LYAPUNOV-BASED CONTROL OF SATURATED AND TIME-DELAYED NONLINEAR SYSTEMS

By

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To my parents Debbie and Matt Fischer for their enduring support

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LIST OF ABBREVIATIONS

a.e.	Almost Everywhere
DCAL	Desired Compensation Adaptation Law
EL	Euler-Lagrange
EMK	Exact Model Knowledge
LK	Lyapunov-Krasovskii
LMI	Linear Matrix Inequality
LP	Linear-in-the-Parameters
LR	Lyapunov-Razumikhin
LYC	LaSalle-Yoshizawa Corollaries
LYT	LaSalle-Yoshizawa Theorem
MVT	Mean Value Theorem
NN	Neural Network
non-LP	Not Linear-in-the-Parameters
PD	Proportional-Derivative
PID	Proportional-Integral-Derivative
RHS	Right-Hand Side
RISE	Robust Integral of the Sign of the Error
RMS	Root Mean Square
SARC	Saturated Adaptive Robust Control
UC	Uniformly Continuous
UUB	Uniformly Ultimately Bounded

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Time delays and actuator saturation are two phenomena which affect the performance of dynamic systems under closed-loop control. Effective compensation mechanisms can be applied to systems with actuator constraints or time delays in either the state or the control. The focus of this dissertation is the design of control strategies for nonlinear systems with combinations of parametric uncertainty, bounded disturbances, actuator saturation, time delays in the state, and/or time delays in the input.

The first contribution of this work is the development of a saturated control strategy based on the Robust Integral of the Sign of the Error (RISE), capable of compensating for system uncertainties and bounded disturbances. To facilitate the design of this controller and analysis, two Lyapunov-based stability corollaries based on the LaSalle-Yoshizawa Theorem (LYT) are introduced using nonsmooth analysis techniques. Leveraging these two results, a RISE-based control design for systems with time-varying state-delays is developed. Since delays can also commonly occur in the control input, a predictor-based control strategy for systems with time-varying input delays is presented. Extending the results for time-delayed systems, a predictor-based controller for uncertain nonlinear systems subject to simultaneous time-varying unknown state and known input delays is introduced. Because errors can build over the deadtime interval when input delays are present leading to large actuator demands, a predictor-based saturated controller for uncertain nonlinear systems with constant input-delays

is developed. Each of the proposed controllers provides advantages over previous literature in their ability to provide smooth, continuous control signals in the presence of exogenous bounded disturbances. Lyapunov-based stability analyses, extensions to Euler-Lagrange (EL) dynamic systems, simulations, and experiments are also provided to demonstrate the performance of each of the control designs throughout the dissertation.

CHAPTER 1
INTRODUCTION

1.1 Motivation and Problem Statement

There exist numerous control solutions for nonlinear systems with additive disturbances. General control literature suggests that robust techniques (such as high gain, sliding mode, or variable structure control) have successfully been developed to accommodate for parametric uncertainties and disturbances in nonlinear plants [1–7]. The coupling of these robust methods with adaptive components has also been shown to improve the overall performance of both regulation and tracking problems for nonlinear systems. Robust control techniques that yield an asymptotic result are typically discontinuous, and often suffer from limitations such as the demand for infinite bandwidth or chatter. Continuous robust control designs such as the RISE strategy [8] have also been developed and have been shown to be effective for systems with bounded disturbances. The RISE strategy works by implicitly learning [9] and compensating for sufficiently smooth bounded disturbances and unstructured parametric uncertainty through the use of a sufficiently large gain multiplied by an integral signum term. RISE techniques are used throughout the dissertation as they present a state-of-the-art approach for control of uncertain nonlinear systems.

Classical stability theory is not applicable for systems described by discontinuous differential equations based on the local Lipschitz assumption (i.e., nonsmooth systems). Examples of such systems include: systems with friction modeled as a force proportional to the sign of a velocity, systems with feedback from a network, digital systems, systems with a discontinuous control law, etc. Differential inclusions are a mathematical tool that can be used to discuss the existence of solutions for nonsmooth systems. Utilizing a differential inclusion framework, numerous Lyapunov methods using generalized notions of solutions have been developed in literature for both autonomous and nonautonomous systems. Of these, several stability theorems have been established which apply to nonsmooth systems for which the derivative of the candidate Lyapunov function can be upper bounded by a negative-definite function: Lyapunov's generalized theorem and finite-time convergence in [10–15] are some examples of such. However, for certain classes of controllers (e.g., adaptive controllers, output feedback controllers, etc.), a negative-definite bound may be difficult (or impossible) to achieve, restricting the use of such methods.

Stability techniques such as the LaSalle-Yoshizawa Theorem (LYT) were introduced for continuous systems to specifically handle the case when the Lyapunov function derivative is bounded by a semi-definite function. Historically, some authors have stated the use of the LYT incorrectly (if the system contains discontinuities, then the locally Lipschitz property required by the theorem does not hold) or have stated that the LYT can applied using nonsmooth techniques without proof. The focus of Chapter 2 is the explicit development of a corollary to the LYT which can be used as an analysis tool for nonsmooth systems with a negative-semi-definite derivative of the candidate Lyapunov function.

While robust control techniques (whether continuous or discontinuous) have been shown to be effective for the compensation of parametric uncertainties and additive disturbances, in general, these techniques (including all previous RISE methods) do not account for the fact that the commanded input may require more actuation than is physically possible by the system (e.g., due to large initial condition offsets, an aggressive desired trajectory, or large perturbations). For example, the typical RISE structure uses a sufficiently large gain multiplied by an integral term, which can potentially lead to a computed control command that exceeds actuator capabilities. Because degraded control performance and the potential risk of thermal or mechanical failure can occur when unmodeled actuator constraints are violated, control schemes which can ensure performance while operating within actuator limitations are motivated.

Leveraging the outcomes developed in Chapter 2, Chapter 3 presents a saturated RISE controller which limits the control authority at or below an adjustable a priori limit. Saturated control designs are available in literature; however, the integration of a saturation scheme into the continuous RISE structure has remained an open problem due, in part to the integrator compensation.

As described in the survey papers [16–19] and relatively recent monographs such as [20–25], time delays are pervasive in nature and engineered systems. A few well-known and documented engineering applications include: digital implementation of a continuous control signal, regenerative chatter in metal cutting (especially prevalent in high speed manufacturing), delays in torque production due to engine cycle delays in internal combustion engines, chemical process control, rolling mills, control over networks, active queue management, financial markets (especially, computer controller exchanges of financial products), etc. Delays are also inherent in many biological process such as: delay in a person's response due to drugs and alcohol, delays in force production in muscle, the cardiovascular control system, etc. Systems that do not compensate for delays can exhibit reduced performance and potential instability.

Since a time delay can be considered another type of disturbance to the system, researchers have investigated adaptive and/or robust techniques to compensate for the undesirable implications delays have on closed-loop control of nonlinear systems. Typical time delayed control results have used novel prediction/compensation techniques (such as Smith predictors or Artstein reduction methods) to handle the delayed terms in closed-loop control; however, methods that achieve asymptotic or exponential results utilizing classic robust techniques suffer from the same discontinuous limitations (e.g., demand for infinite bandwidth and/or chatter) as delay-free control designs. Leveraging a design approach similar to that of the previous chapter, Chapter 4 presents a RISE-based control design for nonlinear systems with time-varying state delays.

While state delays are prevalent in a number of engineered systems, time delays can also occur in the control. Examples of systems with input delays can be found in numerous applications, from teleoperated robotic systems to biological processes. Problems arising from delay corruption of the control input remain unsolved for large classes of practical systems (e.g., uncertain nonlinear systems). While several results have used variations of the Smith and Artstein methods to solve the input delay problem for linear systems (with known and unknown dynamics), and nonlinear systems with exact model knowledge (EMK) (i.e., known forward-complete and strict feedforward systems), few results solve the input delay problem for uncertain nonlinear systems. As stated in the “Beyond this Book” section of the seminal work in [22], Krstic indicates that approaches developed for uncertain linear systems do not extend in an obvious way to nonlinear plants since the linear boundedness of the plant model is explicitly used in the stability proof of such results, and that new methods must be developed for delay-adaptive control for select classes of nonlinear systems with unknown input delays. Methods that solve the input delay problem for uncertain nonlinear systems with known and unknown constant time delays have been studied in [26–32]. However, due to uncertainties in the inherent nature of real world systems, it is often more practical to consider time-varying or state-dependent time delays in the control. Chapter 5 presents a controller for uncertain nonlinear systems with time-varying input delays. Motivated by the same time-varying delay considerations, Chapter 6 integrates the work of Chapters 2, 4 and 5 to design a controller which is capable of handling composite time-varying state delays and time-varying input delays, while achieving better transient and steady state performance and stability.

For systems with input delays, errors can build over the delay interval also leading to large actuator demands, exacerbating potential problems with actuator saturation. Motivated by the same actuator saturation concerns presented in Chapter 3, Chapter 7 develops a control strategy for uncertain input-delayed nonlinear systems with constant time delays and actuator saturation constraints. Previous techniques and outcomes obtained in Chapter 3 are utilized to develop a continuous control design which allows for the bound on the control to be adjusted a priori.

The work in this dissertation is based on Lyapunov stability theory (a common tool in nonlinear control) and presents several control strategies for open problems in nonlinear control literature. Specifically, the work focuses on real-world problems with practical implementation considerations, integrated throughout the individual theoretical contributions.

1.2 Literature Review

A literature review of Chapters 2-7 is presented below.

Chapter 2: Lasalle-Yoshizawa Corollary for Discontinuous Systems: Peano's Theorem states that for a differential equation given by $\dot{x} = f(x,t)$, if $f(x,t)$ is continuous on $\mathbb{R}^n \times [0,+\infty)$, then for each initial pair $(x_0,t_0) \in \mathbb{R}^n \times [0,+\infty)$ there exists at least one local classical solution $x(t)$ such that $x(t_0) = x_0$. When the function $f(x,t)$ is also assumed to be locally Lipschitz continuous, it is possible to prove local uniqueness and continuity of solutions with respect to the initial conditions. In control theory, this assumption is often too restrictive [33]. Thus, it is often more appropriate to pose assumptions on $f(x,t)$ such that the function $f(x,t)$ is essentially locally bounded on $\mathbb{R}^n \times [0,+\infty)$, that is, for each $x \in \mathbb{R}^n$, the function $t \to f(x,t)$ is measurable and for almost every $t \geq 0$, the function is continuous. This simple assumption is the basis for the branch of mathematics (and its extensions into control systems analysis) which includes nonsmooth components of differential equations.

Matrosov Theorems provide a framework for examining the stability of equilibrium points (and sets through various extensions) when a candidate Lyapunov function has negative semi-definite decay. The classical Matrosov Theorem [34] is based on the existence of a differentiable, positive-definite and radially unbounded Lyapunov-like function with a negative semi-definite derivative, where auxiliary functions that sum to be positive-definite are then used to establish stability or asymptotic stability of an equilibrium. Various extensions of this theorem have been developed (cf. [35–39]) to encompass discrete and hybrid systems and to establish stability of closed sets. In particular, [38] (see also the related work in [35] and [36]) extended Matrosov's Theorem to differential inclusions, while also addressing the stability of sets. An extension of Matrosov's Theorem to the stability of sets was also examined in [39], where a weak version of the theorem is developed for autonomous systems in the spirit of LaSalle's Invariance Principle.

In contrast to Matrosov Theorems, LaSalle's Invariance Principle [40] has been widely adopted as a method, for continuous autonomous (time-invariant) systems, to relax the strict negative-definiteness condition on the candidate Lyapunov function derivative while still ensuring asymptotic stability of the origin. Stability of the origin is proven by showing that bounded solutions converge to the largest invariant subset contained in the set of points where the derivative of the candidate Lyapunov function is zero. In [41], LaSalle's Invariance Principle was modified to state that bounded solutions converge to the largest invariant subset of the set where an integrable output function is zero. The integral invariance method was further extended in [42] to differential inclusions. As described in [43], additional extensions of the invariance principle to systems with discontinuous right-hand sides (RHS) were presented in [44–46] for Filippov solutions and [47] for Carathéodory solutions.

Various extensions of LaSalle's Invariance Principle have also been developed for hybrid systems (cf. [43, 48–52]). The results in [48] and [51] focus on switched linear systems, whereas the result in [52] focuses on switched nonlinear systems. In [50], hybrid extensions of LaSalle's Invariance Principle were applied for systems where at least one solution exists for each initial condition, for deterministic systems, and continuous hybrid systems. Left-continuous and impulsive hybrid systems are considered in extensions in [49]. In [43], two invariance principles are developed for hybrid systems: one involves a Lyapunov-like function that is nonincreasing along all trajectories that remain in a given set, and the other considers a pair of auxiliary output functions that satisfy certain conditions only along the hybrid trajectory. A review of invariance principles for hybrid systems is provided in [53].

The challenge for developing invariance-like principles for nonautonomous systems is that it may be unclear how to even define a set where the derivative of the candidate Lyapunov function is stationary since the candidate Lyapunov function is a function of both state and time [54, 55]. By augmenting the state vector with time (cf. [56, 57]), a nonautonomous system can be expressed as an autonomous system: this technique allows autonomous systems results (cf. [58] and [59]) to be extended to nonautonomous systems. While the state augmentation method can be a useful tool, in general, augmenting the state vector yields a non-compact attractor (when the time dependence is not periodic), destroying some of the latent structure of the original equation; for example, the new equation will not have any bounded, periodic, or almost periodic motions. Some results (cf. [60–62]) have explored ways to utilize the augmented system's non-compact attractors by focusing on solution operator decomposition, energy equations or new notions of compactness, but these methods typically require additional regularity conditions (with respect to time) than cases when time is kept as a distinct variable.

The Krasovskii-LaSalle Theorem [63] was originally developed for periodic systems, with several generalizations also existing for not necessarily periodic systems (e.g., see [45, 64–67]). In particular, a (Krasovskii-LaSalle) Extended Invariance Principle is developed in [67] to prove that the origin of a nonautonomous switched system with a piecewise continuous uniformly bounded in time RHS is globally asymptotically stable (or uniformly globally asymptotically stable for autonomous systems). The result in [67] uses a Lipschitz continuous, radially unbounded, positive-definite function with a negative semi-definite derivative (condition C1) along with an auxiliary Lipschitz continuous (possibly indefinite) function whose derivative is upper bounded by terms whose sum are positive-definite (condition C2).

Also for nonautonomous systems, the LaSalle-Yoshizawa Theorem (LYT) (i.e., [55, Theorem 8.4] and [68, Theorem A.8]), based on the work in [40, 69, 70], provides a convenient analysis tool which allows the limiting set (which does not need to be invariant) to be defined where the negative semi-definite bound on the candidate Lyapunov derivative is equal to zero, guaranteeing asymptotic convergence of the state. Given its utility, the LYT has been applied, for example, in adaptive control and in deriving stability from passivity properties such as feedback passivation and backstepping designs of nonlinear systems [40]. Available proofs for the LYT exploit Barbalat's Lemma [71], which is often invoked to show asymptotic convergence for general classes of nonlinear systems. In general, adapting the LYT to systems where the RHS is not locally Lipschitz has remained an open problem. However, using Barbalat's Lemma and the observation that an absolutely continuous function that has a uniformly locally integrable derivative is uniformly continuous, the result in [71] proves asymptotic convergence of an output function for nonlinear systems with L_p disturbances. The result in [71] is developed for differential equations with a continuous right-hand side, but [71, Facts 1-4] provide insights into the application of Barbalat's Lemma to discontinuous systems.

Chapter 3: Saturated RISE Feedback Control: Motivated by issues with actuator constraints for robust control methods, some efforts have focused on developing saturated controllers for the regulation problem (cf. [72–77]) and the more general tracking problem (cf. [78–88]). In [78], the authors developed an adaptive, full-state feedback controller to produce semi-global asymptotic tracking while compensating for unknown parametric uncertainties using multiple embedded hyperbolic saturation functions. The authors of [79] were able to extend the Proportional-Integral-Derivative (PID)-based work of [74] to the tracking control problem by utilizing a general class of saturation functions to achieve a global uniform asymptotic tracking result for a linearly parameterizable (LP) system. This work was based on prior work in [80] and [81] which incorporated hyperbolic saturation functions into the saturated Proportional-Derivative (PD)+ control strategy developed in [82]. The works of [79–81] rely on gains which must abide by a saturation-avoidance inequality (restricting the ability to adjust the performance of the controller) or the characterization of desired trajectories to avoid saturation, both of which limit the domain for which the controller can operate. Anti-windup schemes have been developed [89] to compensate for saturation nonlinearities in nonlinear Euler-Lagrange (EL) systems using PID-like control structures. Results in [90] and [91] achieved global regulation of saturated nonlinear systems using a PID-like control structure and a passivity-based analysis. Each of the saturated PD+ and PID+ based control methods provide an elegant, intuitive structure for which to control an uncertain system; however, due to the inclusion of gravity compensation terms, a priori knowledge of both the model structure and its parameters is required. This assumption is particularly intrusive in the example of systems with added mass such as that of a robot manipulator system with unknown or varying payloads. To compensate for uncertain dynamics and the evaluation of the unknown gravity term, Alvarez-Ramirez, et. al [83] includes an additional saturated integral term and uses energy shaping and damping injection methods to yield a semi-global stability result. More recently in [84], a saturated PID framework controller was proposed which uses sigmoidal functions to achieve global asymptotic regulation to a set-point; however, it is unclear how the result can be extended to the tracking problem due to the control structure.

While each of the mentioned contributions developed saturated controllers with asymptotic stability results, they have not been proven to stabilize systems with both uncertain dynamics and additive unmodeled disturbances. Hong and Yao proposed the development of a continuous saturated adaptive robust control (SARC) algorithm [85] capable of achieving an ultimately bounded tracking result in the presence of an external disturbance. Corradini, et. al proposed a discontinuous saturated sliding mode controller [86] for linear plant models in the presence of bounded matched uncertainties to achieve a semi-global tracking result. In [87], two control algorithms are developed for robust stabilization of spacecraft in the presence of control input saturation, parametric uncertainty, and external disturbances using a discontinuous variable structure control design. In [88], the authors develop a SARC controller a using discontinuous projection method to achieve globally bounded tracking of artificial muscles. However, while each of these saturated robust techniques are able to address uncertain nonlinear systems with additive disturbances, the discontinuous nature of the results motivates the design of continuous saturated robust control techniques. Robust control designs utilizing nested saturation functions for uncertain feedforward nonlinear systems [92–94] have guaranteed global asymptotic stability despite unmodeled dynamic disturbances.

Chapter 4: RISE-Based Control of an Uncertain Nonlinear System With Time-Varying State Delays: Motivated by performance and stability problems with time-delayed systems, solutions typically use appropriate Lyapunov-Razumikhin (LR) or Lyapunov-Krasovskii (LK) functionals to derive bounds on the delay such that the closed-loop system is stable. Numerous methods have been developed throughout literature for time-delayed linear systems and nonlinear systems with known dynamics [16, 18, 21–23]. For uncertain nonlinear systems, techniques have also been developed to compensate for both known and unknown constant state-delays [95–102]. Extensions of these designs to systems with nonlinear, bounded disturbances also exist [100, 102, 103].

For some applications, it is often more practical to consider time-varying or state-dependent time delays. Control methods for uncertain nonlinear systems with time-varying state delays have been studied in results such as [99, 104–107]. However, compensation of time-varying state-delays in systems with both uncertain dynamics and added exogenous disturbances is explored in only a few results. A robust integral sliding mode technique for stochastic systems with time-varying delays and linearly state-bounded nonlinear uncertainties is developed in [108] but depends on convex optimization routines and a Linear Matrix Inequality (LMI) feasibility condition. In [109], an adaptive fuzzy logic control method yielding a semi-global uniformly ultimately bounded (UUB) tracking result is illustrated for a system in Brunovsky form. The authors of [110] utilize the circle criterion and an LMI feasibility condition to design a nonlinear observer for neural-network-based control of a class of uncertain stochastic nonlinear strict-feedback systems. The design proposes a neural network (NN) weight update law that directly cancels the bound on the reconstruction error to yield a globally stable result. Discontinuous model reference adaptive controllers have been designed in [111] and [112] for uncertain nonlinear plants with time-varying delays to achieve asymptotic stability results; however, the discontinuous nature of these results motivates the design of continuous control techniques.

Chapter 5: Lyapunov-Based Control of an Uncertain Nonlinear System with Time-Varying Input Delay: Many of the results for linear systems with constant delays are extensions of classic Smith predictors [113], Artstein model reduction [114], or finite spectrum assignment [115]. Due to uncertainties in the inherent nature of real world systems, it is often more practical to consider time-varying or state-dependent time delays in the control. Extensions of linear control techniques to time-varying input delays are also available [18, 116–121].

For nonlinear systems, controllers considering constant [95–102] and time-varying [99, 104–112, 122, 123] state delays have been recently developed. However, results which consider delayed inputs are far less prevalent, especially for systems with model uncertainties and/or disturbances. Examples of these include constant input delay results in [26–32, 124–129] and time-varying input delay results based on LMI [130, 131] and backstepping [132–134] techniques.

Chapter 6: Time-Varying Input And State Delay Compensation for Uncertain Nonlinear Systems Results: Results which focus on simultaneous constant state and input delays for linear systems are provided in [135–137]. Results which tackle both time-varying state and input delays in uncertain nonlinear systems are rare. The review of literature in Chapter 5 illustrated that few results even exist for nonlinear systems with solely time-varying input delays. Recently in [134], authors extended the predictor-based techniques in [135] and [133] were extended to nonlinear systems with time-varying delays in the state and/or the input utilizing a backstepping transformation to construct a predictor-based compensator. The development in [135] and [133] assumes that the disturbance-free plant is asymptotically stabilizable in the absence of delay, and that the rate of change of the delay is bounded by 1 (a common assumption for predictor-based work). To the author's knowledge, development of a control method for an uncertain nonlinear system with simultaneous time-varying delayed state and actuation with additive bounded disturbances remains as an unsolved problem.

Chapter 7: Saturated Control of an Uncertain Nonlinear System with Input Delay: Saturated controllers for state delay systems have been rigorously studied for both linear and nonlinear systems [138–142]. However, the majority of saturated controllers presently available for systems with input delays are based on linear plant models [141, 143–145] and only a few results are present for nonlinear systems (especially those with uncertainties). The authors of [144] proposed a parametric Lyapunov equation-based low-gain feedback law which guarantees stability of a linear system with delayed and saturated control input. In [146], global uniform asymptotic stabilization is obtained with bounded feedback of a strict-feedforward linear system with delay in the control input. The authors were able to extend the result to an uncertain but disturbance-free strict-feedforward nonlinear system with delays in the control input in [28] using a system of nested saturation functions. The controller requires a nonlinear strict-feedforward dynamic system with parametric uncertainty, $h(t)$, which satisfies the following condition: $|h(x_{i+1}, x_{i+2}, ..., x_n)| \leq M(x_{i+1}^2, x_{i+2}^2, ..., x_n^2)$ where M denotes a positive real number when $|x_j| \leq 1,\ j = i+1, ..., n$. Unlike compensation-based delay methods, the design in [28] cleverly exploits the inherent robustness to delay in the particular structure of the feedback law and the plant. Krstic proposed a saturated compensator-based approach in [30] which results in a nonlinear version of the Smith Predictor [113] with nested saturation functions. The controller is able to achieve quantifiable closed-loop performance by using an infinite dimensional compensator for strict-feedforward nonlinear systems with no uncertainties.

1.3 Contributions

The contributions of Chapters 2-7 are discussed as follows:

Chapter 2: Lasalle-Yoshizawa Corollaries for Discontinuous Systems: Two general Lyapunov-based stability theorems are developed using Filippov solutions for nonautonomous nonlinear systems with RHS discontinuities through locally Lipschitz continuous and regular Lyapunov functions whose time derivatives (in the sense of Filippov) can be bounded by negative semi-definite functions. The chapter also poses as an introduction to Filippov solutions and their use in control design and analysis. Applicability of the corollaries is illustrated with two design examples including an adaptive sliding mode control law and a standard RISE control law.

Chapter 3: Saturated RISE Feedback Control: The main contribution of Chapter 3 is the development of a new RISE-based closed-loop error system that consists of a saturated, continuous tracking controller for a class of uncertain, nonlinear systems which includes time-varying and non-LP functions and unmodeled dynamic effects. Nonsmooth analysis methods introduced in Chapter 2 are used throughout the development. The technical challenge presented by this objective is the need to introduce saturation bounds on the integral signum term while maintaining its functionality to implicitly learn the system disturbances. To achieve the result, a new auxiliary filter structure is designed using hyperbolic functions that work in tandem with the redesigned continuous saturated RISE-like control structure. While the controller is continuous, the closed loop error system contain discontinuities which are examined through a differential inclusion framework. The resulting controller is bounded by the magnitude of an adjustable control gain, and yields asymptotic tracking. The result is extended to general nonlinear systems which can be described by EL dynamics and is illustrated with experimental results to demonstrate the control performance.

Chapter 4: RISE-Based Control of an Uncertain Nonlinear System With Time-Varying State Delays: A continuous controller is developed for uncertain nonlinear systems with an unknown, arbitrarily large, time-varying state delay. Motivated by previous work in [147], a continuous RISE control structure is augmented with a three-layer NN to compensate for time-varying state delays which are arguments of uncertain nonautonomous functions that contain not linear-in-the-parameters (non-LP) uncertainty. Under the assumption that the time delay can be arbitrarily large, bounded and slowly varying, LK functionals are utilized to prove semi-global asymptotic tracking. In comparison to the previous work for constant state delays in [122], new efforts in this chapter required to compensate for time-varying state delays include: strategic grouping of delay-dependent and delay-free terms and a redesigned LK functional. In comparison to [122], NNs are used in the current work to compensate for the non-LP disturbances, and new efforts are required to design the online NN update laws in the presence of the unknown time-varying delay.

Chapter 5: Lyapunov-Based Control of an Uncertain Nonlinear System with Time-Varying Input Delay: Looking instead at time delays which occur in the input instead of the state, Chapter 5 presents a control method to compensate for time-varying input delays in uncertain nonlinear systems with additive disturbances under the assumption that the time delay is bounded and slowly varying. In this result, LK functionals and an innovative PD-like control structure with a predictive integral term of past control values are used to facilitate the design and analysis of a control method that can compensate for the input delay. Since the LK functionals contain time-varying delay terms, additional complexities are introduced into the analysis. Techniques used to compensate for the time-varying delay result in new sufficient control conditions that depend on the length of the delay as well as the rate of delay. The developed controller achieves semi-global UUB tracking despite the time-varying input delay, parametric uncertainties and additive bounded disturbances in the plant dynamics. An extension to general Euler-Lagrange dynamic systems is provided and the resulting controller is numerically simulated for a two-link robot manipulator to examine the performance of the developed controller.

