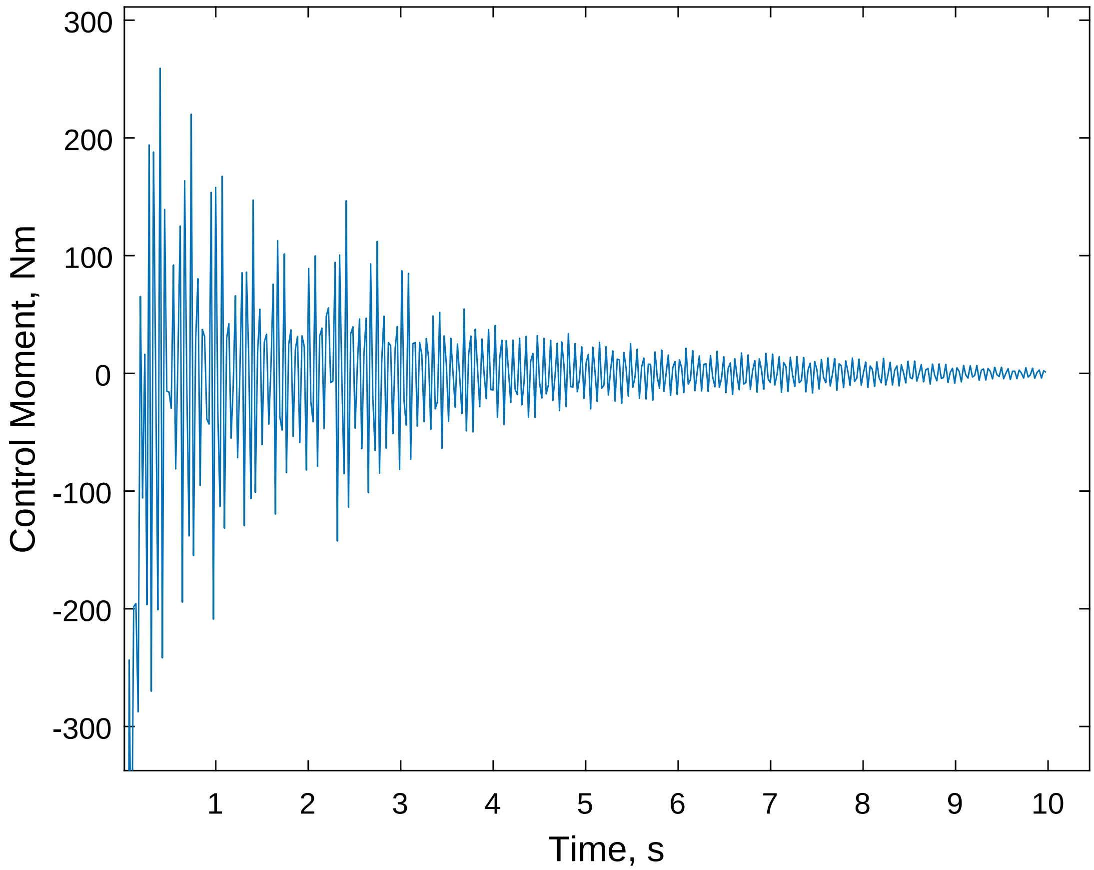

Figure 5-7. Control Moment variation vs. time.

CHAPTER 6
CONCLUSIONS AND FUTURE WORKS

6.1 Conclusions

Input delayed systems are subject of interest of many researchers for past few decades. Although there exists two different kind of control strategies in existing literature, namely robust strategy and prediction-based strategy, amalgamation of both two strategies are not studied. Robust strategy has advantage of not using system model for developing the controller, along with effective application to system with exogenous disturbances. On the other hand, although predictor-based strategy uses model knowledge for system state prediction, it gives a much simpler control gain conditions from stability analysis, unlike robust strategy. These advantages of both two strategies, motivate the development necessity of a noble control strategy for nonlinear unknown dynamical system, subjected to input delay.