Chapter 6: Time-varying Input And State Delay Compensation for Uncertain Nonlinear Systems Results: Motivated by Chapter 5's UUB result, the previous time-varying input delay work is extended in two directions: a) Utilizing techniques for constant input-delayed systems first introduced in [129], time-varying input delays in a nonlinear plant are now considered, b) the ability to compensate for simultaneous unknown time-varying state delays is added, and c) the stability of the closed-loop system is improved to asymptotic tracking. The state delays present in the system are robustly compensated for using a desired compensation adaptation law (DCAL)-based approach. However, this technique is not sufficient to compensate for the system's input delays. A predictor-like error signal based on previous control values provides a delay-free open-loop system, allowing for control design flexibility and the use of more complicated feedback signals over the previous result in Chapter 5. In Chapter 5, complex cross-terms that resulted from the controller inhibited the ability to achieve an asymptotic stability result. In comparison, this result uses a robust technique, termed the robust integral of the sign of the error (RISE) (instead of the previous PD-like compensator) is used, allowing for compensation of the system disturbance and elimination of the ultimate bound on the tracking error. A Lyapunov-based stability analysis utilizing Lyapunov-Krasovskii (LK) functionals demonstrates the ability to achieve semi-global asymptotic tracking in the presence of model uncertainty, additive sufficiently smooth disturbances and simultaneous time-varying state and input delays. The stability analysis considers the effect of arbitrarily small measurement noise and the existence of solutions for discontinuous differential equations. The subsequent development is based on the assumption that the state delay is bounded and slowly varying, but unknown. Improving on the result in Chapter 5, the assumption that the input delays must be sufficiently small is relaxed; instead, the input delays are assumed to be known, bounded and slowly varying. Numerical simulations compare the result to the previous input-delayed control design in Chapter 5 and examine the robustness of the method to various combinations of simultaneous input and state delays.

Chapter 7: Saturated Control of an Uncertain Nonlinear System with Input Delay: To safeguard from the risk of actuator saturation for input-delayed systems, the work presented in Chapter 7 introduces a new saturated control design that can predict/compensate for input delays in uncertain nonlinear systems. Based on the previous non-saturated feedback work and the design structures utilized in Chapters 3 and 5, a continuous saturated controller is developed which allows the bound on the control to be known a priori and to be adjusted by changing the feedback gains. The saturated controller is shown to guarantee UUB tracking despite a known, constant input delay, parametric uncertainties and sufficiently smooth additive disturbances. Efforts focus on developing a delay compensating auxiliary signal to obtain a delay-free open-loop error system and the construction of an LK functional to cancel the time delayed terms. The result is extended to general nonlinear systems which can be described by EL dynamics and is illustrated with experimental results to demonstrate the control performance.

CHAPTER 2
LASALLE-YOSHIZAWA COROLLARY FOR DISCONTINUOUS SYSTEMS

In this chapter, two generalized corollaries to the LYT are presented for nonautonomous nonlinear systems described by differential equations with discontinuous right-hand sides. Lyapunov-based analysis methods which achieve asymptotic convergence when the candidate Lyapunov derivative is upper bounded by a negative semi-definite function in the presence of differential inclusions are presented. Two design examples illustrate the utility of the corollaries.

2.1 Preliminaries

A function f defined on a space X is called essentially locally bounded, if for any $x \in X$ there exists a neighborhood $U \subseteq X$ of x such that $f(U)$ is a bounded set for almost all $u \in U$. The essential supremum is the proper generalization of the maximum to measurable functions, the technical difference is that the values of a function on a set of measure zero[1] do not affect the essential supremum. Given two metric spaces (X, d_X) and (Y, d_Y) the function $f : X \to Y$ is called locally Lipschitz if for any $x \in X$ there exists a neighborhood $U \subseteq X$ of x so that f restricted to U is Lipschitz continuous. As an example, any $\mathcal{C}^1$ continuous function is locally Lipschitz.

Consider the system

$$\dot{x} = f(x, t) \tag{2–1}$$

where $x(t) \in \mathcal{D} \subset \mathbb{R}^n$ denotes the state vector, $f : \mathcal{D} \times [0, \infty) \to \mathbb{R}^n$ is a Lebesgue measurable and essentially locally bounded, uniformly in t function, and $\mathcal{D}$ is some open and connected set. Existence and uniqueness of the continuous solution $x(t)$ are provided under the condition that the function f is Lipschitz continuous [148].

[1] Recall that for sets in the Euclidean n-space ($\mathbb{R}^n$), Lebesgue measure is commonly utilized. For example, any singleton sets, countable sets, or subsets of $\mathbb{R}^n$ whose dimension is less than n are considered Lebesgue measure zero in $\mathbb{R}^n$.

However, if f contains a discontinuity at any point in $\mathcal{D}$, then a solution to (2–1) may not exist in the classical sense. Thus, it is necessary to redefine the concept of a solution. Utilizing differential inclusions, the value of a generalized solution (e.g., Filippov [149] or Krasovskii [150] solutions) at a certain point can be found by interpreting the behavior of its derivative at nearby points. Generalized solutions will be close to the trajectories of the actual system since they are a limit of solutions of ordinary differential equations with a continuous right-hand side [10]. While there exists a Filippov solution for any arbitrary initial condition $x(t_0) \in \mathcal{D}$, the solution is generally not unique [149, 151].

Definition 2.1. (Filippov Solution) [149] A function $x : [0, \infty) \to \mathbb{R}^n$ is called a solution of (2–1) on the interval $[0, \infty)$ if $x(t)$ is absolutely continuous and for almost all $t \in [0, \infty)$,

$$\dot{x} \in K[f](x(t), t)$$

where $K[f](x(t), t)$ is an upper semi-continuous, nonempty, compact and convex valued map on $\mathcal{D}$, defined as

$$K[f](x(t), t) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu N = 0} \overline{co} f(B(x(t), \delta) \setminus N, t), \tag{2–2}$$

$\bigcap_{\mu N = 0}$ denotes the intersection over sets N of Lebesgue measure zero, $\overline{co}$ denotes convex closure, and $B(x(t), \delta) = \{v \in \mathbb{R}^n | \, \|x(t) - v\| < \delta\}$.

Remark 2.1. One can also formulate the solutions of (2–1) in other ways [152]; for instance, using Krasovskii's definition of solutions [150]. The corollaries presented in this work can also be extended to Krasovskii solutions (see [153], for example). In the case of Krasovskii solutions, one would get stronger conclusions (i.e., conclusions for a potentially larger set of solutions) at the cost of slightly stronger assumptions (e.g., local boundedness rather than essentially local boundedness).

Example 2.1. Differential Inclusion Computation

Consider the differential system given by

$$\dot{x} = f(x,t) + g(x,t) \tag{2–3}$$

where $f(x,t) = sgn(x)$ and $g(x,t) = sin(x)$. Based on Definition 2.1, the Filippov solution for the system in (2–3) is given by

$$\dot{x} \in K[f+g](x,t).$$

Based on the calculus for $K[\cdot]$ developed in [154], $K[f+g](x) \subseteq K[f](x,t)+K[g](x,t)$. For continuous functions, the differential inclusion evaluated at every point is equivalent to the continuous function evaluated at that point, i.e, $K[g](x,t) = g(x,t)$. To examine how the differential inclusion is computed, first note that the sets (of Lebesgue measure zero) of discontinuity for $f(x,t)$ include the singleton set $\{0\}$.

When $x > 0$ or $x < 0$, it is straight forward to compute that the expressions for $K[f](\cdot)$ reduce to the singletons $\{1\}$ and $\{-1\}$, respectively. An illustration of the positive case is depicted in Figure 2-1 where $\forall\delta$ (only 3 of the infinite sizes are shown), the function f evaluated at the appropriate reduced set is equivalent to $K[f](x^+) = \overline{co}\{-1,1\} \cap \overline{co}\{-1,1\} \cap \overline{co}\{1\} \cap ....$ Computing the closed convex hull of each intersection reduces the inclusion to $K[f](x^+) = [-1,1] \cap [-1,1] \cap \{1\} \cap ... = \{1\}$. The same arguments can be used to compute the differential inclusion for $x < 0$.

At $x = 0$, the expression for $K[f](x)$ reduces to $K[f](0) = \bigcap_{\delta>0} \overline{co}[sgn(B(0,\delta) - \{0\})]$. Since $B(0,\delta)$, $\delta > 0$, an open interval containing the origin, intersects both $(0,\infty)$ and $(-\infty,0)$ on sets of positive measure, $K[f](0) = \bigcap_{\delta>0} \overline{co}[sgn([x-\delta,\ x+\delta] - \{0\})] = \overline{co}\{-1,\ 1\} = [-1,\ 1]$. This closure is illustrated in Figure 2-2.

Thus it is easy to see that the differential inclusion can be described by $\dot{x} \in SGN(x) + sin(x)$ where $SGN(\cdot)$ is the set-valued sign function defined by $SGN(x) = 1$ if $x > 0$, $[-1,1]$ if $x = 0$, and -1 if $x < 0$. So at $x \neq 0$, $\dot{x} \in K[f+g]$ is a singleton and at $x = 0$, $\dot{x} \in K[f+g]$ is a set.

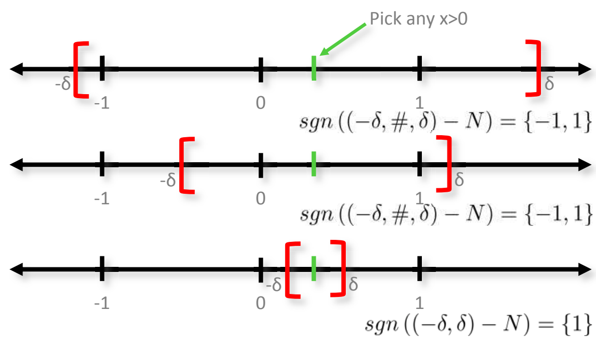

Figure 2-1. Set closure of $K\left[f\right](x)$ for $x>0$ case.

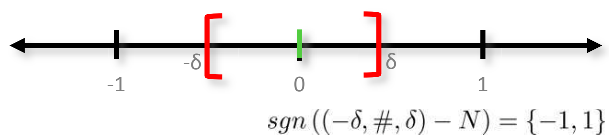

Figure 2-2. Set closure of $K\left[f\right](x)$ for $x=0$ case.

To facilitate the main results, three definitions are provided. Clarke's generalized gradient is used in many Lyapunov-based theorems using nonsmooth analysis. To introduce this idea, the definition of a regular function as defined by Clarke [56] is presented.

Definition 2.2. (Directional Derivative) [155] Given a function $f:\mathbb{R}^m\to\mathbb{R}^n$, the right directional derivative of f at $x\in\mathbb{R}^m$ in the direction of $v\in\mathbb{R}^m$ is defined as

$$f'\left(x,v\right)=\lim_{t\to0^+}\frac{f\left(x+tv\right)-f\left(x\right)}{t}.$$

Additionally, the generalized directional derivative of f at x in the direction of v is defined as

$$f^o\left(x,v\right)=\lim_{y\to x}\sup_{t\to0^+}\frac{f\left(y+tv\right)-f\left(y\right)}{t}.$$

Definition 2.3. (Regular Function) [56] A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is said to be regular at $x \in \mathbb{R}^m$ if for all $v \in \mathbb{R}^m$, the right directional derivative of f at x in the direction of v exists and $f'(x, v) = f^o(x, v)$.[2]

The following Lemma provides a method for computing the time derivative of a regular function V using Clarke's generalized gradient [56] and $K[f](x,t)$ along the solution trajectories of the system in (2–1).

Definition 2.4. (Clarke's Generalized Gradient) [56] For a function $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ that is locally Lipschitz in (x, t), define the generalized gradient of V at (x, t) by

$$\partial V(x,t) = \overline{co}\left\{\lim \nabla V(x,t) \mid (x_i, t_i) \to (x,t), (x_i, t_i) \notin \Omega_V\right\}$$

where Ω_V is the set of measure zero where the gradient of V is not defined.

Definition 2.5. (Locally bounded, uniformly in t) Let $f : \mathcal{D} \times [0, \infty) \to \mathbb{R}$. The map $x \to f(x,t)$ is locally bounded, uniformly in t, if for each compact set $K \subset \mathcal{D}$, there exists $c > 0$ such that $|f(x,t)| \leq c$, $\forall (x,t) \in K \times [0, \infty)$.

Lemma 2.1. *(Chain Rule) [45] Let $x(t)$ be a Filippov solution of system (2–1) and $V : D \times [0, \infty) \to \mathbb{R}$ be a locally Lipschitz, regular function. Then $V(x(t), t)$ is absolutely continuous, $\frac{d}{dt}V(x(t), t)$ exists almost everywhere (a.e.), i.e., for almost all $t \in [0, \infty)$, and $\dot{V}(x(t), t) \overset{a.e.}{\in} \dot{\tilde{V}}(x(t), t)$ where*

$$\dot{\tilde{V}}(x,t) \triangleq \bigcap_{\xi \in \partial V(x,t)} \xi^T \begin{bmatrix} K[f](x,t) \\ 1 \end{bmatrix}.$$

Remark 2.2. Throughout the subsequent discussion, for brevity of notation, let a.e. refer to almost all $t \in [0, \infty)$.

[2] Note that any $\mathcal{C}^1$ continuous function is regular and the sum of regular functions is regular [156].

2.2 Main Result

For the system described in (2–1) with a continuous right-hand side, existing Lyapunov theory can be used to examine the stability of the closed-loop system using continuous techniques such as those described in [148]. However, these theorems must be altered for the set-valued map $\dot{\tilde{V}}(x(t), t)$ for systems with right-hand sides which are not Lipschitz continuous [10, 11, 45]. Lyapunov analysis for nonsmooth systems is analogous to the analysis used for continuous systems. The differences are that differential equations are replaced with inclusions, gradients are replaced with generalized gradients, and points are replaced with sets throughout the analysis. The following presentation and subsequent proofs demonstrate how the LYT can be adapted for such systems.

The following auxiliary lemma from [154] and Barbalat's Lemma are provided to facilitate the proofs of the nonsmooth LYC.

Lemma 2.2. *[154] Let* $x(t)$ *be any Filippov solution to the system in (2–1) and* $V : \mathcal{D} \times [0, \infty) \to \mathbb{R}$ *be a locally Lipschitz, regular function. If* $\dot{V}(x(t), t) \overset{a.e.}{\leq} 0$*, then* $V(x(t), t) \leq V(x(t_0), t_0)$ $\forall t > t_0$.

Proof. For the sake of contradiction, let there exist some $t > t_0$ such that $V(x(t), t) > V(x(t), t_0)$. Then,

$$\int_{t_0}^{t} \dot{V}(x(\sigma), \sigma)\, d\sigma = V(x(t), t) - V(x(t), t_0) > 0.$$

It follows that $\dot{V}(x(t), t) > 0$ on a set of positive measure, which contradicts that $\dot{V}(x(t), t) \leq 0$, a.e. $\square$

The following Lemma recalls Barbalat's lemma for nonautonomous systems, which will be used in the proof of the nonsmooth LYC.

Lemma 2.3. ***(Barbalat's Lemma)*** *[148] Let $\phi:\ \mathbb{R}\to\mathbb{R}$ be a uniformly continuous (UC) function on $[0,\infty)$. Suppose that $\lim\limits_{t\to\infty}\int_0^t\phi(\tau)\,d\tau$ exists and is finite. Then,*

$$\phi(t)\to 0\ \ as\ \ t\to\infty.$$

Based on Lemmas 2.2 and 2.3, nonsmooth corollaries to the LYT (c.f., [55, Theorem 8.4] and [68, Theorem A.8]) are provided in Corollary 2.1 and 2.2.

Corollary 2.1. *For the system in (2–1), let $\mathcal{D}\subset\mathbb{R}^n$ be an open and connected set containing $x=0$ and suppose f is Lebesgue measurable and essentially locally bounded, uniformly in t. Let $V:\mathcal{D}\times[0,\infty)\to\mathbb{R}$ be locally Lipschitz and regular such that*

$$W_1(x)\le V(x,t)\le W_2(x)\quad\forall t\ge 0,\ \ \forall x\in\mathcal{D}\tag{2–4}$$

$$\dot{V}(x(t),t)\overset{a.e.}{\le}-W(x(t))\tag{2–5}$$

where W_1 and W_2 are continuous positive definite functions, and W is a continuous positive semi-definite function on $\mathcal{D}$. Choose $r>0$ and $c>0$ such that $B_r\subset\mathcal{D}$ and $c<\min\limits_{\|x\|=r}W_1(x)$ and $x(t)$ is a Filippov solution to (2–1) where $x(t_0)\in\{x\in B_r\mid W_2(x)\le c\}$. Then $x(t)$ is bounded and satisfies

$$W(x(t))\to 0\ \ as\ \ t\to\infty.$$

Proof. Since $B_r\subset\mathcal{D}$ and $c<\min\limits_{\|x\|=r}W_1(x)$, $\{x\in B_r\mid W_1(x)\le c\}$ is in the interior of B_r. Define a time-dependent set $\Omega_{t,c}$ by

$$\Omega_{t,c}=\{x\in B_r\mid V(x,t)\le c\}.$$

From (2–4), the set $\Omega_{t,c}$ contains $\{x\in B_r\mid W_2(x)\le c\}$ since

$$W_2(x)\le c\Rightarrow V(x,t)\le c.$$

On the other hand, $\Omega_{t,c}$ is a subset of $\{x \in B_r \mid W_1(x) \leq c\}$ since

$$V(x,t) \leq c \Rightarrow W_1(x) \leq c.$$

Thus,

$$\{x \in B_r \mid W_2(x) \leq c\} \subset \Omega_{t,c} \subset \{x \in B_r \mid W_1(x) \leq c\} \subset B_r \subset D.$$

Based on (2–5), $\dot{V}(x(t),t) \overset{a.e.}{\leq} 0$, hence, $V(x(t),t)$ is non-increasing from Lemma 2.2. For any $t_0 \geq 0$ and any $x(t_0) \in \Omega_{t_0,c}$, the solution starting at $(x(t_0),t_0)$ stays in $\Omega_{t,c}$ for every $t \geq t_0$. Therefore, any solution starting in $\{x \in B_r \mid W_2(x) \leq c\}$ stays in $\Omega_{t,c}$, and consequently in $\{x \in B_r \mid W_1(x) \leq c\}$, for all future time. Hence, the Filippov solution $x(t)$ is bounded such that $\|x(t)\| < r$, $\forall t \geq t_0$.

From Lemma 2.2, $V(x(t),t)$ is also bounded such that $V(x(t),t) \leq V(x(t_0),t_0)$. Now, since $\dot{V}(x(t),t)$ is Lebesgue measurable from (2–5),

$$\int_{t_0}^{t} W(x(\tau))\,d\tau \leq -\int_{t_0}^{t} \dot{V}(x(\tau),\tau)\,d\tau = V(x(t_0),t_0) - V(x(t),t) \leq V(x(t_0),t_0). \quad (2\text{–}6)$$

Therefore, $\int_{t_0}^{t} W(x(\tau))\,d\tau$ is bounded $\forall t > t_0$. Existence of $\lim\limits_{t\to\infty} \int_{t_0}^{t} W(x(\tau))\,d\tau$ is guaranteed since the left-hand side of (2–6) is monotonically nondecreasing (based on the definition of $W(x)$ in (2.1)) and bounded above. Since $x(t)$ is locally absolutely continuous and f is essentially locally bounded, uniformly in t, $x(t)$ is uniformly continuous.[3] Because $W(x)$ is continuous in x and x is on the compact set B_r, $W(x(t))$ is uniformly continuous in t on $(t_0, \infty]$. Therefore, by Lemma 2.3, it concludes that

$$W(x(t)) \to 0 \;\; as \;\; t \to \infty. \quad (2\text{–}7)$$

[3] Since $x(t)$ is locally absolutely continuous, $|x(t_2) - x(t_1)| = \left|\int_{t_1}^{t_2} \dot{x}(t)dt\right|$. From the assumption that $x \to f(x,t)$ is essentially locally bounded, uniformly in t and since $x \in \mathcal{L}_\infty$, then, $\dot{x} \in \mathcal{L}_\infty$. Using the fact that defining $\dot{x}(t)$ on a set of zero measure does not change x implies that $\left|\int_{t_1}^{t_2} \dot{x}(t)dt\right| \leq \left|\int_{t_1}^{t_2} Mdt\right|$, where M is a constant. Thus, $\left|\int_{t_1}^{t_2} Mdt\right| = M|t_2 - t_1|$, hence $x(t)$ is uniformly continuous.

□

Remark 2.3. From Def. 2.1, $K[f](x,t)$ is an upper semi-continuous, nonempty, compact and convex valued map. While existence of a Filippov solution for any arbitrary initial condition $x(t_0) \in \mathcal{D}$ is provided by the definition, generally speaking, the solution is non-unique [149, 151].

Note that Corollary 2.1 establishes (2–7) for a specific $x(t)$. Under the stronger condition that[4] $\dot{\tilde{V}}(x,t) \leq W(x)\ \forall x \in \mathcal{D}$, it is possible to show that (2–7) holds for all Filippov solutions of (2–1). The next corollary is presented to illustrate this point.

Corollary 2.2. *For the system given in (2–1), let* $\mathcal{D} \subset \mathbb{R}^n$ *be a domain containing* $x = 0$ *and suppose* f *is Lebesgue measurable and essentially locally bounded, uniformly in* t*. Let* $V : \mathcal{D} \times [0, \infty) \to \mathbb{R}$ *be locally Lipschitz and regular such that*

$$W_1(x) \leq V(x,t) \leq W_2(x) \tag{2–8}$$

$$\dot{\tilde{V}}(x,t) \leq -W(x) \tag{2–9}$$

$\forall t \geq 0,\ \forall x \in \mathcal{D}$ *where* W_1 *and* W_2 *are continuous positive definite functions, and* W *is a continuous positive semi-definite function on* $\mathcal{D}$*. Choose* $r > 0$ *and* $c > 0$ *such that* $B_r \subset \mathcal{D}$ *and* $c < \min\limits_{\|x\|=r} W_1(x)$*. Then, all Filippov solutions of (2–1) such that* $x(t_0) \in \{x \in B_r \mid W_2(x) \leq c\}$ *are bounded and satisfy*

$$W(x(t)) \to 0 \ \ as \ \ t \to \infty. \tag{2–10}$$

Proof. Let $x(t)$ be any arbitrary Filippov solution of (2–1). Then, from Lemma 2.1, and (2–9), $\dot{V}(x(t),t) \overset{a.e.}{\leq} -W(x(t))$, which is precisely the condition (2–5). Since the

[4] The inequality $\dot{\tilde{V}}(x,t) \leq W(x)$ is used to indicate that every element of the set $\dot{\tilde{V}}(x,t)$ is less than or equal to the scalar $W(x)$.

selection of $x(t)$ is arbitrary, Corollary 2.1 can be used to imply that the result in (2–7) holds for each $x(t)$. $\square$

2.3 Design Example 1 (Adaptive + Sliding Mode)

The LYC (and the LaSalle-Yoshizawa Theorem) are useful in its ability to provide boundedness and convergence of solutions, while providing a compact framework to define the region of attraction for which boundedness and convergence results hold. In fact, the region of attraction is provided as part of the corollary structures. In the case of semi-global and local results, these domains and sets are especially useful. It is important to note that Barbalat's Lemma can be used to achieve the same results (in fact, it is used in the proof for Corollary 2.1); however, the use of Barbalat's Lemma would require the identification of the region of attraction for which convergence holds and does not provide boundedness of the trajectories. For illustrative purposes, the following design example targets the regulation of a first order nonlinear system. Corollary 2.1 and 2.2 can also be directly applied to general n^{th} order time-varying nonlinear systems and to tracking control problems.

To illustrate the utility of Corollary 2.2, consider a first order nonlinear differential equation given by

$$\dot{x} = f(x,t) + d(x,t) + u(t) \tag{2–11}$$

where $f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ is an unknown, linear-parameterizable, essentially locally bounded, uniformly in t function that can be expressed as $f(x,t) = Y(x,t)\,\theta$ when $\theta \in \mathbb{R}^p$ is a vector of unknown constant parameters, and $Y : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^{n\times p} \times [0,\infty)$ is the regression matrix for $f(x,t)$, $u : [0,\infty) \to \mathbb{R}^n$ is the control input, $x(t) \in \mathbb{R}^n$ is the measurable system state, and $d(x,t)$ is an essentially locally bounded disturbance which satisfies

$$\|d(x,t)\| \leq c_1 + c_2(\|x\|)\,\|x\| \tag{2–12}$$

where $c_1 \in \mathbb{R}^+$ is a positive constant, and $c_2(\|x\|) : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive, globally invertible, state-dependent function. A regulation controller for (2–11) can be designed as

$$u(x,t) \triangleq -k_1 x - k_2 sgn(x) - Y\hat{\theta} \tag{2–13}$$

where $\hat{\theta}(x,t) \in \mathbb{R}^p$ is the estimate of θ, $k_1, k_2 \in \mathbb{R}^+$ are gain constants, and $sgn(\cdot)$ is defined $\forall \xi \in \mathbb{R}^n = \begin{bmatrix} \xi_1 & \xi_2 & \ldots & \xi_n \end{bmatrix}^T$ as $sgn(\xi) \triangleq \begin{bmatrix} sgn(\xi_1) & sgn(\xi_2) & \ldots & sgn(\xi_n) \end{bmatrix}^T$. Based on the subsequent stability analysis, an adaptive update law can be defined as

$$\dot{\hat{\theta}} = \Gamma Y^T x \tag{2–14}$$

where $\Gamma \in \mathbb{R}^{n \times n}$ is a positive gain matrix. The closed-loop system is given by

$$\dot{x} = Y\tilde{\theta} + d(x,t) - k_1 x - k_2 sgn(x) \tag{2–15}$$

where $\tilde{\theta} \in \mathbb{R}^p$ denotes the mismatch $\tilde{\theta} \triangleq \theta - \hat{\theta}$. In (2–15), it is apparent that the RHS contains a discontinuity in $x(t)$ and requires the use of differential inclusions to provide existence of solutions. Let $y\left(x, \tilde{\theta}\right) \in \mathbb{R}^{n+p}$ denote $y \triangleq \begin{bmatrix} x \\ \tilde{\theta} \end{bmatrix}$ and choose a positive-definite, locally Lipschitz, regular Lyapunov candidate function as

$$V(y) = \frac{1}{2}x^T x + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1} \tilde{\theta}. \tag{2–16}$$

The candidate Lyapunov function in (2–16) satisfies the following inequalities:

$$W_1(y) \le V(y) \le W_2(y) \tag{2–17}$$

where the continuous positive-definite functions $W_1(y), W_2(y) \in \mathbb{R}$ are defined as $W_1(y) \triangleq \lambda_1 \|y\|^2$, $W_2(y) \triangleq \lambda_2 \|y\|^2$ and $\lambda_1, \lambda_2 \in \mathbb{R}^+$ are known constants. Then,

$\dot{V}(y(t),t) \overset{a.e.}{\in} \dot{\tilde{V}}(y(t),t)$ and

$$\dot{\tilde{V}} = \bigcap_{\xi \in \partial V\left(x,\tilde{\theta},t\right)} \xi^T K \begin{bmatrix} \dot{x} \\ \dot{\tilde{\theta}} \\ 1 \end{bmatrix} \left(x, \tilde{\theta}, t\right).$$

Since $V(y,t)$ is $\mathcal{C}^{\infty}$ in y,[5]

$$\dot{\tilde{V}} \subset \nabla V^T K \begin{bmatrix} \dot{x} \\ \dot{\tilde{\theta}} \end{bmatrix} \left(x, \tilde{\theta}\right) \subset \left[x^T, \tilde{\theta}^T \Gamma^{-1}\right] K \begin{bmatrix} \dot{x} \\ \dot{\tilde{\theta}} \end{bmatrix} \left(x, \tilde{\theta}\right). \tag{2–18}$$

After using (2–15), the expression in (2–18) can be written as

$$\dot{\tilde{V}} \subset x^T \left(Y\tilde{\theta} + d(x,t) - k_1 x - k_2 K[sgn(x)]\right) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} \tag{2–19}$$

where $K[sgn(x)] = SGN(x)$ such that $SGN(x_i) = 1$ if $x_i > 0$, $[-1, 1]$ if $x_i = 0$, and -1 if $x_i < 0 \ \forall i = 1, 2, ..., n$.