In Chapter 2 this proposed amalgamation of predictor-based and robust strategy is demonstrated by developing a partial differential equation based controller, for a second order uncertain nonlinear system subjected to known time-varying input delay. This noble control approach utilizes a nonlinear delay dependent transformation, to transform traditional control input to a modified control input, which depends on both time and a dummy spatial variable. Introduction of this new spatial variable not only simplifies the control gain conditions, as demonstrated in the stability analysis, but also takes out time-varying delay term out of the control input, which is advantageous while designing estimator of the unknown time delay, as shown in Chapter 3 and Chapter 4. Finally, application of the developed controller is experimentally demonstrated for a series of dynamic tracking experiments of the knee-joint dynamics. The dynamic tracking experiments show successful implementation of the developed controller on six different healthy individuals.

Chapter 3 extends the concept of partial differential equation based controller, introduced in Chapter 2, for a cascading uncertain dynamical system of Brunovsky canonical form, subjected to an unknown input delay. In order to apply the spatial and time varying transformations for the controller, an estimate of delay is needed. A neural network based estimation strategy is developed for delay estimation, which depends on a nonlinear mapping to transform time into a compact domain, and uses universal functional approximation theorem for the estimation. Simulation is performed for a two link robot dynamics, subjected to a unknown time-varying delay. Simulation results show the performance of the controller, and also an estimate of the unknown delay magnitude using designed neural network.

Chapter 3, demonstrates a neural network based estimation for unknown delay magnitude, although the performance of the estimation depends highly on the choice of activation functions and training data set. As with all the data driven techniques, availability of sufficient data set is much needed, but in order to eliminate the high dependency of estimation performance on choice of activation function, an optimization based strategy is demonstrated in Chapter 4. Nesterov's accelerated gradient descent based algorithm is used for the delay estimation, which uses two previous discrete time steps information for estimating the current time step, instead of one as in traditional gradient descent. Stability analysis also incorporates developed accelerated gradient descent based method for the estimation, and shows an UUB stability of the nonlinear system. Simulation is performed on a two link robot dynamics, and simulation results show a sufficiently smooth delay estimation, and a decay in the objective function.

In Chapter 5, previously developed delay estimation using neural network is applied for a flexible aircraft wing subjected to an unknown time-varying input delay. A boundary control strategy with a delay compensation term is developed to mitigate the limit cycle oscillation of the aircraft wing. The delay compensation term utilizes the neural network based delay estimation strategy developed in 3. Simulation result justifies the necessity of adding a delay compensation term to the boundary controller, by demonstrating the effect of input delay on a flexible aircraft system which is controlled just by an adaptive boundary controller. For the developed delay compensated boundary controller, same system is simulated and the performance of the developed controller significantly improves compare to the case of no delay compensation in the controller.

6.2 Future Works

Image-based control systems rely on feedback from a single or multiple cameras to achieve desired guidance, navigation, and control objectives. The raw images need to be processed, either by a central processing unit or by specialized image processors, to extract and match desired features and patterns. While dedicated systems and graphical processing units provide significant computational resources, potential gains in processing time have been offset by the desire to process higher resolution imagery. Therefore, image-based control systems are inherently susceptible to time delays resulting from image extraction and processing. The delay in image processing to obtain the necessary control signal can be regarded as a time-varying input delay. Additionally, when the camera is not co-located with the system to be controlled, i.e., when using an off-board camera, a network communication channel (wired or wireless) is used to stream images from the camera to the controller. The uncertainties in the communication channel pose another challenge to networked imaging systems as the state received at the controller is delayed, and the delay could be unknown. The developed neural network based delay estimation method can be applied, along with the partial differential equation based controller, for image-based visual servo control problems.

Moreover, another possible application of the developed controller along with delay estimation strategy can be switched systems. Switched systems are hybrid dynamical system, consists of switching between different subsystems, and have strong engineering applications. Similar to linear/nonlinear systems subjected to time delay, switched time-delay systems are studied extensively in existing literature. Although there exists several literature for both continuous, and discrete switched system, subjected to known time-varying delays, continuous uncertain nonlinear switched system subjected to unknown time-varying state delay remains an open problem, based on author's best knowledge. This motivates the necessity of using the

NN functional approximator approach to estimate unknown time-varying state delay, for an uncertain continuous switched system.

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