Remark 2.4. One could also consider the discontinuous function instead of the differential inclusion (i.e., the $sgn(\cdot)$ function can alternatively be defined as $sgn(0) = 0$) using Caratheodory solutions; however, this method lacks would not be an indicator for what happens when measurement noise is present in the system. As described in results such as [157–159], Filippov and Krasovskii solutions for discontinuous differential equations are appropriate for capturing the possible closed-loop system behavior in the presence of arbitrarily small measurement noise. By utilizing the set valued map $SGN(\cdot)$ in the analysis, we account for the possibility that when the true state satisfies

[5] For continuously differentiable Lyapunov candidate functions, the generalized gradient reduces to the standard gradient. However, this is not required by the Corollary itself and only assists in evaluation.

$x = 0$, $sgn(x)$ (of the measured state) falls within the set $[-1, 1]$. Therefore, the presented analysis is more robust to measurement noise than an analysis that depends on $sgn(0)$ to be defined as a known singleton.

Substituting for the adaptive update law in (2–14), canceling terms and utilizing the bound for $d(x, t)$ in (2–12), the expression in (2–18) can be upper bounded as

$$\dot{\tilde{V}} \leq -k_1 \|x\|^2 + c_1 \|x\| + c_2 (\|x\|) \|x\|^2 - k_2 \|x\| . \tag{2–20}$$

The set in (2–19) reduces to the scalar inequality in (2–20) since in the case when $K[sgn(x)]$ is defined as a set, it is multiplied by x, i.e., when $x = 0$, $0 \cdot SGN(0) = \{0\}$. Regrouping similar terms, the expression in (2–20) can be written as

$$\dot{\tilde{V}} \leq -(k_1 - c_2(\|x\|)) \|x\|^2 - (k_2 - c_1) \|x\| . \tag{2–21}$$

Provided $k_2 > c_1$ and $k_1 > c_2(\|x\|)$, the expression in (2–21) can be upper bounded as $\dot{\tilde{V}} \leq -W(y(t))$ where $W(y)$ is a positive semi-definite function defined on the domain $\mathcal{D} \triangleq \left\{ y \mid \|y\| < c_2^{-1}(k_1) \right\}$. The inequalities in (2–17) can be used to show that $V(y(t), t) \in \mathcal{L}_\infty$ in $\mathcal{D}$; hence, $x(t)$ and $\tilde{\theta}(x(t), t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since θ contains the constant unknown system parameters and $\tilde{\theta}(x(t), t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, the definition for $\tilde{\theta}(x(t), t)$ can be used to show that $\hat{\theta}(x(t), t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Given that $x(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, $Y(x(t), t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $x(t), \hat{\theta}(x(t), t)$, and $Y(x(t), t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, the control is bounded from (2–13) and the adaption law in (2–14). The closed-loop dynamics in (2–12) and (2–15) can be used to conclude that $\dot{x}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$; hence, $x(t)$ is uniformly continuous in $\mathcal{D}$.

Choose $0 < r < c_2^{-1}(k_1)$ such that $B_r \subset \mathcal{D}$ denotes a closed ball, and let $\mathcal{S} \subset B_r$ denote the set defined as

$$\mathcal{S} \triangleq \left\{ y \subset B_r \,|\, W_2(y) < \min_{\|y\|=r} W_1(y) = \lambda_1 r^2 \right\}. \tag{2–22}$$

Invoking Corollary 2.2, $W(y(t)) = -(k_1 - c_2(\|x\|))\|x\|^2 \to 0 \;\; as \;\; t \to \infty \;\forall y(0) \in \mathcal{S}$, thus, $x \to 0 \quad as \quad t \to \infty \;\forall y(0) \in \mathcal{S}$. The region of attraction in (2–22) can be made arbitrarily large to include all initial conditions (a semi-global type result) by increasing the gain k_1.

Remark 2.5. For some systems (e.g., closed-loop error systems with sliding mode control laws), it may be possible to show that Corollary 2.2 is more easily applied, as is the focus of the first example. However, in other cases, it may be difficult to satisfy the inequality in (2–9). The usefulness of Corollary 2.1 is demonstrated in those cases where it is difficult or impossible to show that the inequality in (2–9) can be satisfied, but it is possible to show that (2–5) can be satisfied for almost all time, as is the focus of the next example.

2.4 Design Example 2 (RISE)

To illustrate the utility of Corollary 2.1, consider a second order nonlinear differential equation given by

$$\ddot{x} = f(x,t) + d(t) + u(t) \tag{2–23}$$

where $f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ is an unknown essentially locally bounded, uniformly in t function, $u : [0,\infty) \to \mathbb{R}^n$ is the control input, $x(t) \in \mathbb{R}^n$ is the measurable system state, and $d(t) \in \mathbb{R}^n$ is an essentially locally bounded disturbance, which satisfies $d(t), \dot{d}(t), \ddot{d}(t) \in \mathcal{L}_\infty$. A desired trajectory, denoted by $x_d \in \mathbb{R}^n$, satisfies $x_d^{(i)}(t) \in \mathbb{R}^n, \; \forall i = 0, 1, ..., 4$.

To quantify the control objective, a tracking error, denoted by $e_1(\eta, \eta_d) \in \mathbb{R}^6$, is defined as

$$e_1 \triangleq x_d - x, \tag{2–24}$$

and two auxiliary tracking errors denoted by $e_2(e_1, \dot{e}_1), r(e_2, \dot{e}_2) \in \mathbb{R}^6$, are defined as

$$e_2 \;\triangleq\; \dot{e}_1 + \alpha_1 e_1, \tag{2–25}$$

$$r \;\triangleq\; \dot{e}_2 + \alpha_2 e_2 \tag{2–26}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}^+$ are adjustable gains. The auxiliary signal $r(e_2, \dot{e}_2)$ is introduced to facilitate the stability analysis and is not used in the control design since the expression in (2–25) depends on the unmeasurable state $\ddot{x}(t)$.

The open loop error system can be expressed as

$$r = \ddot{x}_d + S - f(x_d, t) - d(t) - u(t) + \alpha_2 e_2 \tag{2–27}$$

where the auxiliary function $S \in \mathbb{R}^n$ is defined as $S \triangleq f(x_d, t) - f(x, t) + \alpha_1 \dot{e}_1 + \alpha_2 e_2$. A RISE-based control structure [8, 160] can be designed as

$$u \triangleq (k_s + 1) e_2 - (k_s + 1) e_2(0) + \upsilon. \tag{2–28}$$

where $\upsilon(e_2) \in \mathbb{R}^n$ is the Filippov solution to the following differential equation

$$\dot{\upsilon} \triangleq (k_s + 1) \alpha e_2 + \beta sgn(e_2), \;\; \upsilon(0) = 0, \tag{2–29}$$

$\beta, k_s \in \mathbb{R}$ are positive, constant control gains and $sgn(\cdot)$ is defined $\forall \xi \in \mathbb{R}^m = \begin{bmatrix} \xi_1 & \xi_2 & ... & \xi_m \end{bmatrix}^T$ as $sgn(\xi) \triangleq \begin{bmatrix} sgn(\xi_1) & sgn(\xi_2) & ... & sgn(\xi_m) \end{bmatrix}^T$. The differential equation given in (2–29) is continuous except when $e_2 = 0$. Using Filippov's theory of differential inclusions [149, 161–163], the existence of solutions can be established for $\dot{\upsilon} \in K[h_1](e_2)$, where $h_1(e_2) \in \mathbb{R}^n$ is defined as the right-hand side of (2–29) and $K[h_1] \triangleq \bigcap_{\delta > 0} \bigcap_{\mu S_m = 0} \overline{co} h_1 (B(e_2, \delta) - S_m)$, where $\bigcap_{\mu S_m = 0}$ denotes the intersection of all sets S_m (of Lebesgue measure zero) of discontinuities, $\overline{co}$ denotes convex closure, and $B(e_2, \delta) = \{\varsigma \in \mathbb{R} | \; \|e_2 - \varsigma\| < \delta\}$ [45, 154].

To facilitate the subsequent analysis, the controller in (2–28) is substituted into (2–27) and the time derivative found by utilizing a DCAL approach to regroup terms as

$$\dot{r} = \tilde{N} + N_d - e_2 - (k_s + 1) r - \beta sgn(e_2) \tag{2–30}$$

where $\tilde{N}(e_2, r, t) \in \mathbb{R}^n$ and $N_d(t) \in \mathbb{R}^n$ are defined as

$$\tilde{N} \triangleq \dot{S} + e_2, \tag{2–31}$$

$$N_d \triangleq \dddot{x}_d - \dot{f}(x_d, \dot{x}_d, t) + \dot{d}(t). \tag{2–32}$$

Using (2–24)-(2–25) and the Mean Value Theorem, the function $\tilde{N}(\cdot)$ in (2–31) can be upper bounded as [164, App A]

$$\left\|\tilde{N}\right\| \leq \rho(\|z\|)\|z\|, \tag{2–33}$$

where $z(e_1, e_2, r) \in \mathbb{R}^{3n}$ is defined as

$$z \triangleq \begin{bmatrix} e_1^T & e_2^T & r^T \end{bmatrix}^T \tag{2–34}$$

and $\rho : \mathbb{R} \to \mathbb{R}$ is a positive, globally invertible, nondecreasing function. Assuming the disturbance and desired trajectory are sufficiently smooth, the following inequalities can be developed:

$$\|N_d\| \leq \zeta_1, \quad \left\|\dot{N}_d\right\| \leq \zeta_2$$

where $\zeta_1, \zeta_2 \in \mathbb{R}^+$ are known constants.

Let $y(z, P) \in \mathbb{R}^{3n+1}$ be defined as

$$y \triangleq \begin{bmatrix} z^T & \sqrt{P} \end{bmatrix}^T \tag{2–35}$$

where the auxiliary function $P(e_2, t) \in \mathbb{R}$ is defined as the Filippov solution to the following differential equation

$$\begin{aligned} \dot{P} &= -r^T(N_d - \beta sgn(e_2)) \\ P(e_2(t_0), t_0) &= \beta \sum_{i=1}^{n} |e_{2i}(t_0)| - e_2(t_0)^T N_d(t_0) \end{aligned} \tag{2–36}$$

where the subscript $i = 1, 2, ..., n$ denotes the ith element of the vector. Similar to the development in (2–29), existence of solutions for P can be established using Filippov's theory of differential inclusions for $\dot{P} \in K[h_2](e_2, r, t)$, where $h_2(e_2, r, t) \in \mathbb{R}$ is defined as $h_2 \triangleq -r^T(N_d - \beta sgn(e_2))$ and $K[h_2] \triangleq \bigcap_{\delta>0} \bigcap_{\mu S_m=0} \overline{co} h_2(B(e_2, \delta) - S_m, r, t)$ as in (2–29). Integrating (2–36) by parts and provided $\beta > \zeta_1 + \frac{\zeta_2}{\alpha_2}$, $P(e_2, t) \geq 0$ (see [165] for details).

Let $V_L : \mathcal{D} \times [0, \infty) \rightarrow \mathbb{R}$ be continuously differentiable in y, locally Lipschitz in t, regular, and defined as

$$V_L = \frac{1}{2}e_1^T e_1 + \frac{1}{2}e_2^T e_2 + \frac{1}{2}r^T r + P \tag{2–37}$$

which satisfies the following inequalities:

$$U_1(y) \leq V_L(y,t) \leq U_2(y), \tag{2–38}$$

where $U_1(y)$, $U_2(y) \in \mathbb{R}$ are positive definite functions defined as $U_1 \triangleq \lambda_1 \|y\|^2$ and $U_2 \triangleq \lambda_2 \|y\|^2$.

Under Filippov's framework, the time derivative of (2–37) exists almost everywhere (a.e.), i.e., for almost all $t \in [0, \infty)$, and $\dot{V}_L(y,t) \overset{a.e.}{\in} \dot{\tilde{V}}_L(y,t)$ where

$$\dot{\tilde{V}}_L = \bigcap_{\xi \in \partial V_L(y,t)} \xi^T K \begin{bmatrix} \dot{e}_1^T & \dot{e}_2^T & \dot{r}^T & \frac{1}{2}P^{-\frac{1}{2}}\dot{P} & 1 \end{bmatrix}^T,$$

where ∂V_L is the generalized gradient of $V_L(y,t)$ [45, 154, 166]. Since $V(y,t)$ is $\mathcal{C}^\infty$ in y,

$$\dot{\tilde{V}}_L \subset \nabla V_L^T K \begin{bmatrix} \dot{e}_1^T & \dot{e}_2^T & \dot{r}^T & \frac{1}{2}P^{-\frac{1}{2}}\dot{P} \end{bmatrix}^T, \tag{2–39}$$

where $\nabla V_L \triangleq \begin{bmatrix} e_1^T & e_2^T & r^T & 2P^{\frac{1}{2}} \end{bmatrix}^T$.

Using the calculus for $K[\cdot]$ from [154], substituting (2–24), (2–25), (2–28), (2–30), and (2–36), and canceling similar terms, the expression in (2–39) becomes

$$\begin{aligned} \dot{\tilde{V}}_L \subset\, & e_1^T e_2 - \alpha_1 e_1^T e_1 - \alpha_2 e_2^T e_2 + r^T \tilde{N} + r^T N_d - (k_s + 1)\, r^T r \\ & - r^T \beta K[sgn(e_2)] - r^T (N_d - \beta K[sgn(e_2)]), \end{aligned} \tag{2–40}$$

where $K\left[sgn(e_2)\right] = SGN\left(e_2\right)$ such that $SGN\left(e_{2_i}\right) = 1$ if $e_{2_i}\left(\cdot\right) > 0$, $\left[-1, 1\right]$ if $e_{2_i}\left(\cdot\right) = 0$, and -1 if $e_{2_i}\left(\cdot\right) < 0$.[6] Utilizing the fact that the set in (2–40) reduces to a scalar equality since the RHS is continuous a.e., i.e, the RHS is continuous except for the Lebesgue negligible set of times when $r^T\beta K\left[sgn\left(e_2\right)\right] - r^T\beta K\left[sgn\left(e_2\right)\right] \neq 0$ [45, 167], an upper bound for $\dot{V}_L$ is given as

$$\begin{aligned}\dot{V}_L \overset{a.e.}{\leq} & -\alpha_1\left\|e_1\right\|^2 + \left\|e_1\right\|\left\|e_2\right\| - \alpha_2\left\|e_2\right\|^2 \\ & +\rho\left(\left\|z\right\|\right)\left\|r\right\|\left\|z\right\| - \left(k_s + 1\right)\left\|r\right\|^2 .\end{aligned} \tag{2–41}$$

To show that the number of times when $r^T\beta K\left[sgn\left(e_2\right)\right] - r^T\beta K\left[sgn\left(e_2\right)\right] \neq 0$ is measure zero, we recall the error system definition in (2–26) and introduce the following lemma.

Lemma 2.4. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuously differentiable function with the property: $f\left(x\right) = 0,\ f'\left(t\right) \neq 0$, then*

$$\mu\left(f^{-1}\left(\{0\}\right)\right) = 0, \tag{2–42}$$

where μ denotes the Lebesgue measure on $[0, \infty)$.

Proof. We will first prove that all the points in the set $f^{-1}\left(\{0\}\right)$ are isolated. That is,

$$\left(\forall a \in f^{-1}\left(\{0\}\right)\right)\left(\exists\epsilon > 0\right) \mid \left(\left(\left(a - \epsilon, a + \epsilon\right) \cap \left(f^{-1}\left(\{0\}\right)\right)\right) \setminus \{a\} = \phi\right). \tag{2–43}$$

To obtain a contradiction, the negation of the statement above is,

$$\left(\exists a \in f^{-1}\left(\{0\}\right)\right) \mid \left(\forall\epsilon > 0\right)\left(\left(\left(a - \epsilon, a + \epsilon\right) \cap \left(f^{-1}\left(\{0\}\right)\right)\right) \setminus \{a\} \neq \phi\right). \tag{2–44}$$

[6] As in the previous example, the $sgn\left(\cdot\right)$ function can alternatively be defined as $sgn\left(0\right) = 0$; however, this restriction lacks robustness with respect to measurement noise.

Assuming (2–44), let $b \in ((a-\epsilon, a+\epsilon) \cap (f^{-1}(\{0\}))) \setminus \{a\}$. Without loss of generality we can assume $b > a$ and $f'(a) > 0$. As f is differentiable and $f(a) = f(b) = 0$, by Rolle's theorem, $\exists c \in (a, b)$ such that

$$f'(c) = 0. \tag{2–45}$$

By continuity of f' at a,

$$(\forall \epsilon_a > 0)(\exists \delta_a > 0) \mid (\forall x \in [0, \infty))(|x-a| < \delta_a \implies f'(a) - \epsilon_a < f'(x) < f'(a) + \epsilon_a).$$

In particular, pick $\epsilon_a = f'(a)$. Then,

$$(\exists \delta_a > 0) \mid (\forall x \in [0, \infty))(|x-a| < \delta_a \implies f'(x) > 0).$$

Now, pick $\epsilon = \delta_a$ in (2–44). Thus from $b \in ((a-\delta_a, a+\delta_a) \cap (f^{-1}(\{0\}))) \setminus \{a\}$ we get $|b-a| < \delta_a$ which from $c \in (a, b)$ implies $|c-a| < \delta_a$ which implies $f'(c) > 0$, which contradicts (2–45).

Thus, all the points in the set $f^{-1}(\{0\})$ are isolated, and hence, $f^{-1}(\{0\})$ is a discrete set. As any discrete subset of Euclidean space is countable, (2–42) is obtained.

□

The set of times

$$\Lambda \triangleq \left\{ t \in [0, \infty) : r(t)^T \beta K [sgn(e_2(t))] - r(t)^T \beta K [sgn(e_2(t))] \neq 0 \right\} \subset [0, \infty)$$

is equivalent to the set of times $\{t : e_2(t) = 0 \ \wedge r(t) \neq 0\}$. From (2–26), this set can also be represented by $\{t : e_2(t) = 0 \ \wedge \dot{e}_2(t) \neq 0\}$. Provided $e_2(t)$ is continuously differentiable (it is in our case), Lemma 2.4 can be used to show that the set of time instances $\{t : e_2(t) = 0 \ \wedge \dot{e}_2(t) \neq 0\}$ is isolated, and thus, measure zero. This implies that the set Λ is measure zero.

Utilizing Young's Inequality, the expression in (2–41) can be reduced to

$$\dot{V}_L \overset{a.e.}{\leq} -\sigma \|z\|^2 - k_s \|r\|^2 + \rho(\|z\|) \|r\| \|z\|, \tag{2–46}$$

where $\sigma = \min\left\{\alpha_1 - \frac{1}{2}, \alpha_2 - \frac{1}{2}, 1\right\}$ and $z\left(e_1, e_2, r\right)$ was defined in (2–34). If the gains are selected such that $\alpha_1 > \frac{1}{2}$ and $\alpha_2 > \frac{1}{2}$, and by completing the squares for $r\left(\cdot\right)$, the expression in (2–46) can be upper bounded as

$$\dot{V}_L \overset{a.e.}{\leq} -\sigma\left\|z\right\|^2 + \frac{\rho^2\left(\left\|z\right\|\right)\left\|z\right\|^2}{4k_s} \leq -U\left(y\right), \tag{2–47}$$

where $U\left(y\right) \triangleq c\left\|z\right\|^2$, for some positive constant $c \in \mathbb{R}$, is a continuous positive semi-definite function such that

$$\mathcal{D} \triangleq \left\{y \in \mathbb{R}^{3n+1} \mid \left\|y\right\| \leq \rho^{-1}\left(2\sqrt{\sigma k_s}\right)\right\}.$$

The size of the domain $\mathcal{D}$ can be enlarged by increasing the gain k_s. The inequalities in (2–38) and (2–47) can be used to show that $V_L \in \mathcal{L}_\infty$ in $\mathcal{D}$. Thus, $e_1\left(\cdot\right), e_2\left(\cdot\right), r\left(\cdot\right) \in \mathcal{L}_\infty$ in $\mathcal{D}$. The closed-loop error system can be used to conclude that the remaining signals are bounded in $\mathcal{D}$, and the definitions for $U\left(y\right)$ and $z\left(\cdot\right)$ can be used to show that $U\left(y\right)$ is uniformly continuous in $\mathcal{D}$. Let $S_{\mathcal{D}} \subset \mathcal{D}$ denote a set defined as

$$\mathcal{S}_{\mathcal{D}} \triangleq \left\{y \in \mathcal{D} \mid U_2\left(y\right) < \lambda_1\left(\rho^{-1}\left(2\sqrt{\sigma k_s}\right)\right)^2\right\}.$$

The region of attraction in $\mathcal{S}_{\mathcal{D}}$ can be made arbitrarily large to include any initial conditions by increasing the control gain k_s. From (2–47), Corollary 2.1 can be invoked to show that $c\left\|z\right\|^2 \rightarrow 0 \;\; as \;\; t \rightarrow \infty \; \forall y\left(0\right) \in \mathcal{S}_{\mathcal{D}}$. Based on the definition of $z\left(\cdot\right)$ in (2–34), $\left\|e_1\right\| \rightarrow 0 \;\; as \;\; t \rightarrow \infty \; \forall y\left(0\right) \in \mathcal{S}_{\mathcal{D}}$.

Remark 2.6. In Example 2.4, we apply Corollary 2.1 when $\dot{V}_L\left(y, t\right) \overset{a.e.}{\leq} -W\left(y\right)$. The difference in this case from Example 2.3, stems from the fact that it is not possible to directly show that all solutions satisfy (2–9). Instead, it is possible to showthat (2–5) can be satisfied in the analysis, and Lemma 2.4 and the associated arguments can be used to prove that this case is satisfied for all time.

2.5 Summary

This chapter introduced the mechanics required to utilize nonsmooth analysis in Lyapunov-based control design and have extended the LYT to differential systems with a discontinuous RHS using Filippov differential inclusions. The result presents theoretical tools applicable to nonlinear systems with discontinuities in the plant dynamics or in the control structure. The generalized Lyapunov-based analysis methods are developed using differential inclusions in the sense of Filippov to achieve asymptotic convergence when the Lyapunov derivative is upper bounded by a negative semi-definite function. Cases when the bound on the Lyapunov derivative holds for all possible Filippov solutions are also considered. An adaptive, sliding mode control example and a RISE control example are provided to illustrate the utility of the main results.

CHAPTER 3
SATURATED RISE FEEDBACK CONTROL

In this chapter, a saturated controller is developed for a class of uncertain, second-order, nonlinear systems which includes time-varying and non-LP functions with bounded disturbances using a continuous control law with smooth saturation functions. Based on the RISE control methodology, the proposed controller is able to utilize the benefits of high gain control strategies while guaranteeing saturation limits are not surpassed. The bounds on the control are known a priori and can be adjusted by changing the feedback gains. The saturated controller yields asymptotic tracking despite uncertainty and added disturbances in the dynamics. Experimental results using a two-link robot manipulator demonstrate the performance of the proposed controller.

3.1 Dynamic Model

Consider a general class of nonlinear systems of the following form:

$$\ddot{x} = f\left(x, \dot{x}, t\right) + u\left(x, \dot{x}, t\right) + d\left(t\right) \tag{3–1}$$

where $x\left(t\right), \dot{x}\left(t\right) \in \mathbb{R}^{n}$ are the generalized system states, $u\left(x, \dot{x}, t\right) \in \mathbb{R}^{n}$ is the generalized control input, $f\left(x, \dot{x}, t\right) : \mathbb{R}^{2n} \times [0, \infty) \to \mathbb{R}^{n}$ is an unknown nonlinear $\mathcal{C}^{2}$ function, and $d\left(t\right) : [0, \infty) \to \mathbb{R}^{n}$ denotes a generalized, sufficiently smooth, non-vanishing nonlinear disturbance (e.g., unmodeled effects).

The subsequent development is based on the assumption that $x\left(t\right)$ and $\dot{x}\left(t\right)$ are measurable outputs. Additionally, the following assumptions will be exploited.

Assumption 3.1. The nonlinear disturbance term and its first two time derivatives (i.e., $d\left(t\right), \dot{d}\left(t\right), \ddot{d}\left(t\right)$) exist and are bounded by known constants [122, 147, 168].[1]

[1] Many practical disturbance terms are continuous including friction (see [169, 170]), wind disturbances, wave/ocean disturbances, unmodeled sufficiently smooth disturbances, etc.).

Assumption 3.2. The desired trajectory $x_d(t) \in \mathbb{R}^n$ is designed such that $x_d^{(i)}(t) \in \mathbb{R}^n$, $\forall i = 0, 1, ..., 4$ exist and are bounded.[2]

Remark 3.1*.* To aid the subsequent control design and analysis, the vector $Tanh(\cdot) \in \mathbb{R}^n$ and the matrix $Cosh(\cdot) \in \mathbb{R}^{n \times n}$ are defined as

$$Tanh(\xi) \triangleq [tanh(\xi_1), ..., tanh(\xi_n)]^T \tag{3–2}$$

$$Cosh(\xi) \triangleq diag\{cosh(\xi_1), ..., cosh(\xi_n)\} \tag{3–3}$$

where $\xi = [\xi_1, ..., \xi_n]^T \in \mathbb{R}^n$. Based on the definition of (3–2), the following inequalities hold $\forall \xi \in \mathbb{R}^n$ [78]:

$$\|\xi\|^2 \geq \sum_{i=1}^{n} ln(cosh(\xi_i)) \geq \frac{1}{2} tanh^2(\|\xi\|), \quad \|\xi\| > \|Tanh(\xi)\|$$

$$\|Tanh(\xi)\|^2 \geq tanh^2(\|\xi\|), \quad \xi^T Tanh(\xi) \geq Tanh^T(\xi) Tanh(\xi). \tag{3–4}$$

Throughout the paper, $\|\cdot\|$ denotes the standard Euclidean norm.

3.2 Control Development

The objective is to design an amplitude-limited, continuous controller which ensures the system state $x(t)$ tracks a desired trajectory $x_d(t)$. To quantify the control objective, a tracking error denoted $e_1(x, x_d) \in \mathbb{R}^n$ is defined as

$$e_1 \triangleq x_d - x. \tag{3–5}$$

Embedding the control in a bounded trigonometric term is an obvious way to limit the control authority below an a priori limit; however, by injecting these terms, difficulty arises in the closed-loop stability analysis. This challenge is exacerbated by the presence of integral control functions that are included to compensate for added

[2] Many guidance and navigation applications utilize smooth, high-order differentiable desired trajectories. Curve fitting methods can also be used to generate sufficiently smooth time-varying trajectories.

disturbances as in this result. Motivated by these stability analysis complexities and through an iterative analysis procedure, two measurable filtered tracking errors are designed which include extra smooth saturation terms. Specifically, the filtered tracking errors $e_2\left(e_1,\dot{e}_1,e_f\right),\ r\left(e_2,\dot{e}_2\right)\in\mathbb{R}^n$, are defined as

$$e_2 \triangleq \dot{e}_1+\alpha_1 Tanh\left(e_1\right)+Tanh\left(e_f\right), \tag{3–6}$$

$$r \triangleq \dot{e}_2+\alpha_2 Tanh\left(e_2\right)+\alpha_3 e_2 \tag{3–7}$$

where $\alpha_1,\alpha_2,\alpha_3 \in \mathbb{R}$ denote constant positive control gains, $Tanh\left(\cdot\right)$ was defined in (3–2), and $e_f\left(e_1,e_2\right)\in\mathbb{R}^n$ is an auxiliary signal whose dynamics are given by

$$\dot{e}_f \triangleq Cosh^2\left(e_f\right)\left\{-\gamma_1 e_2+Tanh\left(e_1\right)-\gamma_2 Tanh\left(e_f\right)\right\} \tag{3–8}$$

and $\gamma_1,\gamma_2 \in \mathbb{R}$ are constant positive control gains. The auxiliary signal $r\left(e_2,\dot{e}_2\right)$ is introduced to facilitate the stability analysis and is not used in the control design since the expression in (3–7) depends on the unmeasurable generalized state $\ddot{x}\left(t\right)$. The structure of the error systems (and included auxiliary signals) is motivated by the need to inject and cancel terms in the subsequent stability analysis, and will become apparent in Section 3.3.

An open-loop tracking error can be obtained by utilizing the filtered tracking error in (3–7) and substituting from (3–1), (3–5), (3–6), and (3–8) to yield

$$r=S-f_d+\ddot{x}_d-d-u\left(t\right)-\gamma_1 e_2 \tag{3–9}$$

where the auxiliary function $S\left(e_1,e_2,e_f,t\right)\in\mathbb{R}^n$ is defined as

$$\begin{aligned} S \triangleq\ & f_d-f-\gamma_2 Tanh\left(e_f\right)+\alpha_1 Cosh^{-2}\left(e_1\right)\left[e_2-\alpha_1 Tanh\left(e_1\right)-Tanh\left(e_f\right)\right] \\ & +\alpha_2 Tanh\left(e_2\right)+\alpha_3 e_2+Tanh\left(e_1\right), \end{aligned} \tag{3–10}$$

and a desired trajectory dependent auxiliary term is defined as $f_d=f\left(x_d,\dot{x}_d,t\right)\in\mathbb{R}^n$.

Based on the form of (3–9) and through an iterative stability analysis, the continuous controller, $u(v)$, is designed as[3]

$$u \triangleq \gamma_1 Tanh(v) \tag{3–11}$$

where $v(e_1, e_2) \in \mathbb{R}^n$ is the generalized Filippov solution to the following differential equation

$$\dot{v} = Cosh^2(v)\left[\alpha_2 Tanh(e_2) + \alpha_3 e_2 + \beta sgn(e_2) - \alpha_1 Cosh^{-2}(e_1)\, e_2 + \gamma_2 e_2\right], \quad v(0) = 0 \tag{3–12}$$

where $\beta \in \mathbb{R}$ is a positive constant control gain and $sgn(\cdot)$ is defined $\forall \xi \in \mathbb{R}^m = \begin{bmatrix} \xi_1 & \xi_2 & ... & \xi_m \end{bmatrix}^T$ as $sgn(\xi) \triangleq \begin{bmatrix} sgn(\xi_1) & sgn(\xi_2) & ... & sgn(\xi_m) \end{bmatrix}^T$.[4] Using Filippov's theory of differential inclusions [149, 161–163], the existence of solutions can be established for $\dot{v} \in K[h_1](e_1, e_2)$, where $h_1(e_1, e_2) \in \mathbb{R}^n$ is defined as the RHS of (3–12) and $K[h_1] \triangleq \bigcap_{\delta>0} \bigcap_{\mu S_m=0} \overline{co} h_1(e_1, B(e_2, \delta) - S_m)$, where $\bigcap_{\mu S_m=0}$ denotes the intersection of all sets S_m of Lebesgue measure zero, $\overline{co}$ denotes convex closure, and $B(e_2, \delta) = \{\varsigma \in \mathbb{R}^n |\ \|e_2 - \varsigma\| < \delta\}$ [45, 154].

In review of (3–5)-(3–10), the control strategy in (3–11) and (3–12) entails several components including the development of the filtered error systems in (3–6) and (3–7), which are composed of saturated hyperbolic tangent functions designed from the Lyapunov analysis to cancel cross terms. The motivation for the design of (3–8) stems from the need to inject a $-\gamma_1 e_2$ signal into the closed-loop error system and to cancel cross terms in the analysis. Based on the stability analysis methods associated with

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[3] An important feature of the controller in (3–11) is its applicability to the case where constraints exist on the available control. Note that the control law is upper bounded by the adjustable control gain γ_1 as $\|u\| \leq \sqrt{n} \cdot \gamma_1$ where n is the dimension of u.

[4] The initial condition for $v(0, 0)$ is selected such that $u(0) = 0$.

the RISE control strategy [8, 147, 169, 171], an extra derivative is applied to the closed-loop error system. The time derivative of (3–11) will include a $Cosh^{-2}(v)$ term. The design of (3–12) is motivated by the desire to cancel the $Cosh^{-2}(v)$ term, enabling the remaining terms to provide the desired feedback and cancel nonconstructive terms and disturbances as dictated by the subsequent stability analysis.

The closed-loop tracking error system can be developed by taking the time derivative of (3–9), and using the time derivative of (3–11) to yield

$$\dot{r} = \tilde{N} + N_d - \gamma_1 r - \gamma_1 \beta sgn\left(e_2\right) - Tanh\left(e_2\right) - e_2 \tag{3–13}$$

where $\tilde{N}\left(e_1, e_2, r, e_f\right) \in \mathbb{R}^n$ and $N_d\left(x_d, \dot{x}_d, \ddot{x}_d, t\right) \in \mathbb{R}^n$ are defined as

$$\tilde{N} \triangleq \dot{S} + \gamma_1 \alpha_1 Cosh^{-2}\left(e_1\right) e_2 - \gamma_1 \gamma_2 e_2 + Tanh\left(e_2\right) + e_2, \tag{3–14}$$

$$N_d \triangleq \dddot{x}_d - \dot{f}_d - \dot{d}. \tag{3–15}$$

The structure of (3–13) is motivated by the desire to segregate terms that can be upper bounded by state-dependent terms and terms that can be upper bounded by constants. By applying the Mean Value Theorem (MVT), an upper bound can be developed for the expression in (3–14) as [164, App A]

$$\left\|\tilde{N}\right\| \leq \rho\left(\|w\|\right)\|w\| \tag{3–16}$$

where the bounding function $\rho\left(\cdot\right) \in \mathbb{R}$ is a positive, globally invertible, nondecreasing function, and
$w\left(e_1, e_2, r, e_f\right) \in \mathbb{R}^{5n}$ is defined as

$$w \triangleq \left[Tanh^T\left(e_1\right),\ e_2^T,\ r^T,\ Tanh^T\left(e_f\right)\right]^T. \tag{3–17}$$

From Assumptions 3.1 and 3.2, the following inequality can be developed based on the expression in (3–15):

$$\|N_d\| \leq \zeta_{N_{d1}}, \quad \left\|\dot{N}_d\right\| \leq \zeta_{N_{d2}} \tag{3–18}$$

where $\zeta_{N_{d1}}, \zeta_{N_{d2}} \in \mathbb{R}$, are known positive constants.

3.3 Stability Analysis

Theorem 3.1. *Given the dynamics in (3–1), the controller given by (3–11) and (3–12) ensures asymptotic tracking in the sense that*

$$\|e_1\| \to 0 \quad as \quad t \to \infty$$

provided the control gains are selected sufficiently large based on the initial conditions of the states (see the subsequent stability analysis) and the following sufficient conditions

$$\alpha_1 > \frac{1}{2}, \qquad \alpha_2 > 0, \qquad \alpha_3 > \frac{1}{2} + \frac{\gamma_1^2 \zeta^2}{4}, \qquad \gamma_2 > \frac{1}{\zeta^2}, \qquad \beta\gamma_1 > \zeta_{N_{d1}} + \frac{\zeta_{N_{d2}}}{\alpha_3}. \tag{3–19}$$

where $\alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2$ and β were introduced in (3–6)-(3–8) and (3–12), respectively, and $\zeta \in \mathbb{R}$ is a subsequently defined adjustable positive constant.

Proof. Let $z\left(e_1, e_2, r, e_f\right) \in \mathbb{R}^{4n}$ be defined as

$$z \triangleq \left[e_1^T, \;\; e_2^T, \;\; r^T, \;\; Tanh^T\left(e_f\right)\right]^T \tag{3–20}$$

and $y\left(z, P\right) \in \mathbb{R}^{4n+1}$ be defined as

$$y \triangleq \left[\begin{array}{cc} z^T & \sqrt{P} \end{array} \right]^T. \tag{3–21}$$

In (3–21), the auxiliary function $P\left(e_2, t\right) \in \mathbb{R}$ is defined as the generalized Filippov solution to the following differential equation

$$\dot{P} = -r^T\left(N_d - \beta\gamma_1 sgn\left(e_2\right)\right), \quad P\left(e_2\left(t_0\right), t_0\right) = \beta\gamma_1 \sum_{i=1}^{n} \left|e_{2i}\left(t_0\right)\right| - e_2\left(t_0\right)^T N_d\left(t_0\right) \tag{3–22}$$

where the subscript $i = 1, 2, ..., n$ denotes the ith element of the vector. Similar to the development in (3–12), existence of solutions for $P\left(e_2, t\right)$ can be established using Filippov's theory of differential inclusions for $\dot{P} \in K\left[h_2\right]\left(e_2, r, t\right)$, where $h_2\left(e_2, r, t\right) \in \mathbb{R}$ is defined as $h_2 \triangleq -r^T\left(N_d - \beta\gamma_1 sgn\left(e_2\right)\right)$ and $K\left[h_2\right] \triangleq \bigcap_{\delta>0} \bigcap_{\mu S_m=0} \overline{co} h_2\left(B\left(e_2, \delta\right) - S_m, r, t\right)$ as in (3–41). Provided the sufficient condition for β in (3–19) is satisfied, $P(e_2,t)\geq 0$ (See the Appendix A for details).

Let $V_L(y,t):\mathcal{D}\times[0,\infty)\to\mathbb{R}$ be a positive-definite regular function defined as

$$\begin{aligned} V_L \ &\triangleq\ \sum_{i=1}^{n} ln\left(cosh\left(e_{1i}\right)\right)+\sum_{i=1}^{n} ln\left(cosh\left(e_{2i}\right)\right)+\frac{1}{2}e_2^Te_2+\frac{1}{2}r^Tr \\ &\quad +\frac{1}{2}Tanh^T\left(e_f\right)Tanh\left(e_f\right)+P \end{aligned} \tag{3–23}$$

where $e_{1i}(\cdot)$ and $e_{2i}(\cdot)$ denote the ith element of the vector $e_1(x,x_d)$ and $e_2(e_1,\dot{e}_1,e_f)$, respectively. The Lyapunov function candidate in (3–23) satisfies the following inequalities:

$$\phi_1(y)\leq V_L(y,t)\leq\phi_2(y). \tag{3–24}$$

Based on (3–4) and (3–23), the continuous positive definite functions $\phi_1(y),\phi_2(y)\in\mathbb{R}$ in (3–24) are defined as $\phi_1(y)\triangleq\frac{1}{2}tanh^2(\|y\|)$, $\phi_2(y)\triangleq\frac{3}{2}\|y\|^2$.

Under Filippov's framework, the time derivative of (3–23) exists almost everywhere, i.e., for almost all $t\in[t_0,t_f]$, and $\dot{V}(y,t)\overset{a.e.}{\in}\dot{\tilde{V}}(y,t)$ where

$$\dot{\tilde{V}}_L=\bigcap_{\xi\in\partial V_L(y,t)}\xi^TK\left[\dot{e}_1^T\ \ \dot{e}_2^T\ \ \dot{r}^T\ \ Cosh^{-2}\left(e_f\right)\dot{e}_f^T\ \ \frac{1}{2}P^{-\frac{1}{2}}\dot{P}\ \ 1\right]^T,$$

∂V_L is the generalized gradient of $V_L(y,t)$ [166]. Since $V_L(y,t)$ is a Lipschitz continuous regular function,

$$\dot{\tilde{V}}_L\ \subset\ \nabla V_L^TK\left[\dot{e}_1^T\ \ \dot{e}_2^T\ \ \dot{r}^T\ \ Cosh^{-2}\left(e_f\right)\dot{e}_f^T\ \ \frac{1}{2}P^{-\frac{1}{2}}\dot{P}\right]^T \tag{3–25}$$

where $\nabla V_L\triangleq\left[Tanh^T\left(e_1\right),\ \left(Tanh^T\left(e_2\right)+e_2^T\right),\ r^T,\ Tanh^T\left(e_f\right),\ 2P^{\frac{1}{2}}\right]^T$.

Using the calculus for $K\left[\cdot\right]$ from [154], and substituting (3–5)-(3–8), and (3–13) into (3–25), yields

$$\begin{aligned}\dot{\tilde{V}}_L \subset\ & r^T\left(\tilde{N}+N_d-\gamma_1 r-Tanh\left(e_2\right)-e_2-\gamma_1\beta K\left[sgn\left(e_2\right)\right]\right)\\ &+Tanh^T\left(e_1\right)\left(e_2-\alpha_1 Tanh\left(e_1\right)-Tanh\left(e_f\right)\right)+Tanh^T\left(e_2\right)\left(r-\alpha_2 Tanh\left(e_2\right)-\alpha_3 e_2\right)\\ &+e_2^T\left(r-\alpha_2 Tanh\left(e_2\right)-\alpha_3 e_2\right)+Tanh^T\left(e_f\right)\left(-\gamma_1 e_2+Tanh\left(e_1\right)\right)\\ &+Tanh^T\left(e_f\right)\left(-\gamma_2 Tanh\left(e_f\right)\right)+\dot{P} \qquad (3\text{–}26)\end{aligned}$$

where $K\left[sgn(e_2)\right]=SGN\left(e_2\right)$ [154] such that $SGN\left(e_{2_i}\right)=1$ if $e_{2_i}\left(\cdot\right)>0$, $\left[-1,1\right]$ if $e_{2_i}\left(\cdot\right)=0$, and -1 if $e_{2_i}\left(\cdot\right)<0$. Substituting (3–22), canceling common terms and rearranging the resulting expression yields

$$\begin{aligned}\dot{V}_L \overset{a.e.}{=}\ & -\alpha_1 Tanh^T\left(e_1\right)Tanh\left(e_1\right)-\alpha_2 Tanh^T\left(e_2\right)Tanh\left(e_2\right)-\alpha_3 e_2^T e_2\\ &-\gamma_2 Tanh^T\left(e_f\right)Tanh\left(e_f\right)-\gamma_1 r^T r+r^T\tilde{N}+Tanh^T\left(e_1\right)e_2-Tanh^T\left(e_2\right)\alpha_3 e_2\\ &-\gamma_1 Tanh^T\left(e_f\right)e_2-\alpha_2 e_2^T Tanh\left(e_2\right) \qquad (3\text{–}27)\end{aligned}$$

where the set in (3–26) reduces to the scalar equality in (3–27) since the RHS is continuous a.e., i.e, the RHS is continuous except for the Lebesgue measure zero set of times when $r^T\gamma_1\beta K\left[sgn\left(e_2\right)\right]-r^T\gamma_1\beta K\left[sgn\left(e_2\right)\right]\neq 0$.[5] Utilizing the definition of (3–7), (3–16), and (3–18), the expression in (3–27) can be upper bounded as

$$\begin{aligned}\dot{V}_L \overset{a.e.}{\leq}\ & -\alpha_1\left\|Tanh\left(e_1\right)\right\|^2-\left(2\alpha_2+\alpha_3\right)\left\|Tanh\left(e_2\right)\right\|^2-\alpha_3\left\|e_2\right\|^2-\gamma_2\left\|Tanh\left(e_f\right)\right\|^2-\gamma_1\left\|r\right\|^2\\ &+\rho\left\|w\right\|\left\|r\right\|+\left\|Tanh\left(e_1\right)\right\|\left\|e_2\right\|+\gamma_1\left\|Tanh\left(e_f\right)\right\|\left\|e_2\right\|. \qquad (3\text{–}28)\end{aligned}$$

[5] The set of times $\Lambda\triangleq\left\{t\in\left[0,\infty\right):r\left(t\right)^T\gamma_1\beta K\left[sgn\left(e_2\left(t\right)\right)\right]-r\left(t\right)^T\gamma_1\beta K\left[sgn\left(e_2\left(t\right)\right)\right]\neq 0\right\}\subset\left[0,\infty\right)$ is equivalent to the set of times $\left\{t:e_2\left(t\right)=0\ \wedge r\left(t\right)\neq 0\right\}$. From (3–7), this set can also be represented by $\left\{t:e_2\left(t\right)=0\ \wedge\dot{e}_2\left(t\right)\neq 0\right\}$. Provided $e_2\left(t\right)$ is continuously differentiable, it can be shown that the set of time instances $\left\{t:e_2\left(t\right)=0\ \wedge\dot{e}_2\left(t\right)\neq 0\right\}$ is isolated, and thus, measure zero. This implies that the set Λ is measure zero.

Young Inequality can be applied to select terms in (3–28) as

$$\begin{aligned}\|Tanh(e_1)\|\,\|e_2\| &\leq \frac{1}{2}\|Tanh(e_1)\|^2+\frac{1}{2}\|e_2\|^2 \\ \gamma_1\|Tanh(e_f)\|\,\|e_2\| &\leq \frac{1}{\zeta^2}\|Tanh(e_f)\|^2+\frac{\gamma_1^2\zeta^2}{4}\|e_2\|^2 .\end{aligned} \tag{3–29}$$

To facilitate the subsequent stability analysis, let γ_1 be selected as $\gamma_1=\gamma_a+\gamma_b$ where $\gamma_a,\gamma_b\in\mathbb{R}$ are positive gain constants. Utilizing (3–29), completing the squares on $r(\cdot)$ and grouping terms, the expression in (3–28) can be upper bounded by

$$\begin{aligned}\dot{V}_L \overset{a.e.}{\leq}& -\left(\alpha_1-\frac{1}{2}\right)\|Tanh(e_1)\|^2-(2\alpha_2+\alpha_3)\|Tanh(e_2)\|^2-\left(\alpha_3-\frac{1}{2}-\frac{\gamma_1^2\zeta^2}{4}\right)\|e_2\|^2 \\ &-\left(\gamma_2-\frac{1}{\zeta^2}\right)\|Tanh(e_f)\|^2-\gamma_a\|r\|^2+\frac{\rho^2(\|w\|)\|w\|^2}{4\gamma_b}.\end{aligned} \tag{3–30}$$

Provided the sufficient conditions in (3–19) are satisfied, (3–17) and (3–20) can be used to conclude that

$$\dot{V}_L \overset{a.e.}{\leq} -\phi_3(\|z\|)\leq -U(y) \tag{3–31}$$

where $\phi_3(\|z\|)\in\mathbb{R}$ is defined as $\phi_3\triangleq\left(\lambda-\frac{\rho^2(\|w\|)}{4\gamma_b}\right)\tanh^2(\|z\|)$, $\lambda\in\mathbb{R}^+$ is defined as $\lambda=min\left\{\alpha_1-\frac{1}{2},\ 2\alpha_2+\alpha_3,\ \alpha_3-\frac{1}{2}-\frac{\gamma_1^2\zeta^2}{4},\ \gamma_2-\frac{1}{\zeta^2},\ \gamma_a\right\}$, and $U(y)\triangleq c\tanh^2(\|z\|)\,\forall y\subset\mathcal{D}$ is a continuous, positive semi-definite function for some positive constant $c\in\mathbb{R}$, where

$$\mathcal{D} \triangleq \left\{y\in\mathbb{R}^{4n+1}\mid \|y\|\leq\rho^{-1}\left(2\sqrt{\lambda\gamma_b}\right)\right\}. \tag{3–32}$$

The inequalities in (3–24) and (3–31) can be used to show that $V_L(y,t)\in\mathcal{L}_\infty$ in $\mathcal{D}$, hence,

$e_1(\cdot),e_2(\cdot),r(\cdot),Tanh(e_f(\cdot))\in\mathcal{L}_\infty$ in $\mathcal{D}$. From (3–2), $Tanh(e_1),Tanh(e_2)\in\mathcal{L}_\infty$ in $\mathcal{D}$. Thus, from (3–6) and (3–7), $\dot{e}_1(\cdot),\dot{e}_2(\cdot)\in\mathcal{L}_\infty$ in $\mathcal{D}$. From (3–11) and (3–4), $u(\cdot)\in\mathcal{L}_\infty$ in $\mathcal{D}$. From Assumption 3.2 and by utilizing the fact that $e_1(\cdot),\dot{e}_1(\cdot)\in\mathcal{L}_\infty$, $q(t),\dot{q}(t)\in\mathcal{L}_\infty$ in $\mathcal{D}$. From the above statements, (3–13) can be used to show that $\dot{r}(\cdot)\in\mathcal{L}_\infty$ in $\mathcal{D}$. Since f is $\mathcal{C}^2$ and $q(t),\dot{q}(t)\in\mathcal{L}_\infty$, $f(q,\dot{q},t)\in\mathcal{L}_\infty$. Utilizing the derivative of (3–7), Assumption 3.1 and the facts that $e_2(\cdot),\dot{e}_2(\cdot),\dot{r}(\cdot),f(\cdot),u(\cdot)\in\mathcal{L}_\infty$, the product

$Cosh^{-2}(e_f)\,\dot{e}_f \in \mathcal{L}_\infty$. Thus, $\dot{z}(\cdot) \in \mathcal{L}_\infty$ in $\mathcal{D}$ and it can be shown that $z(\cdot)$ is uniformly continuous (UC) in $\mathcal{D}$. Since $z(\cdot)$ is UC, $tanh(\|z\|)$ is UC. The definitions of $U(y)$ and $z(\cdot)$ can be used to prove that $U(y)$ is UC in $\mathcal{D}$. Let $\mathcal{S} \subset \mathcal{D}$ denote a set defined as

$$\mathcal{S} \triangleq \left\{ y \in \mathcal{D} \mid \phi_2(y) < \left(\rho^{-1}\left(2\sqrt{\lambda\gamma_b} \right) \right)^2 \right\}. \tag{3–33}$$

The region of attraction in (3–33) can be made larger by increasing the control gain γ_b. For arbitrarily large initial conditions or arbitrarily large disturbances, the control gains required to satisfy the sufficient gain conditions in (3–19) may demand an input that is not physically deliverable by the system (i.e., the gain γ_1 may be required to be larger than the saturation limit of the actuator). Despite gain dependency on the system's initial conditions, this result does not satisfy the standard semi-global result because under the consideration of input constraints, γ_b cannot be arbitrarily increased and consequently the region of attraction cannot be arbitrarily enlarged to include all initial conditions.[6]

From (3–31), Corollary 2.1 can be invoked to show that $tanh(\|z\|) \rightarrow 0 \;\; as \;\; t \rightarrow \infty\,\forall y(0) \in \mathcal{S}$. Based on the definition of $z(\cdot)$ in (3–20), $\|e_1\| \rightarrow 0 \; as \; t \rightarrow \infty\,\forall y(0) \in \mathcal{S}$. $\square$

3.4 Euler-Lagrange Extension

The results presented in Chapter 3 can be extended to general systems which can be described by EL equations of motion. Specifically, consider a nonlinear system of the form

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + G(q) + F(\dot{q}) + \tau_d(t) = u(t) \tag{3–34}$$

where $M(q) \in \mathbb{R}^{n\times n}$ denotes the generalized, state-dependent inertia, $V_m(q,\dot{q}) \in \mathbb{R}^{n\times n}$ denotes the generalized centrifugal and Coriolis forces, $G(q) \in \mathbb{R}^n$ denotes the

[6] This outcome is not surprising from a physical perspective in the sense that such demands may yield cases where the actuation is insufficient to stabilize the system.

generalized gravity, $F(\dot{q}) \in \mathbb{R}^n$ denotes the generalized friction, $\tau_d \in \mathbb{R}^n$ denotes a general nonlinear disturbance (e.g., unmodeled effects), $q(t), \dot{q}(t), \ddot{q}(t) \in \mathbb{R}^n$ denote the generalized states and $u(t) \in \mathbb{R}^n$ denotes the generalized control force.

The design of the error systems and controller follow similarly to the method presented in Section 3.2. Utilizing standard properties of the inertia and centrifugal/Coriolis matrices, and the assumptions from Section 3.1, the control development can be extended to achieve a similar result as in Section 3.3.

Assumption 3.3. The inertia matrix $M(q)$ is symmetric positive-definite, and satisfies the following inequality $\forall y(t) \in \mathbb{R}^n$:

$$\underline{m}\,\|y\|^2 \leq y^T M y \leq \bar{m}(q)\,\|y\|^2 \tag{3–35}$$

where $\underline{m} \in \mathbb{R}$ is a known positive constant, $\bar{m}(q) \in \mathbb{R}$ is a known positive function, and $\|\cdot\|$ denotes the standard Euclidean norm.

The error systems $e_1(\cdot)$, $e_2(\cdot)$, $r(\cdot)$, $e_f(\cdot)$ are designed as in (3–5)-(3–8), respectively. An open-loop error system similar to (3–9) is developed as

$$Mr = S + R - u(t) - M\gamma_1 e_2 \tag{3–36}$$

where the auxiliary functions $S(e_1, e_2, e_f, t) \in \mathbb{R}^n$ and $R(q_d, \dot{q}_d, \ddot{q}_d, t) \in \mathbb{R}^n$ are defined as

$$\begin{aligned} S \triangleq\ & M\ddot{q}_d + V_m\dot{q} + G + F - S_d + M\alpha_1 Cosh^{-2}(e_1)\left[e_2 - \alpha_1 Tanh(e_1) - Tanh(e_f)\right] \\ & - M\gamma_2 Tanh(e_f) + M\alpha_2 Tanh(e_2) + M\alpha_3 e_2 + MTanh(e_1), \end{aligned} \tag{3–37}$$

$$R \triangleq S_d + \tau_d \tag{3–38}$$

and a desired trajectory dependent auxiliary term, $S_d(q_d, \dot{q}_d, \ddot{q}_d, t) \in \mathbb{R}^n$, defined as

$$S_d \triangleq M_d\ddot{q}_d + V_{md}\dot{q}_d + G_d + F_d, \tag{3–39}$$

is added and subtracted. In (3–39), M_d, V_{md}, G_d, F_d denote $M(q_d) \in \mathbb{R}^{n\times n}$, $V_m(q_d, \dot{q}_d) \in \mathbb{R}^{n\times n}$, $G(q_d) \in \mathbb{R}^n$, $F(\dot{q}_d) \in \mathbb{R}^n$, respectively.

The control is designed similarly to that of (3–11) and (3–12), however several additional terms must be included to handle terms associated with the inertia matrix. Because of this, the EL extension result requires the inertia matrix to be known. Based on the form of (3–36) and through an iterative stability analysis, the continuous controller, $u(t)$, is designed as

$$u \triangleq \gamma_1 Tanh(v) \tag{3–40}$$

where $v(e_1, e_2) \in \mathbb{R}^n$ is the generalized Filippov solution to the following differential equation

$$\begin{aligned}\dot{v} = & Cosh^2(v)\left[M\alpha_2 Tanh(e_2) + M\alpha_3 e_2 + \beta sgn(e_2)\right. \\ & \left.-\alpha_1 Cosh^{-2}(e_1)\, e_2 - \dot{M}e_2 + \gamma_2 M e_2\right],\end{aligned} \tag{3–41}$$

$v(0) = 0$, where $\beta \in \mathbb{R}$ is a positive constant control gain. Because the RHS of $\dot{v}$ contains the $sgn(\cdot)$ function, Filippov theory must again be used to prove existence of solutions. Because the details follow from the previous development, they are omitted in this extension.

The closed-loop tracking error system can be developed by inserting the control into (3–36), taking the time derivative, and by adding and subtracting $Tanh(e_2)$ and e_2 to yield

$$M\dot{r} = -\frac{1}{2}\dot{M}r + \tilde{N} + N_d - M\gamma_1 r - \gamma_1\beta sgn(e_2) - Tanh(e_2) - e_2 \tag{3–42}$$

where $\tilde{N}(e_1, e_2, r, e_f) \in \mathbb{R}^n$ and $N_d\left(q_d, \dot{q}_d, \ddot{q}_d, q_d^{(3)}, t\right) \in \mathbb{R}^n$ are defined as

$$\tilde{N} \triangleq -\frac{1}{2}\dot{M}r + \dot{\tilde{S}} + Tanh(e_2) + e_2 \tag{3–43}$$

and

$$N_d \triangleq \dot{R}. \tag{3–44}$$

In (3–43), $\dot{\tilde{S}}(e_1, e_2, e_f, t) \in \mathbb{R}^n$ is defined as $\dot{\tilde{S}} \triangleq \dot{S} - \alpha_1\gamma_1 M Cosh^{-2}(e_1)\, e_2 + \gamma_1\gamma_2 M e_2$ where the last two terms are from the time derivative of (3–40) and cancel with inverse terms inside of $\dot{S}$ (which arise due to $Tanh\left(e_{f}\right)$ terms inside S) to yield $\dot{\bar{S}}$ free of direct use of the gain parameter γ_{1}. Remaining γ_{1} terms in $\dot{\bar{S}}$ are encapsulated within $Tanh\left(\cdot\right)$ functions and thus can be upper bounded by 1.

Utilizing a similar Lyapunov candidate function $V_{L}\left(y,t\right):\mathcal{D}\times\left[0,\infty\right)\rightarrow\mathbb{R}$ defined as

$$V_{L}\triangleq\sum_{i=1}^{n}ln\left(cosh\left(e_{1i}\right)\right)+\sum_{i=1}^{n}ln\left(cosh\left(e_{2i}\right)\right)+\frac{1}{2}e_{2}^{T}e_{2}+\frac{1}{2}r^{T}Mr \\ +\frac{1}{2}Tanh^{T}\left(e_{f}\right)Tanh\left(e_{f}\right)+P, \tag{3–45}$$

and Corollary 2.1, it can be shown that

$$\dot{V}_{L}\overset{a.e.}{\leq}-\phi_{3}\left(\left\Vert z\right\Vert\right)\leq-U\left(y\right) \tag{3–46}$$

where $\phi_{3}\left(\left\Vert z\right\Vert\right)\in\mathbb{R}$ is defined as $\phi_{3}\triangleq\left(\lambda-\frac{\rho^{2}\left(\left\Vert x\right\Vert\right)}{4\gamma_{b}}\right)tanh^{2}\left(\left\Vert z\right\Vert\right)$, λ is defined similar to in (3–31), and $U\left(y\right)\triangleq c\cdot tanh^{2}\left(\left\Vert z\right\Vert\right)$ for some positive constant c, is a continuous, positive semi-definite function defined on $\mathcal{D}$ (defined in (3–32)). Additionally, γ_{1} is designed such that $\gamma_{1}\triangleq\frac{\gamma_{a}+\gamma_{b}}{\underline{m}}$ where $\gamma_{a},\gamma_{b}\in\mathbb{R}$ are positive gain constants and $x\left(e_{1},e_{2},r,e_{f}\right)\in\mathbb{R}^{5n}$ is defined the same as w in (3–17). From (3–46), $tanh\left(\left\Vert z\right\Vert\right)\rightarrow0\ \ as\ \ t\rightarrow\infty\ \forall y\left(0\right)\in\mathcal{S}$ where $\mathcal{S}$ is defined as

$$\mathcal{S}\triangleq\left\{y\in\mathcal{D}\mid max\left\{\frac{1}{2}\bar{m}\left(q\right),\frac{3}{2}\right\}\left\Vert y\right\Vert^{2}<\frac{1}{2}min\left\{1,\underline{m}\right\}\left(\rho^{-1}\left(2\sqrt{\lambda\gamma_{b}}\right)\right)^{2}\right\}.$$

Based on the definition of $z\left(\cdot\right)$ in (3–20), it can be shown that

$$\left\Vert e_{1}\left(t\right)\right\Vert\rightarrow0\ \ as\ \ t\rightarrow\infty\ \forall y\left(0\right)\in\mathcal{S}.$$

Additional details regarding the EL extension of this chapter can be found in [172].

3.5 Experimental Results

To examine the performance of the saturated RISE approach, the controller in (3–40) and (3–41) was implemented on a planar manipulator testbed.[7] The manipulator can be modeled as an EL system with the following dynamics

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + F(\dot{q}) + \tau_d(t) = \tau(t) \tag{3–47}$$

where $M(q) \in \mathbb{R}^{2\times 2}$, $V_m(q,\dot{q}) \in \mathbb{R}^{2\times 2}$, $F(\dot{q}) \in \mathbb{R}^2$, and $\tau_d(t) \in \mathbb{R}^2$ were defined in (3–34), $q(t), \dot{q}(t), \ddot{q}(t) \in \mathbb{R}^2$ denote the link position, velocity and acceleration and $\tau(t) \in \mathbb{R}^2$ denotes the control torque.

The control objective is to track a desired link trajectory, selected as $q_{d1}(t) = q_{d2}(t) = (45 + 60sin(2t))\left(1 - e^{-0.01t^3}\right)\ deg$. The initial conditions for the manipulator were selected a complete rotation away from the initial conditions of the desired trajectory as $q_1(0) = 360\ deg$ and $q_2(0) = -180\ deg$. The control torque was arbitrarily selected to be artificially limited (well-within the capabilities of the actuator) to $|\tau_1| \leq 60\ N-m$, $|\tau_2| \leq 15\ N-m$, thus, γ_1 was chosen accordingly. Specifically, the feedback gains for the proposed controller were selected as $\gamma_1 = diag(52, 13)$, $\gamma_2 = diag(22, 19)$, $\beta = diag(3.8, 3.8)$, $\alpha_1 = diag(6.2, 6.0)$, $\alpha_2 = diag(8, 11)$, $\alpha_3 = diag(45, 45)$ and $e_f(0,0)$ is selected as $e_f(0,0) = 0$.[8]

[7] The manipulator consists of a two-link direct drive revolute robot consisting of two aluminum links, mounted on 240.0 N-m (base joint) and 20.0 N-m (second joint) switched reluctance motors. The motor resolvers provide rotor position measurements with a resolution of 614,400 pulses/revolution, and a standard backwards difference algorithm is used to numerically determine velocity from the encoder readings. Data acquisition and control implementation were performed in real-time using QNX at a frequency of 1.0 kHz.

[8] It is important to note that for the given Euler-Lagrange system, the implemented controller is $\tau = M(q)u$. Thus, the bound on the implemented control will include the (known) bound on the inertia matrix. For this experiment, the inertia matrix can be bounded by $\|M(q)\| \leq 1.15$.

Table 3-1. Steady-state RMS error and torque for each of the analyzed control designs.

	RMS Error 1	RMS Error 2	RMS Torque 1	RMS Torque 2
PID w/ AW [173]	8.3857	1.4096	14.5630	1.8732
Adaptive FSFB [174]	3.8232	1.6538	14.9367	1.2360
Saturated RISE (Proposed)	0.1607	0.2889	14.3363	1.1883

The performance of the saturated RISE control design was compared against two controllers available in literature: a classical PID controller with conditional integral clamping anti-windup [173] and an adaptive full-state feedback controller with bounded inputs [174]. Each controller was tuned to achieve the best possible performance, given the saturation bounds. Since each controller has a different structure, it is difficult to comment on the comparative nature of the gains which were implemented. Starting with the same large initial condition offset, the tracking errors of each controller are depicted in Figure 3-1. The actual trajectories with respect to the desired trajectories are shown in Figure 3-2. The control torques for each controller are shown in Figure 3-3 and each remain within the prescribed bounds. To quantify the steady-state performance, root mean square (RMS) errors are listed in Table 3-1. The table illustrates that for comparable RMS torque values, the saturated RISE controller exhibits improved steady-state performance when compared to the other control designs.

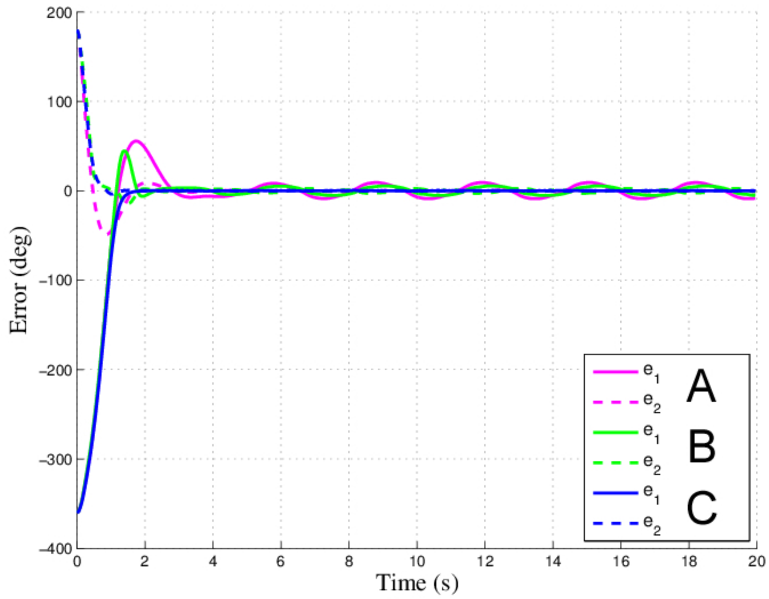

Figure 3-1. Tracking errors vs. time for A) classical PID with integral clamping anti-windup, B) adaptive full-state feedback controller, and C) the proposed saturated RISE controller.

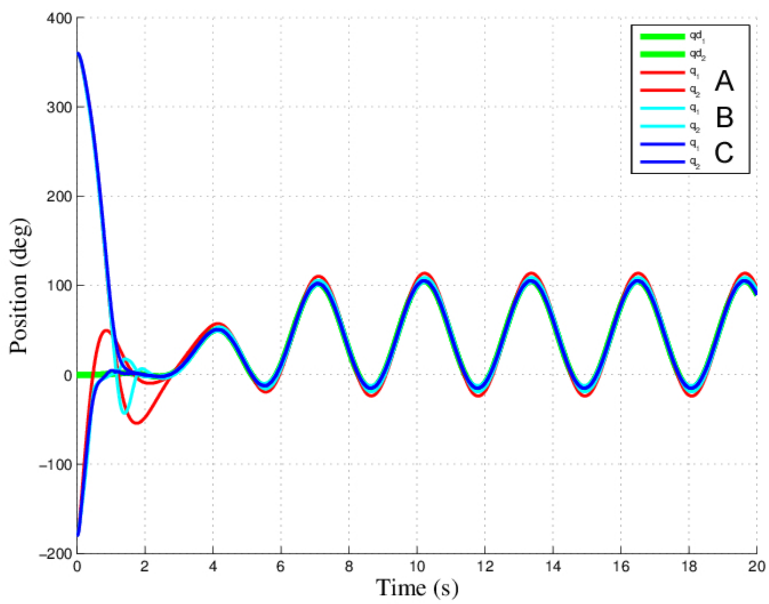

Figure 3-2. Desired and actual trajectories vs. time for A) classical PID with integral clamping anti-windup, B) adaptive full-state feedback controller, and C) the proposed saturated RISE controller.

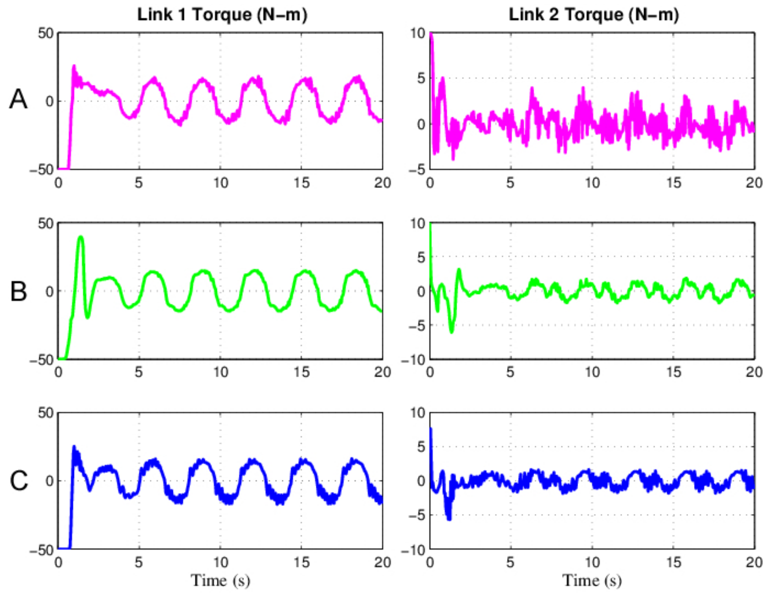

Figure 3-3. Control torque vs. time for A) classical PID with integral clamping anti-windup, B) adaptive full-state feedback controller, and C) the proposed saturated RISE controller.

3.6 Summary

A continuous saturated controller is developed for a class of uncertain nonlinear systems which includes time-varying and non-LP functions and additive bounded disturbances. The bound on the control is known a priori and can be adjusted by changing the feedback gains. The saturated controller is shown to guarantee asymptotic tracking using smooth hyperbolic functions. An extension to EL systems is presented and illustrated via experimental results using a two-link robot manipulator to demonstrate the performance of the control design.

CHAPTER 4
RISE-BASED CONTROL OF AN UNCERTAIN NONLINEAR SYSTEM WITH TIME-VARYING STATE DELAYS

This chapter considers a continuous control design for second-order control affine nonlinear systems with time-varying state delays. Building on previous work in Chapters 2 and 3, a NN is augmented with a RISE control structure to achieve semi-global asymptotic tracking in the presence of unknown, arbitrarily large, time-varying delays, non-LP uncertainty and additive bounded disturbances. By expressing unknown functions in terms of the desired trajectories and through strategic grouping of delay-free and delay-dependent terms, LK functionals are utilized to cancel the delayed terms in the analysis and obtain delay-free NN update laws.

4.1 Dynamic Model

Consider a class of uncertain second-order control affine nonlinear systems with an unknown time-varying state delay described by

$$\ddot{x} = f\left(x, \dot{x}, t\right) + g\left(x\left(t - \tau\right), \dot{x}\left(t - \tau\right), t\right) + d\left(t\right) + u\left(t\right). \tag{4–1}$$

In (4–1), $f\left(x, \dot{x}, t\right) : \mathbb{R}^{2n} \times [0, \infty) \to \mathbb{R}^n$ is an unknown function, $g\left(x\left(t - \tau\right), \dot{x}\left(t - \tau\right), t\right) : \mathbb{R}^{2n} \times [0, \infty) \to \mathbb{R}^n$ is an unknown time-delayed function, $\tau\left(t\right) \in \mathbb{R}$ is an unknown, time-varying, arbitrarily large time-delay, $d\left(t\right) : [0, \infty) \to \mathbb{R}^n$ is a sufficiently smooth bounded disturbance (e.g., unmodeled effects), $u\left(t\right) \in \mathbb{R}^n$ is the control input, and $x\left(t\right), \dot{x}\left(t\right) \in \mathbb{R}^n$ are measurable system states. Throughout the chapter, a time-dependent delayed function is denoted as $\zeta\left(t - \tau\right)$ or ζ_τ, and $\left\|\cdot\right\|$ denotes the Euclidean norm of a vector.

Following the work of Chapter 3, Assumptions 3.1 and 3.2 are utilized for the system in (4–1). Additionally, the following assumptions are applicable:

Assumption 4.1. The unknown time delay is bounded such that $0 \leq \tau\left(t\right) \leq \varphi_1$ and the rate of change of the delay is bounded such that $\left|\dot{\tau}\left(t\right)\right| \leq \varphi_2 < 1$ where $\varphi_1, \varphi_2 \in \mathbb{R}^+$ are known constants.

Assumption 4.2. The functions $f(\cdot), g(\cdot)$ and their first and second derivatives with respect to their arguments are Lipschitz continuous.

4.2 Control Development

The control objective is to design a continuous controller that will ensure $x(t)$ tracks a desired trajectory. As in Chapter 3, a tracking error denoted $e_1(x,t) \in \mathbb{R}^n$ is defined as

$$e_1 \triangleq x_d - x. \tag{4–2}$$

To facilitate the subsequent analysis, two filtered tracking errors, denoted by $e_2(e_1, \dot{e}_1, t), r(e_2, \dot{e}_2, t) \in \mathbb{R}^n$, are defined as

$$e_2 \triangleq \dot{e}_1 + \alpha_1 e_1 \tag{4–3}$$

$$r \triangleq \dot{e}_2 + \alpha_2 e_2 \tag{4–4}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}^+$ are known gain constants. The auxiliary signal $r(e_2, \dot{e}_2, t)$ is introduced to facilitate the stability analysis and is not used in the control design since the expression in (4–4) depends on the unmeasurable state $\ddot{x}(t)$.

An open-loop tracking error can be obtained by substituting for (4–1)-(4–4) to yield

$$\begin{aligned} r = {} & \alpha_1 e_1 + \alpha_2 e_2 + \ddot{x}_d - d \\ & - f(x, \dot{x}, t) - g(x_\tau, \dot{x}_\tau, t) - u. \end{aligned} \tag{4–5}$$

Using a desired compensation adaptation law (DCAL)-based design approach [175], (4–5) can be written as

$$\begin{aligned} r = {} & \alpha_1 e_1 + \alpha_2 e_2 + S_1 + S_d + \ddot{x}_d - d \\ & + g(x_d, \dot{x}_d) - g(x_{d\tau}, \dot{x}_{d\tau}) - u \end{aligned} \tag{4–6}$$

where the auxiliary functions $S_1(x, x_d, \dot{x}, \dot{x}_d, x_\tau, \dot{x}_\tau, x_{d\tau}, \dot{x}_{d\tau}, t)$, $S_d(x_d, \dot{x}_d) \in \mathbb{R}^n$ are defined as

$$S_1 \triangleq -f(x, \dot{x}, t) + f(x_d, \dot{x}_d) - g(x_\tau, \dot{x}_\tau, t) + g(x_{d\tau}, \dot{x}_{d\tau}),$$
$$S_d \triangleq -f(x_d, \dot{x}_d) - g(x_d, \dot{x}_d).$$

The grouping of terms in (4–5) is motivated by the desire to segregate terms that can be upper bounded by state-dependent terms (whether delayed or delay-free) from the terms that can be upper bounded by constants.

The Universal Approximation Theorem can be used to represent the auxiliary function $S_d(\cdot)$ by a three-layer NN as

$$S_d \triangleq W^T \sigma\left(V^T x_{nn}\right) + \varepsilon \tag{4–7}$$

where $V(t) \in \mathbb{R}^{(N_1+1)\times N_2}$ and $W(t) \in \mathbb{R}^{(N_2+1)\times n}$ are bounded constant ideal weights for the first-to-second and second-to-third layers, respectively, N_1 is the number of neurons in the input layer, N_2 is the number of neurons in the hidden layer, n is the number of neurons in the output layer, $\sigma(\cdot) \in \mathbb{R}^{N_2+1}$ is an activation function, $x_{nn}(t) \in \mathbb{R}^{N_1+1}$ denotes the input to the NN defined on a compact set containing the known bounded desired trajectories as $x_{nn} = \left[1, x_d^T, \dot{x}_d^T\right]^T$, and $\varepsilon(x_{nn}) \in \mathbb{R}^n$ denotes the functional reconstruction errors.

Assumption 4.3. The ideal NN weights are assumed to exist and be bounded by known positive constants, i.e. $\|V\|_F^2 \leq V_B$, $\|W\|_F^2 \leq W_B$ where $\|\cdot\|_F$ is the Frobenius norm for a matrix.

Assumption 4.4. The functional reconstruction errors $\varepsilon(\cdot)$ and their first derivative with respect to their arguments are bounded such that $\|\varepsilon(x_{nn})\| \leq \varepsilon_{b1}$, $\|\dot{\varepsilon}(x_{nn}, \dot{x}_{nn})\| \leq \varepsilon_{b2}$, where $\varepsilon_{b_1}, \varepsilon_{b_2} \in \mathbb{R}$ are known positive constants.

Assumption 4.5. The activation function $\sigma(\cdot)$ and its derivative, $\sigma'(\cdot)$ are bounded.

Remark 4.1. Assumptions 4.3-4.4 are standard assumptions in NN control literature (cf. [176]). The ideal weight upper bounds are assumed to be known to facilitate the use of the projection algorithm to ensure the weight estimates are always bounded. There are numerous activations functions which satisfy Assumption 4.5, e.g., sigmoidal or hyperbolic tangent functions.

The controller is designed using a three-layer NN feedforward term augmented by a RISE feedback term as

$$u \triangleq \hat{S}_d + \mu. \tag{4–8}$$

The RISE feedback term $\mu\left(e_2, \upsilon\right) \in \mathbb{R}^n$ is defined as [8, 160]

$$\mu \triangleq \left(k_s + 1\right) e_2 - \left(k_s + 1\right) e_2\left(0\right) + \upsilon \tag{4–9}$$

where $\upsilon\left(e_2\right) \in \mathbb{R}^n$ is the generalized Filippov solution to the following differential equation

$$\dot{\upsilon} \triangleq \left(k_s + 1\right) \alpha_2 e_2 + \beta sgn\left(e_2\right), \tag{4–10}$$

$\beta, k_s \in \mathbb{R}$ are positive, constant control gains, and $sgn\left(\cdot\right)$ is defined $\forall \xi \in \mathbb{R}^n = \begin{bmatrix} \xi_1 & \xi_2 & ... & \xi_n \end{bmatrix}^T$ as $sgn\left(\xi\right) \triangleq \begin{bmatrix} sgn\left(\xi_1\right) & sgn\left(\xi_2\right) & ... & sgn\left(\xi_n\right) \end{bmatrix}^T$.[1]

Using Filippov's theory of differential inclusions [149, 161–163], the existence of solutions can be established for $\dot{\upsilon} \in K\left[h_1\right]\left(e_2\right)$, where $h_1\left(e_2\right) \in \mathbb{R}^n$ is defined as the RHS of $\dot{\upsilon}$ in (4–10) and $K\left[h_1\right] \triangleq \bigcap_{\delta>0} \bigcap_{\mu S_m=0} \overline{co} h_1\left(B\left(e_2, \delta\right) - S_m\right)$, where $\bigcap_{\mu S_m=0}$ denotes the intersection over all sets S_m of Lebesgue measure zero, $\overline{co}$ denotes convex closure, and $B\left(e_2, \delta\right) = \left\{\varsigma \in \mathbb{R}^n | \ \left\|e_2 - \varsigma\right\| < \delta\right\}$ [45, 154].

The NN feedforward term $\hat{S}_d\left(t\right) \in \mathbb{R}^n$ in (4–8) is designed as

$$\hat{S}_d \triangleq \hat{W}^T \sigma\left(\hat{V}^T x_{nn}\right) \tag{4–11}$$

[1] The initial condition for $\upsilon\left(0\right)$ is selected such that $u\left(0\right) = 0$.

where $\hat{V}(t) \in \mathbb{R}^{(N_1+1)\times N_2}$ and $\hat{W}(t) \in \mathbb{R}^{(N_2+1)\times n}$ are estimates of the ideal weights. Based on the subsequent stability analysis, the DCAL-based weight update laws for the NN in (4–11) are generated online as

$$\dot{\hat{W}} \triangleq proj\left(\Gamma_1 \hat{\sigma}' \hat{V}^T \dot{x}_{nn} e_2^T\right) \tag{4–12}$$

$$\dot{\hat{V}} \triangleq proj\left(\Gamma_2 \dot{x}_{nn} \left(\hat{\sigma}'^T \hat{W} e_2\right)^T\right), \tag{4–13}$$

where $\Gamma_1 \in \mathbb{R}^{(N_2+1)\times(N_2+1)}$ and $\Gamma_2 \in \mathbb{R}^{(N_1+1)\times(N_1+1)}$ are positive-definite, constant symmetric control gain matrices, and $\hat{\sigma}'(\cdot) \in \mathbb{R}^{N_2+1}$ denotes the partial derivative of $\hat{\sigma} \triangleq \sigma\left(\hat{V}^T x_{nn}\right)$.

The closed-loop dynamics are developed by substituting (4–8)-(4–11) into (4–6), taking the time derivative, and adding and subtracting $W^T \hat{\sigma}' \hat{V}^T \dot{x}_{nn} + \hat{W}^T \hat{\sigma}' \tilde{V}^T \dot{x}_{nn}$ to yield

$$\begin{aligned}
\dot{r}=&\alpha_1 \dot{e}_1 + \alpha_2 \dot{e}_2 + \dot{S}_1 + \dddot{x}_d - \dot{d} \\
&-\dot{g}\left(x_{d\tau}, \dot{x}_{d\tau}, \ddot{x}_{d\tau}\right) + \dot{g}\left(x_d, \dot{x}_d, \ddot{x}_d\right) \\
&-\left(k_s+1\right) r - \beta sgn\left(e_2\right) + \hat{W}^T \hat{\sigma}' \tilde{V}^T \dot{x}_{nn} \\
&+\tilde{W}^T \hat{\sigma}' \hat{V}^T \dot{x}_{nn} + W^T \sigma' V^T \dot{x}_{nn} - W^T \hat{\sigma}' \hat{V}^T \dot{x}_{nn} \\
&-\hat{W}^T \hat{\sigma}' \tilde{V}^T \dot{x}_{nn} - \dot{\hat{W}}^T \hat{\sigma} - \hat{W}^T \hat{\sigma}' \dot{\hat{V}}^T x_{nn} + \dot{\varepsilon}
\end{aligned} \tag{4–14}$$

where estimate mismatches for the ideal weights, denoted $\tilde{V}(t) \in \mathbb{R}^{(N_1+1)\times N_2}$ and $\tilde{W}(t) \in \mathbb{R}^{(N_2+1)\times n}$, are defined as $\tilde{V}(t) = V(t) - \hat{V}(t)$ and $\tilde{W}(t) = W(t) - \hat{W}(t)$. Using the NN weight update laws from (4–12) and (4–13), the expression in (4–14) can be rewritten as

$$\dot{r} = \tilde{N} + N + e_2 - \left(k_s+1\right) r - \beta sgn\left(e_2\right) \tag{4–15}$$

where $\tilde{N}\left(\hat{W},\hat{V},e_1,e_2,\dot{e}_1,\dot{e}_2,t\right)\in\mathbb{R}^n$ and $N\left(\hat{W},\hat{V},t\right)\in\mathbb{R}^n$ are defined as

$$\begin{aligned}\tilde{N}\triangleq&\alpha_1\dot{e}_1+\alpha_2\dot{e}_2+\dot{S}_1-e_2-proj\left(\Gamma_1\hat{\sigma}'\hat{V}^T\dot{x}_{nn}e_2^T\right)^T\hat{\sigma}\\&-\hat{W}^T\hat{\sigma}'proj\left(\Gamma_2\dot{x}_{nn}\left(\hat{\sigma}'^T\hat{W}e_2\right)^T\right),\end{aligned}\tag{4–16}$$

$$N\triangleq N_D+N_B.\tag{4–17}$$

In (4–17), $N_D\left(x_d,\dot{x}_d,\ddot{x}_d,\dddot{x}_d,t\right)\in\mathbb{R}^n$ is defined as

$$\begin{aligned}N_D\triangleq&\;W^T\sigma'V^T\dot{x}_{nn}+\dot{\varepsilon}+\dddot{x}_d-\dot{d}\\&-\dot{g}\left(x_{d\tau},\dot{x}_{d\tau},\ddot{x}_{d\tau}\right)+\dot{g}\left(x_d,\dot{x}_d,\ddot{x}_d\right)\end{aligned}\tag{4–18}$$

and $N_B\left(\hat{W},\hat{V},x_d,\dot{x}_d,\ddot{x}_d,t\right)\in\mathbb{R}^n$ is separated such that

$$N_B\triangleq N_{B_1}+N_{B_2}\tag{4–19}$$

where $N_{B_1}\left(\hat{W},\hat{V},x_d,\dot{x}_d,\ddot{x}_d,t\right),\ N_{B_2}\left(\hat{W},\hat{V},x_d,\dot{x}_d,\ddot{x}_d,t\right)\in\mathbb{R}^n$ are defined as

$$N_{B_1}\triangleq-W^T\hat{\sigma}'\hat{V}^T\dot{x}_{nn}-\hat{W}^T\hat{\sigma}'\tilde{V}^T\dot{x}_{nn},$$

$$N_{B_2}\triangleq\hat{W}^T\hat{\sigma}'\tilde{V}^T\dot{x}_{nn}+\tilde{W}^T\hat{\sigma}'\hat{V}^T\dot{x}_{nn}.$$

Separating the terms in (4–17) is motivated by the fact that the different components have different bounds [147].

Using Assumptions 3.1 and 4.1-4.5, $N_D(\cdot)$ from (4–18) and $N_B(\cdot)$ from (4–19) and their time derivatives can be upper bounded as

$$\|N_D\|\le\zeta_1,\quad\|N_B\|\le\zeta_2,\quad\left\|\dot{N}_D\right\|\le\zeta_3,\quad\left\|\dot{N}_B\right\|\le\zeta_4+\zeta_5\|e_2\|,$$

where $\zeta_i\in\mathbb{R}^+,\ \forall i=1,...,5$ are known constants. Additionally, $\tilde{N}(\cdot)$ from (4–16) can be upper bounded as

$$\left\|\tilde{N}\right\|\le\rho_1(\|z\|)\|z\|+\rho_2(\|z_\tau\|)\|z_\tau\|\tag{4–20}$$

where $z(e_1, e_2, r) \in \mathbb{R}^{3n}$ denotes the vector $z = \begin{bmatrix} e_1^T & e_2^T & r^T \end{bmatrix}^T$ and $\rho_1(\cdot), \rho_2(\cdot) : \mathbb{R} \to \mathbb{R}$ are positive, globally invertible, nondecreasing functions. The upper bound for the auxiliary function $\tilde{N}(\cdot)$ is segregated into delay-free and delay-dependent bounding functions to eliminate the delayed terms with the use of an LK functional in the stability analysis. Specifically, let $R_{LK}(z, t) \in \mathbb{R}$ denote an LK functional defined as

$$R_{LK} \triangleq \frac{\gamma}{2k_s} \int_{t-\tau(t)}^{t} \rho_2^2 (\|z(\sigma)\|) \|z(\sigma)\|^2 d\sigma \tag{4–21}$$

where $\gamma \in \mathbb{R}^+$ is an adjustable constant, and k_s and $\rho_2(\cdot)$ were introduced in (4–9) and (4–20), respectively.

4.3 Stability Analysis

Theorem 4.1. *The controller proposed in (4–8) and the weight update laws designed in (4–12)-(4–13) ensure that the states and controller are bounded and the tracking errors are regulated in the sense that*

$$\|e_1\| \to 0 \quad as \quad t \to \infty$$

provided the control gain k_s introduced in (4–9) is selected sufficiently large based on the initial conditions of the states, and the remaining control gains are selected based on the following sufficient conditions

$$\alpha_1 > \frac{1}{2}, \qquad \alpha_2 > \beta_2 + \frac{1}{2}, \qquad \beta_2 > \zeta_5,$$
$$\beta > \zeta_1 + \zeta_2 + \frac{1}{\alpha_2}\zeta_3 + \frac{1}{\alpha_2}\zeta_4, \qquad (1 - \varphi_2)\gamma > 1 \tag{4–22}$$

where $\alpha_1, \alpha_2, \beta, \gamma$ were introduced in (4–2), (4–3), (4–10) and (4–21), φ_2 was introduced in Assumption 4.1 and β_2 is a subsequently defined gain constant.

Proof. Let $\mathscr{D} \subset \mathbb{R}^{3n+3}$ be a domain containing $y(e_1, e_2, r, P, Q, R_{LK}) \in \mathbb{R}^{3n+3}$, defined as

$$y \triangleq \begin{bmatrix} z & \sqrt{P} & \sqrt{Q} & \sqrt{R_{LK}} \end{bmatrix}. \tag{4–23}$$

Similar to (3–22), the auxiliary function $P(e_2,t) \in \mathbb{R}$ is defined as the generalized Filippov solution to the following differential equation

$$\dot{P} \triangleq -r^T \left(N_{B_1} + N_D - \beta sgn\left(e_2\right)\right) - \dot{e}_2^T N_{B_2} + \beta_2 \left\|e_2\right\|^2,$$
$$P\left(e_2\left(t_0\right), t_0\right) \triangleq \beta \sum_{i=1}^{n} \left|e_{2i}\left(t_0\right)\right| - e_2\left(t_0\right)^T N_D\left(t_0\right) \tag{4–24}$$

where the subscript $i = 1, 2, ..., n$ denotes the ith element of the vector. Similar to the development in (4–10), existence of solutions for $P(e_2,t)$ can be established using Filippov's theory of differential inclusions for $\dot{P} \in K[h_2](r, \dot{e}_2, e_2, t)$, where $h_2(r, \dot{e}_2, e_2, t) \in \mathbb{R}$ is defined as the RHS of $\dot{P}$. Provided the sufficient conditions in (4–22) are satisfied, $P(e_2,t) \geq 0$ (See [147] for proof). Additionally, the auxiliary function $Q\left(\tilde{W}, \tilde{V}, t\right) \in \mathbb{R}$ in (4–23) is defined as

$$Q \triangleq \frac{\alpha_2}{2} tr\left(\tilde{W}^T \Gamma_1^{-1} \tilde{W}\right) + \frac{\alpha_2}{2} tr\left(\tilde{V}^T \Gamma_2^{-1} \tilde{V}\right) \tag{4–25}$$

where $Q \geq 0$ since Γ_1 and Γ_2 are constant, symmetric, and positive definite matrices and $\alpha_2 \in \mathbb{R}^+$.

Let $V(y,t) : \mathscr{D} \times [0,\infty) \to \mathbb{R}$ be a continuously differentiable in y, locally Lipschitz in t, regular function defined as

$$V \triangleq \frac{1}{2} e_1^T e_1 + \frac{1}{2} e_2^T e_2 + \frac{1}{2} r^T r + P + Q + R_{LK} \tag{4–26}$$

which satisfies the following inequalities

$$\phi_1(y) \leq V(t) \leq \phi_2(y) \tag{4–27}$$

where the continuous positive-definite functions $\phi_1(y), \phi_2(y) \in \mathbb{R}$ are defined as $\phi_1(y) \triangleq \lambda_1 \|y\|^2$, $\phi_2(y) \triangleq \lambda_2 \|y\|^2$ and $\lambda_1, \lambda_2 \in \mathbb{R}^+$ are known constants. Under Filippov's framework, the time derivative of (4–26) exists almost everywhere, i.e., for almost all

$t \in [t_0, t_f]$, and $\dot{V}(y,t) \overset{a.e.}{\in} \dot{\tilde{V}}(y,t)$ where

$$\dot{\tilde{V}} = \bigcap_{\xi \in \partial V(y,t)} \xi^T K[\varrho]$$

where $\varrho \in \mathbb{R}^{3n+4}$ is defined as $\varrho \triangleq \begin{bmatrix} \dot{e}_1^T & \dot{e}_2^T & \dot{r}^T & \frac{1}{2}P^{-\frac{1}{2}}\dot{P} & \frac{1}{2}Q^{-\frac{1}{2}}\dot{Q} & \frac{1}{2}R_{LK}^{-\frac{1}{2}}\dot{R}_{LK} & 1 \end{bmatrix}^T$, and ∂V is the generalized gradient of $V(y,t)$ [166]. Since $V(y,t)$ is $\mathcal{C}^\infty$,

$$\dot{\tilde{V}} \subset \nabla V K[\cdot]^T \tag{4–28}$$

where

$$\nabla V \triangleq \begin{bmatrix} e_1^T & e_2^T & r^T & 2P^{\frac{1}{2}} & 2Q^{\frac{1}{2}} & 2R_{LK}^{\frac{1}{2}} \end{bmatrix}.$$

Using the calculus for $K[\cdot]$ from [154], and substituting (4–2)-(4–4), and (4–15), (4–24), the time derivatives of (4–21), and (4–25) into (4–28), yields

$$\begin{aligned} \dot{\tilde{V}} \subset\ & e_1^T(e_2 - \alpha_1 e_1) + e_2^T(r - \alpha_2 e_2) + r^T\left(\tilde{N} + N_D + N_{B_1} + N_{B_2} + e_2 - (k_s+1)r\right) \\ & + r^T(-\beta K[sgn(e_2)]) + \beta_2\|e_2\|^2 - r^T(N_{B_1} + N_D - \beta K[sgn(e_2)]) - \dot{e}_2^T N_{B_2} \\ & + \frac{\gamma}{2k_s}\rho_2^2(\|z\|)\|z\|^2 - \frac{\gamma(1-\dot{\tau})}{2k_s}\rho_2^2(\|z_\tau\|)\|z_\tau\|^2 + tr\left(\alpha_2\tilde{W}^T\Gamma_1^{-1}\dot{\hat{W}}\right) + tr\left(\alpha_2\tilde{V}^T\Gamma_2^{-1}\dot{\hat{V}}\right) \end{aligned} \tag{4–29}$$

where $K[sgn(e_2)] = SGN(e_2)$ [154] such that $SGN(e_{2_i}) = 1$ if $e_{2_i}(\cdot) > 0$, $[-1,1]$ if $e_{2_i}(\cdot) = 0$, and -1 if $e_{2_i}(\cdot) < 0$. Canceling terms and utilizing the bounds from (4–20) and Assumption 4.1, (4–29) can upper bounded as

$$\begin{aligned} \dot{V} \overset{a.e.}{\leq}\ & \|e_1\|\|e_2\| - \alpha_1\|e_1\|^2 - \alpha_2\|e_2\|^2 + \|r\|\rho_1(\|z\|)\|z\| + \|r\|\rho_2(\|z_\tau\|)\|z_\tau\| \\ & - (k_s+1)\|r\|^2 + \beta_2\|e_2\|^2 + \frac{\gamma}{2k_s}\rho_2^2(\|z\|)\|z\|^2 - \frac{\gamma(1-\varphi_2)}{2k_s}\rho_2^2(\|z_\tau\|)\|z_\tau\|^2 \end{aligned} \tag{4–30}$$

where the set in (4–29) reduces to the scalar inequality in (4–30) since the RHS is continuous a.e., i.e, the RHS is continuous except for the Lebesgue negligible set of times when $r^T\beta K[sgn(e_2)] - r^T\beta K[sgn(e_2)] \neq 0$. Young's Inequality can be used to show that $\|e_1\|\|e_2\| \leq \frac{1}{2}\|e_1\|^2 + \frac{1}{2}\|e_2\|^2$ and $\|r\|\rho_2(\|z_\tau\|)\|z_\tau\| \leq \frac{k_s}{2}\|r\|^2 + \frac{1}{2k_s}\rho_2^2(\|z_\tau\|)\|z_\tau\|^2$, which allows for the following upper bound for (4–30)

$$\dot{V} \overset{a.e.}{\leq} \frac{1}{2}\|e_1\|^2 + \frac{1}{2}\|e_2\|^2 - \alpha_1\|e_1\|^2 - \alpha_2\|e_2\|^2 - \frac{k_s}{2}\|r\|^2 - \|r\|^2 + \beta_2\|e_2\|^2$$
$$+ \|r\|\rho_1(\|z\|)\|z\| + \frac{1}{2k_s}\rho_2^2(\|z_\tau\|)\|z_\tau\|^2 + \frac{\gamma}{2k_s}\rho_2^2(\|z\|)\|z\|^2$$
$$-\frac{\gamma(1-\varphi_2)}{2k_s}\rho_2^2(\|z_\tau\|)\|z_\tau\|. \quad (4–31)$$

If $(1-\varphi_2)\gamma > 1$, and by completing the squares for $r(e_2, \dot{e}_2, t)$, (4–31) becomes

$$\dot{V} \overset{a.e.}{\leq} -\left(\alpha_1 - \frac{1}{2}\right)\|e_1\|^2 - \left(\alpha_2 - \beta_2 - \frac{1}{2}\right)\|e_2\|^2 - \|r\|^2$$
$$+\frac{1}{2k_s}\rho_1^2(\|z\|)\|z\|^2 + \frac{\gamma}{2k_s}\rho_2^2(\|z\|)\|z\|^2. \quad (4–32)$$

Regrouping similar terms, the expression can be upper bounded by

$$\dot{V} \overset{a.e.}{\leq} -\left(\lambda_3 - \frac{\rho^2(\|z\|)}{2k_s}\right)\|z\|^2 \quad (4–33)$$

where $\rho^2(\|z\|) \triangleq \rho_1^2(\|z\|) + \gamma\rho_2^2(\|z\|)$ and $\lambda_3 \triangleq min\left\{\alpha_1 - \frac{1}{2},\ \alpha_2 - \beta_2 - \frac{1}{2},\ 1\right\}$. The bounding function $\rho(\|z\|) : \mathbb{R} \to \mathbb{R}$ is a positive-definite, globally invertible, nondecreasing function. The expression in (4–33) can be further upper bounded by a continuous, positive semi-definite function

$$\dot{V} \overset{a.e.}{\leq} -\phi_3(y) = -c\|z\|^2 \qquad \forall y \in \mathscr{D} \quad (4–34)$$

for some positive constant $c \in \mathbb{R}^+$ and domain $\mathscr{D} = \left\{y \in \mathbb{R}^{3n+3} \mid \|y\| < \rho^{-1}\left(\sqrt{2\lambda_3 k_s}\right)\right\}$. Larger values of k_s will expand the size of the domain $\mathscr{D}$. The inequalities in (4–27) and (4–34) can be used to show that $V \in \mathcal{L}_\infty$ in $\mathscr{D}$. Thus, $e_1(\cdot), e_2(\cdot), r(\cdot) \in \mathcal{L}_\infty$ in $\mathscr{D}$. The closed-loop error system can be used to conclude that the remaining signals are bounded in $\mathscr{D}$, and the definitions for $\phi_1(\cdot)$ and $z(\cdot)$ can be used to show that $\phi_1(\cdot)$ is uniformly continuous in $\mathscr{D}$. Let $S_{\mathscr{D}} \subset \mathscr{D}$ denote a set defined as

$$\mathcal{S}_{\mathscr{D}} \triangleq \left\{y \in \mathscr{D} \mid \phi_2 < \lambda_1\left(\rho^{-1}\left(\sqrt{2\lambda_3 k_s}\right)\right)^2\right\}. \quad (4–35)$$

The region of attraction in (4–35) can be made arbitrarily large to include any initial conditions by increasing the control gain k_s. From (4–34), 2.1 can be invoked to show that $c\left\|z\right\|^2 \rightarrow 0 \;\; as \;\; t \rightarrow \infty \, \forall y\left(0\right) \in \mathcal{S}_{\mathscr{D}}$. Based on the definition of $z\left(\cdot\right)$ in (4–20), $\left\|e_1\right\| \rightarrow 0 \;\; as \;\; t \rightarrow \infty \, \forall y\left(0\right) \in \mathcal{S}_{\mathscr{D}}$. $\square$

4.4 Summary

A continuous, neural network augmented, RISE controller is utilized for uncertain nonlinear systems which include unknown, arbitrarily large, time-varying state delays and additive bounded disturbances. The controller assumes the time-delay is bounded and slowly varying. Time-varying LK functionals are utilized to prove semi-global asymptotic tracking of the closed-loop system in the presence of time-varying and non-LP functions and sufficiently smooth unmodeled dynamic effects.

CHAPTER 5
LYAPUNOV-BASED CONTROL OF AN UNCERTAIN NONLINEAR SYSTEM WITH TIME-VARYING INPUT DELAY

A predictor-based controller is developed for uncertain second-order nonlinear systems subject to time-varying input delay and additive bounded disturbances. A Lyapunov-based stability analysis utilizing LK functionals is provided to prove semi-global uniformly ultimately bounded tracking assuming the input delay is known, sufficiently small, and slowly varying. Simulation results demonstrate the robustness of the control design with respect to uncertainties in the magnitude and time-variation of the delay.

5.1 Dynamic Model

Consider a class of control affine, second-order[1] nonlinear systems described by

$$\ddot{x} = f\left(x, \dot{x}, t\right) + u\left(t - \tau\left(t\right)\right) + d\left(x, t\right) \tag{5–1}$$

where $x\left(t\right), \dot{x}\left(t\right) \in \mathbb{R}^n$ are the generalized system states, $u\left(t-\tau\right) \in \mathbb{R}^n$ represents the generalized delayed control input vector, where $\tau\left(t\right) \in \mathbb{R}^+$ is a known non-negative time-varying delay, $f\left(x, \dot{x}, t\right) : \mathbb{R}^{2n} \times [0, \infty) \to \mathbb{R}^n$ is an unknown nonlinear $\mathcal{C}^2$ function, uniformly bounded in t, and $d\left(x, t\right) \in \mathbb{R}^n$ denotes a sufficiently smooth exogenous disturbance (e.g., unmodeled effects).

The subsequent development is based on the assumption that $x\left(t\right)$ and $\dot{x}\left(t\right)$ are measurable outputs, and the time delay and control input vector and its past values (i.e., $u\left(t-\theta\right) \forall\theta \in [0 \;\; \tau\left(t\right)]$) are measurable. Throughout the chapter, a time-dependent-delayed function is denoted as $\zeta\left(t - \tau\left(t\right)\right)$ or ζ_τ.

Additionally, the following assumptions and properties will be exploited. Note that these assumptions have been adjusted slightly from the assumptions in the previous

[1] The result in this chapter can be extended to n^{th}-order nonlinear systems following a similar development to those presented in [122, 177].

chapters. For example, this chapter considers a more general disturbance (with possible state-dependencies) and a more restrictive bound on the delay.

Assumption 5.1. The nonlinear disturbance term and its first time derivative (i.e., $d(x,t), \dot{d}(x,\dot{x},t)$) exist and are bounded such that $\|d(x,t)\| \leq \bar{d}_1 \|x\| + \bar{d}_2$ and $\left\|\dot{d}(x,\dot{x},t)\right\| \in \mathcal{L}_\infty$, where $\bar{d}_1, \bar{d}_2 \in \mathbb{R}$ are nonnegative constants.

Assumption 5.2. The time delay is bounded such that $0 \leq \tau(t) \leq \varphi_1$, where $\varphi_1 \in \mathbb{R}^+$ is a sufficiently small (see subsequent stability analysis) known constant and the rate of change of the delay is bounded such that $|\dot{\tau}(t)| < \varepsilon < \frac{1}{2}$, where $\varepsilon \in \mathbb{R}^+$ is a known constant.

Assumption 5.3. The desired trajectory $x_d(t) \in \mathbb{R}^n$ is designed to be sufficiently smooth such that $x_d(t), \dot{x}_d(t), \ddot{x}_d(t) \in \mathcal{L}_\infty$.

Remark 5.1. In Assumption 5.2, the slowly time-varying constraint (i.e., $|\dot{\tau}(t)| < \varepsilon < \frac{1}{2}$) is common (though slightly more restrictive) to results which utilize classical LK functionals to compensate for time-varying time-delays [18]. The input delay is required to be known since past values of the control are used in the control structure.

5.2 Control Development

The objective is to design a continuous controller that will ensure the generalized state $x(t)$ of the input-delayed system in (5–1) tracks $x_d(t)$. To quantify the control objective, a tracking error denoted by $e(x,t) \in \mathbb{R}^n$, is defined as

$$e \triangleq x_d - x. \tag{5–2}$$

To facilitate the subsequent analysis, a measurable auxiliary tracking error, denoted by $r(e,\dot{e},e_z) \in \mathbb{R}^n$, is defined as

$$r \triangleq \dot{e} + \alpha e - e_z \tag{5–3}$$

where $\alpha \in \mathbb{R}^{+}$ is a known gain constant, and $e_z(t) \in \mathbb{R}^n$ is an auxiliary signal containing the time-delays in the system, defined as

$$e_z \triangleq \int_{t-\tau(t)}^{t} u(\theta)\, \mathrm{d}\theta. \tag{5–4}$$

The $e_z(t)$ component of (5–3) is motivated by the desire to inject a predictor-like term in the error system development. By injecting the integral of the control effort over the delay interval, the open-loop error system for the auxiliary error can be expressed in terms of a delay-free control input.

The open-loop error system can be obtained taking the time derivative of (5–3) and utilizing the expressions in (5–1), (5–2) and (5–4) to yield

$$\dot{r} \;=\; \ddot{x}_d - f(x, \dot{x}, t) - d(x, t) - u - u_\tau \dot{\tau} + \alpha \dot{e}. \tag{5–5}$$

From (5–5) and the subsequent stability analysis, the control input $u(r)$ is designed as [31]

$$u = k_b r \tag{5–6}$$

where $k_b \in \mathbb{R}^{+}$ is a known constant control gain. The closed-loop error system is obtained utilizing (5–3), (5–5) and (5–6) to yield

$$\dot{r} = N_d + \chi - k_b r - k_b r_\tau \dot{\tau} - e - d(x, t) \tag{5–7}$$

where the auxiliary terms $\chi(e, r, e_z)$, $N_d(x_d, \dot{x}_d, \ddot{x}_d, t) \in \mathbb{R}^n$ are defined as

$$\chi \;\triangleq\; -f(x, \dot{x}, t) + f(x_d, \dot{x}_d, t) + \alpha r - \alpha^2 e + \alpha e_z + e, \tag{5–8}$$

$$N_d \triangleq \ddot{x}_d - f(x_d, \dot{x}_d, t). \tag{5–9}$$

Assumptions 5.1 and 5.3 are used to develop the following inequality based on the expression in (5–9)

$$\|N_d\| \leq \bar{n}_d \tag{5–10}$$

where $\bar{n}_d \in \mathbb{R}^+$ is a known constant. The structure of (5–7) is motivated by the desire to segregate terms that can be upper bounded by state-dependent terms and terms that can be upper bounded by constants. Using the MVT, the expression in (5–8) can be upper bounded as [164, App A] (also similar to the presentation in Appendix B)

$$\|\chi\| \leq \rho\left(\|z\|\right)\|z\| \tag{5–11}$$

where $\rho\left(\|z\|\right)$ is a positive, globally invertible, nondecreasing function, and $z\left(e, r, e_z\right) \in \mathbb{R}^{3n}$ is defined as

$$z \triangleq \left[\begin{array}{ccc} e^T & r^T & e_z^T \end{array}\right]^T. \tag{5–12}$$

To facilitate the subsequent stability analysis, let $y\left(e, r, P, Q\right) \in \mathbb{R}^{2n+2}$ be defined as

$$y \triangleq \left[\begin{array}{cccc} e^T & r^T & \sqrt{P} & \sqrt{Q} \end{array}\right]^T \tag{5–13}$$

where $P\left(t, \tau\right), Q\left(r, t, \tau\right) \in \mathbb{R}$ denote LK functionals defined as

$$P \triangleq \omega \int_{t-\tau(t)}^{t} \left(\int_{s}^{t} \left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta \right) \mathrm{d}s, \tag{5–14}$$

$$Q \triangleq \frac{k_b}{2} \int_{t-\tau(t)}^{t} \left\|r\left(\theta\right)\right\|^2 \mathrm{d}\theta \tag{5–15}$$

and $\omega \in \mathbb{R}^+$ is a known, adjustable constant. Additionally, let $k_b = k_{b1} + k_{b2} + k_{b3}$, where $k_{bi} \in \mathbb{R}^+,\ i = 1, 2, 3$ are adjustable constants.

5.3 Stability Analysis

Theorem 5.1. *Given the dynamics in (5–1), the controller in (5–6) ensures semi-global uniformly ultimately bounded tracking in the sense that*

$$\left\|e\left(t\right)\right\| \leq \epsilon_0 exp\left(-\epsilon_1 t\right) + \epsilon_2 \tag{5–16}$$

where $\epsilon_0, \epsilon_1, \epsilon_2 \in \mathbb{R}^+$ denote constants, provided the time-delay is sufficiently small, the rate of change of the time-delay is sufficiently slow (see Assumption 5.2) and the *following sufficient gain conditions are satisfied*

$$\alpha > \frac{\zeta^2}{4}, \quad k_{b1} > \frac{1}{3}\left(k_{b2} + k_{b3} + 4k_b^2\omega\tau\right), \quad \omega\zeta^2 > \frac{2\tau}{1-\dot{\tau}},$$

$$4\beta k_{b3} > \rho_2^2\left(\sqrt{\frac{2\left\|y\left(0\right)\right\|^2 + \frac{\bar{n}^2}{4k_{b2}\delta}}{\min\left\{1, \frac{1}{2k_b\tau}\right\}}}\right) \tag{5–17}$$

where $\beta \triangleq min\left\{\ \alpha - \frac{\zeta^2}{4},\ \ k_{b1} - \frac{k_b}{2}\left(1-\dot{\tau}\right) - k_b^2\omega\tau,\ \ \frac{1}{\tau}\left(\frac{\omega}{2}\left(1-\dot{\tau}\right) - \frac{\tau}{\zeta^2}\right)\ \right\}$ *and* $\zeta, \delta \in \mathbb{R}^+$ *are subsequently defined constants.*

Proof. Let $V_L\left(y,t\right) : \mathcal{D}\times\left[0,\ \infty\right) \to \mathbb{R}$ be a continuously differentiable, positive-definite functional on a domain $\mathcal{D} \subseteq \mathbb{R}^{2n+2}$, defined as

$$V_L \triangleq \frac{1}{2}e^Te + \frac{1}{2}r^Tr + P + Q \tag{5–18}$$

which can be bounded as

$$\frac{1}{2}\left\|y\right\|^2 \leq V_L \leq \left\|y\right\|^2. \tag{5–19}$$

Utilizing (5–3) and (5–7), applying the Leibniz Rule to determine the time derivative of (5–14) and (5–15), and by canceling similar terms, the time derivative of (5–18) can be expressed as

$$\begin{aligned}\dot{V}_L \;=\; & -\alpha e^Te + e^Te_z + r^TN_d + r^T\chi - r^Td - k_br^Tr - k_b\dot{\tau}r^Tr_\tau + \omega\tau\left\|u\right\|^2 \\ & -\omega\left(1-\dot{\tau}\right)\int_{t-\tau(t)}^{t}\left\|u\left(\theta\right)\right\|^2d\theta + \frac{k_b}{2}\left\|r\right\|^2 - \frac{k_b\left(1-\dot{\tau}\right)}{2}\left\|r_\tau\right\|^2.\end{aligned} \tag{5–20}$$

Young's Inequality can be used to upper bound select terms in (5–20) as

$$\left\|e\right\|\left\|e_z\right\| \;\leq\; \frac{\zeta^2}{4}\left\|e\right\|^2 + \frac{1}{\zeta^2}\left\|e_z\right\|^2, \tag{5–21}$$

$$\left\|r\right\|\left\|r_\tau\right\| \;\leq\; \frac{1}{2}\left\|r\right\|^2 + \frac{1}{2}\left\|r_\tau\right\|^2. \tag{5–22}$$

Utilizing Assumption 5.2, (5–6), (5–10), (5–11), (5–21) and (5–22), (5–20) can be expanded, regrouped and upper bounded as

$$\begin{aligned}\dot{V}_L \leq\ & -\alpha \|e\|^2 + \frac{\zeta^2}{4}\|e\|^2 + \frac{1}{\zeta^2}\|e_z\|^2 + \bar{n}_d\|r\| + \rho(\|z\|)\|z\|\|r\| + \bar{d}_1\|x\|\|r\| + \bar{d}_2\|r\| \\ & -k_b\|r\|^2 + \frac{k_b\dot{\tau}}{2}\|r\|^2 + \frac{k_b\dot{\tau}}{2}\|r_\tau\|^2 + k_b^2\omega\tau\|r\|^2 - \omega(1-\dot{\tau})\int_{t-\tau(t)}^{t}\|u(\theta)\|^2 d\theta \\ & +\frac{k_b}{2}\|r\|^2 - \frac{k_b(1-\dot{\tau})}{2}\|r_\tau\|^2 . \end{aligned} \quad (5\text{–}23)$$

Utilizing the Cauchy-Schwarz inequality, the integral in (5–23) can be upper bounded as

$$-\omega(1-\dot{\tau})\int_{t-\tau(t)}^{t}\|u(\theta)\|^2 d\theta \leq -\frac{\omega(1-\dot{\tau})}{2\tau}\|e_z\|^2 - \frac{\omega(1-\dot{\tau})}{2}\int_{t-\tau(t)}^{t}\|u(\theta)\|^2 d\theta. \quad (5\text{–}24)$$

Substituting (5–24) into (5–23) yields

$$\begin{aligned}\dot{V}_L \leq\ & -\left(\alpha - \frac{\zeta^2}{4}\right)\|e\|^2 - \frac{1}{\tau}\left(\frac{\omega}{2}(1-\dot{\tau}) - \frac{\tau}{\zeta^2}\right)\|e_z\|^2 - \left(k_{b1} - \frac{k_b\dot{\tau}}{2} - \frac{k_b}{2} - k_b^2\omega\tau\right)\|r\|^2 \\ & +\frac{k_b\dot{\tau}}{2}\|r_\tau\|^2 - \frac{k_b(1-\dot{\tau})}{2}\|r_\tau\|^2 - k_{b2}\|r\|^2 + \bar{n}\|r\| - k_{b3}\|r\|^2 + \rho_2(\|z\|)\|z\|\|r\| \\ & -\frac{\omega(1-\dot{\tau})}{2}\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta \end{aligned} \quad (5\text{–}25)$$

where $k_{bi}\ \forall i = 1,2,3$ were defined after (5–15), $\bar{n} \in \mathbb{R}^+$ is defined as $\bar{n} \triangleq \bar{n}_d + \bar{d}_2$, and $\rho_2(\|z\|) \in \mathbb{R}$ is a positive, globally invertible, nondecreasing function defined as $\rho_2(\|z\|) \triangleq \rho(\|z\|) + \bar{d}_1\|x\|$. Based on Assumption 5.2, $\frac{k_b(1-\dot{\tau})}{2}\|r_\tau\|^2 > \frac{k_b\dot{\tau}}{2}\|r_\tau\|^2$. Utilizing this fact and completing the squares for $\|r\|$, the expression in (5–25) can be upper bounded as

$$\dot{V}_L \leq -\left(\beta - \frac{\rho_2^2(\|z\|)}{4k_{b3}}\right)\|z\|^2 - \frac{\omega(1-\dot{\tau})}{2}\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta + \frac{\bar{n}^2}{4k_{b2}} \quad (5\text{–}26)$$

where β is an auxiliary constant defined in Theorem 5.1. If the sufficient conditions in (5–17) are satisfied, then $\beta > 0$. The inequality [31]

$$\int_{t-\tau(t)}^{t}\left(\int_{s}^{t}\|u(\theta)\|^2 \mathrm{d}\theta\right)\mathrm{d}s \leq \tau \sup_{s\in[t,\, t-\tau]}\left[\int_{s}^{t}\|u(\theta)\|^2 \mathrm{d}\theta\right] = \tau\int_{t-\tau(t)}^{t}\|u(\theta)\|^2 \mathrm{d}\theta$$

can be used to upper bound (5–26) as

$$\begin{aligned}\dot{V}_L \leq & -\left(\beta - \frac{\rho_2^2(\|z\|)}{4k_{b3}}\right)\|z\|^2 + \frac{\bar{n}^2}{4k_{b2}} - \frac{\omega}{4\tau}(1-\dot{\tau})\int_{t-\tau(t)}^{t}\left(\int_s^t \|u(\theta)\|^2 \mathrm{d}\theta\right) \\ & -\frac{k_b^2\omega}{4}(1-\dot{\tau})\int_{t-\tau(t)}^{t}\|r(\theta)\|^2 \mathrm{d}\theta. \end{aligned} \tag{5–27}$$

Based on (5–12)-(5–15), an upper bound for (5–27) can be developed as

$$\dot{V}_L \leq -\beta_2\|y\|^2 - \left(\beta - \frac{\rho_2^2(\|z\|)}{4k_{b3}}\right)\|e_z\|^2 + \frac{\bar{n}^2}{4k_{b2}} \tag{5–28}$$

where $\beta_2(\|z\|, \tau, \dot{\tau}) \in \mathbb{R}^+$ is defined as

$$\beta_2 = \underset{\tau,\dot{\tau}}{inf}\left\{\beta - \frac{\rho_2^2(\|z\|)}{4k_{b3}},\ \frac{k_b\omega(1-\dot{\tau})}{2},\ \frac{(1-\dot{\tau})}{4\tau}\right\}.$$

Provided the following inequality is satisfied,

$$\beta - \frac{\rho_2^2(\|z\|)}{4k_{b3}} > 0, \tag{5–29}$$

the bound in (5–19) can be used to upper bound the inequality in (5–28) as

$$\dot{V}_L \leq -\beta_2 V_L + \frac{\bar{n}^2}{4k_{b2}}. \tag{5–30}$$

Based on Assumption 5.2, and provided the inequality in (5–29) is satisfied, $\beta_2(\|z\|, \tau, \dot{\tau})$ can be lower bounded by a constant, δ, introduced in Theorem 5.1; hence, the linear differential equation in (5–30) satisfies

$$V_L \leq V_L(0)e^{-\delta t} + \frac{\bar{n}^2}{4k_{b2}\delta}\left[1 - e^{-\delta t}\right]. \tag{5–31}$$

From (5–19) and (5–31),

$$\|y(t)\|^2 \leq 2\|y(0)\|^2 + \frac{\bar{n}^2}{4k_{b2}\delta}. \tag{5–32}$$

Based on (5–13) and (5–32),

$$\|e(t)\|^2 + \|r(t)\|^2 + Q(t) \leq 2\|y(0)\|^2 + \frac{\bar{n}^2}{4k_{b2}\delta}. \tag{5–33}$$

Based on (5–3), (5–4), and (5–15), $\|e_z(t)\|^2 \leq 2k_b\tau(t)\,Q(t)$; hence from (5–33),

$$\min\left\{1, \frac{1}{2k_b\tau}\right\}\|z(t)\|^2 \leq \|e(t)\|^2 + \|r(t)\|^2 + \frac{\|e_z(t)\|^2}{2k_b\tau} \leq 2\|y(0)\|^2 + \frac{\bar{n}^2}{4k_{b2}\delta}. \tag{5–34}$$

From (5–34), a final sufficient condition for (5–29) can be obtained as

$$4\beta k_{b3} > \rho_2^2\left(\sqrt{\frac{2\|y(0)\|^2 + \frac{\bar{n}^2}{4k_{b2}\delta}}{\min\left\{1, \frac{1}{2k_b\tau}\right\}}}\right). \tag{5–35}$$

Consider a set $\mathcal{S}$ defined as

$$\mathcal{S} \triangleq \left\{y(t) \in \mathbb{R}^{2n+2} \mid \|y(0)\| < \sqrt{\frac{1}{2}\min\left\{1, \frac{1}{2k_b\tau}\right\}\rho_2^{-1}\left(2\sqrt{\beta k_{b3}}\right) - \frac{\bar{n}^2}{4k_{b2}\delta}}\right\}. \tag{5–36}$$

From (5–31), given $y(0)$, k_{b3} can be selected such that $y(0) \in \mathcal{S}$ (i.e. a semi-global result) to yield the result in (5–16) when τ is sufficiently small. $\square$

Remark 5.2. Given an initial condition $y(0)$, k_{b3} can be selected large enough to satisfy the sufficient condition in (5–35), provided τ is sufficiently small (i.e., $|\tau(t)| < \varepsilon_\tau < 1$). If k_{b3} is selected arbitrarily large, then ω, introduced in (5–14), needs to be selected arbitrarily small so that k_{b1} can be selected large enough to satisfy the second sufficient condition given in (5–17). If ω is selected arbitrarily small, then ζ, from Theorem 5.1 and (5–21), needs to be selected sufficiently large to satisfy the third sufficient condition given in (5–17), provided $\dot{\tau}(t)$ satisfies Assumption 5.2. The constants ω and ζ are not present in the controller. However, the ramifications of the fact that ζ must be selected large enough, are that the control gain α, defined in (5–3), must be selected sufficiently large based on the first inequality in (5–17).

5.4 Euler-Lagrange Extension

Although the work in [28, 30, 146, 178] provides fundamental contributions to the input delay problem for feedforward systems, the applicability of these methods to general uncertain electromechanical systems (e.g., modeled by EL dynamics) is not clear. A transformation is provided in [179] to convert an EL system into a feedforward system, but the transformation requires EMK. It is not apparent how to transform a feedforward system into an EL system when the system parameters are unknown or the dynamics are uncertain, which implies that methods developed for feedforward systems with input delays may not be applicable to uncertain EL dynamics.

The results in this chapter can be extended to general, nonlinear EL dynamics. To illustrate this, consider an input-delayed Euler-Lagrange system of the form

$$M(q)\,\ddot{q}+V_m(q,\dot{q})\,\dot{q}+G(q)+F(\dot{q})+d\,(t)=u\,(t-\tau\,(t)) \tag{5–37}$$

where $M\,(q)\,\in\,\mathbb{R}^{n\times n}$ denotes a generalized inertia matrix, $V_m\,(q,\dot{q})\,\in\,\mathbb{R}^{n\times n}$ denotes a generalized centripetal-Coriolis matrix, $G\,(q)\,\in\,\mathbb{R}^{n}$ denotes a generalized gravity vector, $F\,(\dot{q})\,\in\,\mathbb{R}^{n}$ denotes generalized friction, $d\,(t)\,\in\,\mathbb{R}^{n}$ denotes an exogenous disturbance, $u\,(t-\tau\,(t))\in\mathbb{R}^{n}$ represents the generalized delayed input control vector, where $\tau\,(t)\in\mathbb{R}$ is a non-negative time-varying delay, and $q\,(t)\,,\dot{q}\,(t)\,,\ddot{q}\,(t)\in\mathbb{R}^{n}$ denote the generalized states.

The subsequent development is based on the assumption that $q\,(t)\,,\dot{q}\,(t)$ are measurable outputs, $M\,(q)$, $V_m\,(q,\dot{q})$, $G\,(q)$, $F\,(\dot{q})$, $d\,(t)$ are unknown, the time-varying input delay is known. Knowledge of past values of the control input are again assumed. Additionally, the following property is used:

Property 1. The inertia matrix $M\,(q)$ is symmetric positive-definite, and satisfies the following inequality:

$$\underline{m}\,\|\xi\|^2\leq\xi^TM\xi\leq\bar{m}\,\|\xi\|^2\,,\quad\forall\xi\in\mathbb{R}^n$$

where $\underline{m},\bar{m}\in\mathbb{R}^{+}$ are known constants and $\|\cdot\|$ denotes the standard Euclidean norm.

The inclusion of this EL extension can be considered as an application to the method presented in this chapter. The design of the error systems and controller follow similarly to the development in Section 5.2. Specifically, (5–3) can be modified slightly to accommodate the uncertain inertia effects in the dynamics and (5–4) and (5–6) can be designed as in Section 5.2.

The filtered tracking error, $r(e, e_z, t)$ is redefined as

$$r \triangleq \dot{e} + \alpha e - B e_z \tag{5–38}$$

where $B \in \mathbb{R}^{n \times n}$ is a symmetric, positive-definite constant gain matrix that satisfies the following inequality $\|B\|_\infty \leq \bar{b}$ where $\bar{b} \in \mathbb{R}^+$ is a known constant. The error between B and $M^{-1}(q)$ is denoted by $\eta(q) \in \mathbb{R}^{n \times n}$ and is defined as

$$\eta \triangleq B - M^{-1}$$

and satisfies the following inequality $\|\eta\|_\infty \leq \bar{\eta}$ where $\bar{\eta} \in \mathbb{R}^+$ is a known constant.

Due to the inclusion of B, the open loop error system will contain an additive time-delayed term: $M\eta(u - u_\tau)$. Motivated by the need to cancel this term in the stability analysis, based on the structure of (5–6) and inspired by the work in [180], the LK functional $Q(e_f) \in \mathbb{R}$ is redefined as

$$Q \triangleq \frac{k_b(2\bar{m}\bar{\eta} + 1)}{2(1 - \dot{\tau})} \int_{t-\tau(t)}^{t} \|r(\theta)\|^2 \, \mathrm{d}\theta. \tag{5–39}$$

A Lyapunov-based stability analysis similar to the one presented for the general second-order nonlinear system in Section 5.3 is used to derive sufficient conditions for UUB tracking. Due to the form of (5–37) and the inclusion of B, The Euler-Lagrange extension analysis results in altered sufficient conditions:

$$\alpha > \frac{\bar{b}^2\psi^2}{4}, \qquad k_b > \sup_{\tau,\dot{\tau}} \left(\frac{\varphi_2(2\bar{m}\bar{\eta} + 1)}{2(1 - \dot{\tau})^2 (\psi^2\omega(1 - \dot{\tau}) + \tau)} \right),$$

$$k_3 > \sup_{\tau,\dot{\tau}} \left(k_b^2 \omega \tau + \frac{2k_b\bar{m}\bar{\eta}}{1 - \dot{\tau}} \right) \tag{5–40}$$

where $\varphi_2 \in \mathbb{R}^+$ is a known constant bound on the second derivative of the delay. For more details of the extension and a complete stability analysis, see [181].

5.5 Simulation Results

The controller in (5–6) was simulated using two-link planar robot manipulator dynamics to examine the performance and robustness to variations in the input delay. Motivation for using robot dynamics stems from the fact that the dynamics can be expressed as an EL system (as provided in the extension) that is common to a large class of practical engineering systems. In (5–37), $M(q) \triangleq \begin{bmatrix} p_1 + 2p_3c_2 & p_2 + p_3c_2 \\ p_2 + p_3c_2 & p_2 \end{bmatrix}$, $V_m(q,\dot{q}) \triangleq \begin{bmatrix} -p_3s_2\dot{q}_2 & -p_3s_2(\dot{q}_1 + \dot{q}_2) \\ p_3s_2\dot{q}_1 & 0 \end{bmatrix}$, and $F(\dot{q}) \triangleq \begin{bmatrix} f_{d1} & 0 \\ 0 & f_{d2} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$, where $p_1 = 3.473\, kg \cdot m^2$, $p_2 = 0.196\, kg \cdot m^2$, $p_3 = 0.242\, kg \cdot m^2$, $f_{d1} = 5.3\, Nm\, sec$, $f_{d2} = 1,1\, Nm\, sec$, c_2 denotes $cos(q_2)$, and s_2 denotes $sin(q_2)$. An additive non-vanishing exogenous disturbance was applied as $d_1(t) = 0.2sin\left(\frac{t}{2}\right)$, and $d_2(t) = 0.1sin\left(\frac{t}{4}\right)$. The initial conditions for the manipulator were selected as $q_1, q_2 = 0\ deg$. The desired trajectories were selected as

$$\begin{aligned} q_{d1}(t) &= 20sin(1.5t)\left(1 - e^{-0.01t^3}\right)\ deg, \\ q_{d2}(t) &= 10sin(1.5t)\left(1 - e^{-0.01t^3}\right)\ deg. \end{aligned}$$

To illustrate robustness to the input delay, simulations were completed using various time-varying delays. For each case, the RMS errors are shown in Table 5-1. The results indicate that the performance of the system is relatively less sensitive to the delay frequency and more sensitive to the delay magnitude. This outcome agrees with previous input delay results where the tracking performance reduces as larger constant delays are applied to the system [31].

Results in Table 5-1 indicate that the performance degradation resulting from the frequency of the delay appeared to be minimal. Thus, analysis was also conducted to further examine the robustness of the controller with respect to unknown variances in the frequency and magnitude of the delay. In each case, the actual input

Table 5-1. RMS errors for time-varying time-delay rates and magnitudes.

Time-Delay	$\tau(t)$ (ms)	RMS Error Link 1	RMS Error Link 2
Fast, Small	$2 \cdot sin\left(\frac{t}{2}\right) + 3$	0.0524^o	0.0363^o
Fast, Large	$20 \cdot sin\left(\frac{t}{2}\right) + 30$	0.4913^o	0.5687^o
Slow, Small	$2 \cdot sin\left(\frac{t}{10}\right) + 3$	0.0521^o	0.0341^o
Slow, Large	$20 \cdot sin\left(\frac{t}{10}\right) + 30$	0.5179^o	0.6970^o

Table 5-2. RMS errors when the controller is applied with a mismatch between the assumed time delay and the actual delay. The Time-Delay Variance column indicates the % difference of the magnitude and frequency of the actual input delay in the system.

Time-Delay Variance	RMS Error Link 1	RMS Error Link 2
-30% magnitude	0.0633^o	0.0766^o
-10% magnitude	0.0497^o	0.0662^o
0% magnitude	0.0394^o	0.0605^o
+10% magnitude	0.0495^o	0.0764^o
+30% magnitude	0.0628^o	0.1069^o
+10% frequency	0.0393^o	0.0605^o
+30% frequency	0.0394^o	0.0604^o
+50% frequency	0.0405^o	0.0619^o

delay was varied from the assumed known delay used in the controller. The controller was implemented assuming a sinusoidal time-varying input delay given as $\tau(t) = \left(1 + \frac{m_v}{100}\right) m_a sin\left(\frac{t}{\left(1+\frac{f_v}{100}\right) f_a}\right) + \phi$ s, where m_a denotes the baseline magnitude coefficient (3 ms in this case), m_v denotes the magnitude variance, f_a denotes the baseline frequency coefficient (6 in this case), f_v denotes the frequency coefficient variance, and ϕ denotes the delay offset (7 ms in this case), resulting in a baseline delay signal with a peak magnitude of 10 ms. The results in Table 5-2 suggest that the controller is robust to variances in the delay magnitude and frequency. Figure 5-1 illustrates the time-delay and the tracking errors associated with the +50% frequency variance case.

5.6 Summary

A continuous predictor-based controller is developed for uncertain nonlinear systems which include time-varying input delays and sufficiently smooth additive bounded disturbances. The controller guarantees UUB tracking provided the delay is sufficiently small and slowly varying. An extension illustrates the controller's applicability

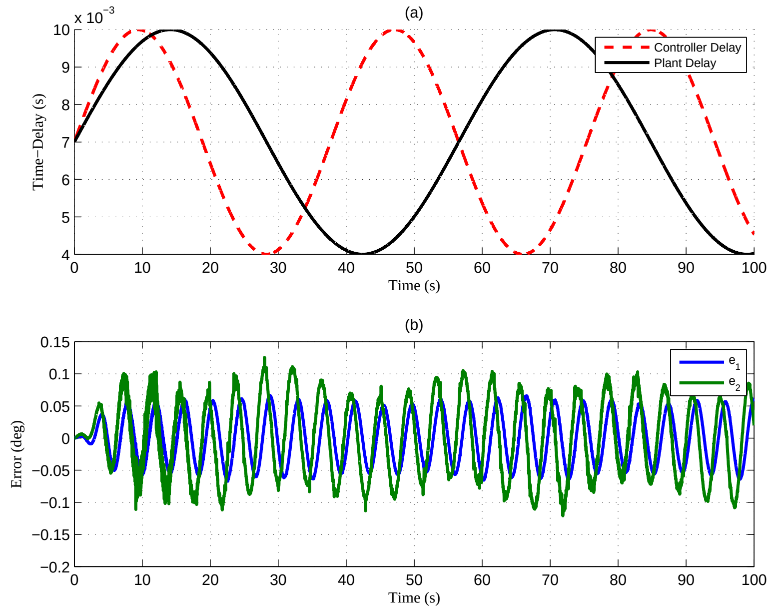

Figure 5-1. Tracking errors vs. time for controller proposed in (5–6) with +50% frequency variance in input-delay: A) Time-delay in seconds, B) Tracking error in degrees.

to a wide array of electromechanical systems that can be described by EL dynamics. While the control development can be applied when there is uncertainty in the system dynamics, the controller is based on the assumption that the time-varying delay is known. However, the simulation results indicate some robustness to uncertainty in the delay magnitude and frequency. Various practical scenarios motivate the need to relax the assumption that the delay profile is known. Future efforts will focus on eliminating this assumption, which presents a significant challenge, since the inherent structure of the predictor depends on integrating the control effort over the known delay interval.

The stability analysis also indicates (in a conservative manner through sufficient gain conditions) an expected link between the initial conditions, the delay magnitude, the delay rate, and the domain of attraction. A favorable outcome of the developed controller is that given any finite initial condition and finite amount of delay (though sufficiently small), the control gains can be selected to ensure the tracking error is regulated, assuming arbitrarily large control authority.

CHAPTER 6
TIME-VARYING INPUT AND STATE DELAY COMPENSATION FOR UNCERTAIN NONLINEAR SYSTEMS

Chapter 6 combines the work of Chapters 4 and 5 by considering a nonlinear system with both time-varying input and state delays. A continuous, robust, predictor-based controller is developed for uncertain, second-order nonlinear systems subject to simultaneous time-varying (unknown) state and (known) input delays in addition to additive bounded disturbances. A DCAL-based predictor structure of previous control values facilitates a delay-free open-loop error system and the design of a controller based on the RISE control technique. A stability analysis utilizing LK functionals guarantees semi-global asymptotic tracking (thanks in part to a new error system development and the inclusion of the RISE controller) assuming the delays are bounded and slowly varying. Numerical simulations illustrate improved performance over Chapter 5's time-varying input delay control design and robustness of the developed method to various combinations of simultaneous input and state delays.

6.1 Dynamic Model

Consider a class of second order nonlinear systems of the following form[1] :

$$\ddot{x} = f\left(x, \dot{x}, t\right) + g\left(x\left(t - \tau_s\left(t\right)\right), \dot{x}\left(t - \tau_s\left(t\right)\right), t\right) + d\left(t\right) + u\left(x, \dot{x}, t - \tau_i\left(t\right)\right) \tag{6–1}$$

where $x\left(t\right), \dot{x}\left(t\right) \in \mathbb{R}^n$ are the generalized system states, $u\left(x, \dot{x}, t - \tau_i\right) \in \mathbb{R}^n$ is the generalized control input, $f\left(x, \dot{x}, t\right) : \mathbb{R}^{2n} \times [0, \infty) \to \mathbb{R}^n$ is an unknown nonlinear $\mathcal{C}^2$ function, $g\left(x\left(t - \tau_s\right), \dot{x}\left(t - \tau_s\right), t\right) : \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ is an unknown nonlinear $\mathcal{C}^2$ time-delayed function, $d\left(t\right) \in \mathbb{R}^n$ denotes a generalized, sufficiently smooth,

[1] The result in this chapter can be extended to n^{th}-order nonlinear systems following a similar development to those presented in [122, 129, 177].

nonvanishing nonlinear disturbance (e.g., unmodeled effects) and $\tau_i(t), \tau_s(t) \in \mathbb{R}^+$ denote non-negative input and state delays, respectively.

The subsequent development is based on the assumption that $x(t)$ and $\dot{x}(t)$ are measurable outputs. Throughout the chapter, a time-dependent delayed function is denoted as $\zeta(t-\tau)$ or ζ_τ, and $\|\cdot\|$ denotes the Euclidean norm of a vector. As in Chapter 4, Assumption 3.1 is used to describe the disturbance term, and Assumption 3.2 is used to describe the desired trajectory. Additionally, the following assumptions on the delays will be exploited.

Assumption 6.1. The input and state delays are bounded such that $0 \le \tau_i(t) \le \varphi_{i_1}$ and $0 \le \tau_s(t) \le \varphi_{s_1}$ and the rate of change of the delays are bounded such that $|\dot{\tau}_i(t)| \le \varphi_{i_2} < 1$ and $|\dot{\tau}_s(t)| \le \varphi_{s_2} < 1$ where $\varphi_j \in \mathbb{R}^+\ \forall j = i_1, i_2, s_1, s_2$ are known constants. The state delay is assumed to be unknown, while the input delay is assumed to be known.

Remark 6.1. In Assumption 6.1, the slowly time-varying constraint (i.e., $|\dot{\tau}_{i,s}(t)| \le \varphi_{i_2,s_2} < 1$) is common to results which utilize classical LK functionals to compensate for time-varying time-delays [18]. Knowledge of the state delays in the system is not required since compensation for the state delays is addressed through the use of a DCAL-based robust control approach. This technique is not sufficient to address the input delays in the system, presenting a more significant challenge to develop a technique to compensate for the input delays separately from the state delay compensation. The input delay is assumed to be known since the interval of past control values in the predictor structure depends on the delay. The simulation results illustrate robustness to how well this value needs to be known.

6.2 Control Development

The objective is to design a continuous controller that will ensure the system state $x(t)$ of the delayed system in (6–1) tracks the desired state trajectory. To quantify the control objective, a tracking error denoted by $e_1(x,t) \in \mathbb{R}^n$, is defined as

$$e_1 \triangleq x_d - x. \tag{6–2}$$

To facilitate the subsequent analysis, two auxiliary tracking errors $e_2(e_1, \dot{e}_1), r(e_2, \dot{e}_2, e_u) \in \mathbb{R}^n$ are defined as [129]

$$e_2 \triangleq \dot{e}_1 + \alpha_1 e_1 \tag{6–3}$$

$$r \triangleq \dot{e}_2 + \alpha_2 e_2 + e_u \tag{6–4}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ denote constant positive control gains and $e_u(\tau_i(t), t) \in \mathbb{R}^n$ denotes the mismatch between the delayed control input and the actual control input, defined as

$$e_u \triangleq u(t - \tau_i(t)) - u(t). \tag{6–5}$$

The auxiliary signal e_u facilitates the ability to inject a delay-free control input into the error system development. In contrast to the development in Chapter 5, the predictor-like term in (6–5) is contained within the $r(\cdot)$ auxiliary term instead of the $e_2(\cdot)$ signal. Functionally, the e_u term still injects an integral of past control values into the open-loop system in this form; however, the development in this chapter introduces fewer cross-terms, allowing for more control design flexibility in the subsequent analysis. The auxiliary signal $r(e_2, \dot{e}_2)$ is introduced to facilitate the stability analysis and is not used in the control design since the expression in (6–4) depends on the unmeasurable state $\ddot{x}(t)$. The structure of the error systems (and included auxiliary signals) is motivated by the need to inject and cancel terms in the subsequent stability analysis and will become apparent in Section 6.3.

An auxiliary filter for (6–5), denoted by $e_{uf}(e_u) \in \mathbb{R}^n$, is defined as the solution to the differential equation

$$\dot{e}_{uf} \triangleq -\alpha_2 e_{uf} + e_u. \tag{6–6}$$

Substituting (6–6) into (6–4), yields

$$r = \dot{e}_2 + \alpha_2 e_2 + \dot{e}_{uf} + \alpha_2 e_{uf}. \tag{6–7}$$

Utilizing an auxiliary signal, $\eta\left(e_2, e_{uf}\right) \in \mathbb{R}^n$, defined as

$$\eta = e_2 + e_{uf}, \tag{6–8}$$

the expression in (6–7) can be rewritten as

$$r = \dot{\eta} + \alpha_2 \eta. \tag{6–9}$$

Equation (6–9) is an intermediate step in the development of the open loop error system that will be explicitly used in the subsequent stability analysis. Substituting (6–1)-(6–3) into (6–9) yields

$$r = \ddot{x}_d - f\left(x, \dot{x}, t\right) - g\left(x_{\tau_s}, \dot{x}_{\tau_s}, t\right) - d\left(t\right) - u_{\tau_i} + \alpha_1 \dot{e}_1 + \dot{e}_{uf} + \alpha_2 \eta. \tag{6–10}$$

Using a DCAL-based design approach [175], an open-loop tracking error can be obtained by substituting (6–6) and (6–8) into (6–10), allowing the time-delayed control inputs to cancel as

$$r = S_1 + S_2 - u \tag{6–11}$$

where the auxiliary function $S_1\left(x, x_d, \dot{x}, \dot{x}_d, x_{\tau_s}, \dot{x}_{\tau_s}, x_{d\tau_s}, \dot{x}_{d\tau_s}, \dot{e}_1, e_2, t\right) \in \mathbb{R}^n$ and $S_2\left(x_d, \dot{x}_d, \ddot{x}_d, x_{d\tau_s}, \dot{x}_{d\tau_s}, t\right) \in \mathbb{R}^n$ are defined as

$$S_1 \triangleq f\left(x_d, \dot{x}_d, t\right) - f\left(x, \dot{x}, t\right) + g\left(x_{d\tau_s}, \dot{x}_{d\tau_s}, t\right) - g\left(x_{\tau_s}, \dot{x}_{\tau_s}, t\right) + \alpha_1 \dot{e}_1 + \alpha_2 e_2$$

$$S_2 \triangleq \ddot{x}_d - f\left(x_d, \dot{x}_d, t\right) - g\left(x_{d\tau_s}, \dot{x}_{d\tau_s}, t\right) - d.$$

Based on the form of (6–11) and the subsequent stability analysis, the controller, $u\left(e_2, v\right)$, is designed as

$$u \triangleq \left(k_s + 1\right) e_2 - \left(k_s + 1\right) e_2\left(t_0\right) + v \tag{6–12}$$

where $v(e_2, e_u, \eta) \in \mathbb{R}^n$ is the Filippov solution to the following differential equation

$$\dot{v} \triangleq (k_s + 1)\,\alpha_2 e_2 + (k_s + 1)\,e_u + \beta sgn(\eta) \tag{6–13}$$

where $k_s, \beta \in \mathbb{R}$ are positive constant control gains, and $sgn(\cdot)$ is defined $\forall \xi \in \mathbb{R}^m = \begin{bmatrix} \xi_1 & \xi_2 & \ldots & \xi_m \end{bmatrix}^T$ as $sgn(\xi) \triangleq \begin{bmatrix} sgn(\xi_1) & sgn(\xi_2) & \ldots & sgn(\xi_m) \end{bmatrix}^T$.[2] The existence of Filippov solutions can be established for $\dot{v} \in K[h_1](e_2, e_u, \eta)$, where $h_1(e_2, e_u, \eta) \in \mathbb{R}^n$ is defined as the right-hand side of (6–13), and $K[h_1] \triangleq \bigcap_{\delta>0} \bigcap_{\mu S_m=0} \overline{co} h_1(e_2, e_u, B(\eta, \delta) - S_m)$, where $\bigcap_{\mu S_m=0}$ denotes the intersection of all sets S_m of Lebesgue measure zero, $\overline{co}$ denotes convex closure, and $B(\eta, \delta) = \{\varsigma \in \mathbb{R}^n |\ \|\eta - \varsigma\| < \delta\}$ [45, 154]. In this case, S_m is expressed by the singleton set $\{\eta = 0\}$.

The closed-loop tracking error system can be developed by taking the time derivative of (6–11) and using the time derivative of (6–12) to yield

$$\dot{r} = \tilde{N} + N_d - e_2 - (k_s + 1)\,r - \beta sgn(\eta) \tag{6–14}$$

where $\tilde{N}(S_1, e_2) \in \mathbb{R}^n$ and $N_d(S_2) \in \mathbb{R}^n$ are defined as

$$\tilde{N} \triangleq \dot{S}_1 + e_2, \tag{6–15}$$

$$N_d \triangleq \dot{S}_2. \tag{6–16}$$

The structure of (6–14) is motivated by the desire to segregate terms that can be upper bounded by a state-dependent signal and terms that can be upper bounded by constants. Based on Assumptions 3.1 and 3.2, the following inequalities can be developed from the expression in (6–16):

$$\|N_d\| \leq \zeta_{N_{d1}}, \quad \left\|\dot{N}_d\right\| \leq \zeta_{N_{d2}} \tag{6–17}$$

[2] The initial condition for $v(0)$ is selected such that $u(0) = 0$.

where $\zeta_{N_{d1}}, \zeta_{N_{d2}} \in \mathbb{R}$, are known positive constants. Applying the MVT, an upper bound can be developed for the expression in (6–15) as [164, Appendix A]

$$\left\|\tilde{N}\right\| \leq \rho_1\left(\|z\|\right)\|z\| + \rho_2\left(\|z_{\tau_s}\|\right)\|z_{\tau_s}\| \tag{6–18}$$

where $z\left(e_1, e_2, r\right) \in \mathbb{R}^{3n}$ denotes the vector

$$z \triangleq \left[e_1^T,\ e_2^T,\ r^T\right]^T \tag{6–19}$$

and the bounding terms $\rho_1\left(\cdot\right), \rho_2\left(\cdot\right) \in \mathbb{R}$ are a positive, globally invertible, nondecreasing functions. The upper bound for the auxiliary function $\tilde{N}\left(\cdot\right)$ is segregated into delay-free and delay-dependent bounding functions to eliminate the delayed terms with the use of an LK functional in the stability analysis.

6.3 Stability Analysis

Theorem 6.1. *Given the dynamics in (6–1), the controller given in (6–12) and (6–13) ensures asymptotic tracking in the sense that*

$$\left\|e_1\left(t\right)\right\| \to 0 \quad as \quad t \to \infty$$

provided the control gains are selected based on the following sufficient conditions

$$\alpha_1 > \frac{1}{2}, \qquad \alpha_2 > 1, \qquad \beta > \zeta_{N_{d1}} + \frac{\zeta_{N_{d2}}}{\alpha_2},$$

$$\frac{2\omega\left(1-\dot{\tau}_i\right)}{2\omega+1} > \tau_i, \qquad 2\sigma k_s > \rho^2\left(\left\|z\left(0\right)\right\|\right) \tag{6–20}$$

where $\sigma \triangleq \min\left\{\alpha_1 - \frac{1}{2},\ \alpha_2 - 1,\ 1\right\} \in \mathbb{R}$, $\rho\left(\|z\|\right) : \mathbb{R} \to \mathbb{R}$ *is a subsequently defined positive-definite, globally invertible, nondecreasing function, and* $\left\|z\left(0\right)\right\|$ *contains the initial conditions of the state.*

Proof. Let $\mathscr{D} \subset \mathbb{R}^{3n+3}$ be a domain containing $y\left(z, P, Q, R_{LK}\right) \in \mathbb{R}^{3n+3}$, defined as

$$y \triangleq \begin{bmatrix} z^T & \sqrt{P} & \sqrt{Q} & \sqrt{R_{LK}} \end{bmatrix}^T. \tag{6–21}$$

In (6–21), the auxiliary function $P(\eta, t) \in \mathbb{R}$ is defined as the Filippov solution to the following differential equation

$$\dot{P} = -r^T (N_d - \beta sgn(\eta)), \quad P(\eta(t_0), t_0) = \beta \sum_{i=1}^{n} |\eta_i(t_0)| - \eta_i(t_0)^T N_d(t_0) \tag{6–22}$$

where the subscript $i = 1, 2, ..., n$ denotes the ith element of a vector. Similar to the development in (6–13), existence of solutions $P(\eta, t)$ for (6–22) can be established. Provided the sufficient condition for β in (6–20) is satisfied, $P(\eta, t) \geq 0$ (See Appendix B for details). Additionally, let $Q(\dot{u}, \tau_i, t), R_{LK}(z, \tau_s, t) \in \mathbb{R}$ denote LK functionals, defined as

$$Q \triangleq \omega \int_{t-\tau_i(t)}^{t} \left(\int_{s}^{t} \|\dot{u}(\theta)\|^2 d\theta \right) ds \tag{6–23}$$

$$R_{LK} \triangleq \frac{\gamma}{2k_s} \int_{t-\tau_s(t)}^{t} \rho_2^2(\|z(\sigma)\|) \|z(\sigma)\|^2 d\sigma \tag{6–24}$$

where $\omega, \gamma \in \mathbb{R}$ are known, positive, adjustable constants, and k_s and $\rho_2(\cdot)$ were introduced in (6–12) and (6–18), respectively.

Let $V : \mathscr{D} \times [0, \infty) \to \mathbb{R}$ be a continuously differentiable in $y(\cdot)$, locally Lipschitz in t, regular function defined as

$$V \triangleq \frac{1}{2} e_1^T e_1 + \frac{1}{2} e_2^T e_2 + \frac{1}{2} r^T r + P + Q + R_{LK} \tag{6–25}$$

which satisfies the following inequalities:

$$\phi_1(y) \leq V(y, t) \leq \phi_2(y). \tag{6–26}$$

where the continuous, positive-definite functions $\phi_1(y), \phi_2(y) \in \mathbb{R}$ in (6–26) are defined as $\phi_1(y) \triangleq \lambda_1 \|y\|^2$, $\phi_2(y) \triangleq \lambda_2 \|y\|^2$ and $\lambda_1, \lambda_2 \in \mathbb{R}^+$ are positive constants.

Under Filippov's framework, the time derivative of (6–25) exists almost everywhere, i.e., for almost all $t \in [t_0, t_f]$, and $\dot{V}(y, t) \overset{a.e.}{\in} \dot{\tilde{V}}(y, t)$ where

$$\dot{\tilde{V}} = \bigcap_{\xi \in \partial V_L(y,t)} \xi^T K \left[\dot{e}_1^T \ \dot{e}_2^T \ \dot{r}^T \ \frac{1}{2} P^{-\frac{1}{2}} \dot{P} \ \frac{1}{2} Q^{-\frac{1}{2}} \dot{Q} \ \frac{1}{2} R_{LK}^{-\frac{1}{2}} \dot{R}_{LK} \ 1 \right]^T,$$

and ∂V is the generalized gradient of $V(y,t)$ [166]. Since $V(y,t)$ is a Lipschitz continuous regular function

$$\dot{\tilde{V}} \subset \nabla V^T K\left[\dot{e}_1^T \;\; \dot{e}_2^T \;\; \dot{r}^T \;\; \frac{1}{2}P^{-\frac{1}{2}}\dot{P} \;\; \frac{1}{2}Q^{-\frac{1}{2}}\dot{Q} \;\; \frac{1}{2}R_{LK}^{-\frac{1}{2}}\dot{R}_{LK}\right]^T \tag{6–27}$$

where $\nabla V \triangleq \left[e_1^T,\; e_2^T,\; r^T,\; 2P^{\frac{1}{2}},\; 2Q^{\frac{1}{2}},\; 2R_{LK}^{\frac{1}{2}}\right]^T$.

Using the calculus for $K[\cdot]$ from [154], applying the Leibniz Rule to determine the time derivative of (6–23) and (6–24), and substituting (6–2)-(6–4), (6–14), and (6–22) into (6–27), yields

$$\begin{aligned}\dot{\tilde{V}} \subset\ & e_1^T(e_2-\alpha_1 e_1)+e_2^T(r-\alpha_2 e_2-e_u)+r^T\left(\tilde{N}+N_d-e_2-(k_s+1)r-\beta K[sgn(\eta)]\right)\\ & -r^T(N_d-\beta K[sgn(\eta)])+\omega\tau_i\|\dot{u}\|^2-\omega(1-\dot{\tau}_i)\int_{t-\tau_i(t)}^{t}\|\dot{u}(\theta)\|^2 d\theta\\ & +\frac{\gamma}{2k_s}\rho_2^2(\|z\|)\|z\|^2-\frac{\gamma(1-\dot{\tau}_s)}{2k_s}\rho_2^2(\|z_{\tau_s}\|)\|z_{\tau_s}\|^2\end{aligned} \tag{6–28}$$

where $K[sgn(e_2)] = SGN(e_2)$ such that $SGN(e_{2_i}) = 1$ if $e_{2_i}(\cdot) > 0$, $[-1,1]$ if $e_{2_i}(\cdot) = 0$, and -1 if $e_{2_i}(\cdot) < 0$ [154].

Canceling common terms, the expression in (6–28) can be upper bounded as

$$\begin{aligned}\dot{\tilde{V}} \overset{a.e.}{\leq} & -\alpha_1\|e_1\|^2-\alpha_2\|e_2\|^2-(k_s+1)\|r\|^2+\|e_1\|\|e_2\|+\|e_2\|\|e_u\|\\ & +\|r\|\rho_1(\|z\|)\|z\|+\|r\|\rho_2(\|z_{\tau_s}\|)\|z_{\tau_s}\|+\omega\tau_i\|\dot{u}\|^2-\omega(1-\dot{\tau}_i)\int_{t-\tau_i(t)}^{t}\|\dot{u}(\theta)\|^2 d\theta\\ & +\frac{\gamma}{2k_s}\rho_2^2(\|z\|)\|z\|^2-\frac{\gamma(1-\dot{\tau}_s)}{2k_s}\rho_2^2(\|z_{\tau_s}\|)\|z_{\tau_s}\|^2\end{aligned} \tag{6–29}$$

where the set in (6–28) reduces to the scalar inequality in (6–29) since the RHS is continuous a.e., i.e, the RHS is continuous except for the Lebesgue negligible set of times when $r^T\beta K[sgn(\eta)] - r^T\beta K[sgn(\eta)] \neq 0$ [45, 167].[3] Utilizing the definition of

[3] The set of times $\Lambda \triangleq \left\{t \in [0,\infty) : r(t)^T\beta K[sgn(\eta(t))] - r(t)^T\beta K[sgn(\eta(t))] \neq 0\right\} \subset [0,\infty)$ is equivalent to the set of times $\{t : \eta(t) = 0 \wedge r(t) \neq 0\}$. From (6–9), this set can

(6–4), (6–17) and (6–18), and Young's Inequality to show that $\left\|e_1^T e_2\right\| \leq \frac{1}{2}\left\|e_1\right\|^2+\frac{1}{2}\left\|e_2\right\|^2$, $\left\|e_2^T e_u\right\| \leq \frac{1}{2}\left\|e_2\right\|^2+\frac{1}{2}\left\|e_u\right\|^2$ and $\|r\| \rho_2\left(\left\|z_{\tau_s}\right\|\right)\left\|z_{\tau_s}\right\| \leq \frac{k_s}{2}\|r\|^2+\frac{1}{2k_s} \rho_2^2\left(\left\|z_{\tau_s}\right\|\right)\left\|z_{\tau_s}\right\|^2$, the expression in (6–29) can be upper bounded as

$$\begin{aligned}
\dot{\tilde{V}} \overset{a.e.}{\leq} & -\left(\alpha_1-\frac{1}{2}\right)\left\|e_1\right\|^2-\left(\alpha_2-1\right)\left\|e_2\right\|^2-\left(k_s+1-\frac{k_s}{2}\right)\|r\|^2+\frac{1}{2}\left\|e_u\right\|^2+\omega \tau_i\|\dot{u}\|^2 \\
& -\omega\left(1-\dot{\tau}_i\right) \int_{t-\tau_i(t)}^t\|\dot{u}(\theta)\|^2 d\theta+\frac{\gamma}{2k_s} \rho_2^2(\|z\|)\|z\|^2-\frac{\gamma\left(1-\dot{\tau}_s\right)}{2k_s} \rho_2^2\left(\left\|z_{\tau_s}\right\|\right)\left\|z_{\tau_s}\right\|^2 \\
& +\|r\| \rho_1(\|z\|)\|z\|+\frac{1}{2k_s} \rho_2^2\left(\left\|z_{\tau_s}\right\|\right)\left\|z_{\tau_s}\right\|^2 .
\end{aligned} \tag{6–30}$$

If $\left(1-\varphi_{s_2}\right)\gamma>1$, by completing the squares for $\|r\|$ and by utilizing the fact that

$$\|\dot{u}(t)\|^2 \leq \int_{t-\tau_i(t)}^t\|\dot{u}(\theta)\|^2 d\theta, \qquad \left\|e_u\right\|^2 \leq \tau_i \int_{t-\tau_i(t)}^t\|\dot{u}(\theta)\|^2 d\theta,$$

the expression in (6–30) can be upper bounded as

$$\begin{aligned}
\dot{\tilde{V}} \overset{a.e.}{\leq} & -\left(\alpha_1-\frac{1}{2}\right)\left\|e_1\right\|^2-\left(\alpha_2-1\right)\left\|e_2\right\|^2-\|r\|^2+\frac{\rho_1^2(\|z\|)\|z\|^2}{2k_s} \\
& +\gamma \frac{\rho_2^2(\|z\|)\|z\|^2}{2k_s}-\left(\omega\left(1-\dot{\tau}_i\right)-\omega \tau_i-\frac{\tau_i}{2}\right) \int_{t-\tau_i(t)}^t\|\dot{u}(\theta)\|^2 d\theta .
\end{aligned} \tag{6–31}$$

If the conditions in (6–20) are satisfied, the expression in (6–31) reduces to

$$\dot{\tilde{V}} \overset{a.e.}{\leq} -\left(\sigma-\frac{\rho^2(\|z\|)}{2k_s}\right)\|z\|^2 \leq -\phi_3(y)=-c\|z\|^2 \qquad \forall y \in \mathscr{D}$$

for some positive constant $c \in \mathbb{R}^{+}$ and domain $\mathscr{D}=\left\{y \in \mathbb{R}^{3n+3} \mid \|y\|<\rho^{-1}\left(\sqrt{2\sigma k_s}\right)\right\}$, where σ was introduced in (6–20), and the bounding function $\rho(\|z\|)$ from (6–20) is defined as $\rho^2(\|z\|) \triangleq \rho_1^2(\|z\|)+\gamma \rho_2^2(\|z\|)$. Larger values of k_s will expand the size of the domain $\mathscr{D}$. The inequalities in (6–26) and (6–31) can be used to show that $V \in \mathcal{L}_{\infty}$ in $\mathscr{D}$. Thus, $e_1(\cdot), e_2(\cdot), r(\cdot) \in \mathcal{L}_{\infty}$ in $\mathscr{D}$. The closed-loop error system can be used to

also be represented by $\{t: \eta(t)=0 \ \wedge \dot{\eta}(t) \neq 0\}$. Provided $\eta(t)$ is continuously differentiable, it can be shown that the set of time instances $\{t: \eta(t)=0 \ \wedge \dot{\eta}(t) \neq 0\}$ is isolated, and thus, measure zero. This implies that the set Λ is measure zero.

conclude that the remaining signals are bounded in $\mathscr{D}$, and the definitions for $\phi_3(\cdot)$ and $z(\cdot)$ can be used to show that $\phi_3(\cdot)$ is uniformly continuous in $\mathscr{D}$. Let $S_{\mathscr{D}} \subset \mathscr{D}$ denote a set defined as

$$\mathcal{S}_{\mathscr{D}} \triangleq \left\{ y \in \mathscr{D} \mid \phi_2(y) < \lambda_1 \left(\rho^{-1} \left(\sqrt{2\sigma k_s} \right) \right)^2 \right\}. \tag{6–32}$$

The region of attraction in (6–32) can be made arbitrarily large to include any initial conditions by increasing the control gain k_s. From (6–28), [182, Corollary 1] can be invoked to show that $c\|z\|^2 \to 0 \ as \ t \to \infty \, \forall y(0) \in \mathcal{S}_{\mathscr{D}}$. Based on the definition of $z(\cdot)$ in (6–19), $\|e_1\| \to 0 \ as \ t \to \infty \, \forall y(0) \in \mathcal{S}_{\mathscr{D}}$. □

6.4 Simulation Results

The controller in (6–12) was simulated to examine the performance and robustness to variations in both the state and input delay. Specifically the dynamics from (6–1) are utilized where $n = 2$, $f(x, \dot{x}, t) \triangleq \begin{bmatrix} -p_4 s_2 \\ p_5 s_2 \dot{x}_2 \end{bmatrix}$, $g(x_{\tau_s}, \dot{x}_{\tau_s}, t) \triangleq \begin{bmatrix} -p_3 s_2 \dot{x}_2 & -p_3 s_2 (\dot{x}_1 + \dot{x}_2) \\ p_3 s_2 \dot{x}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_{1\tau_s} \\ \dot{x}_{2\tau_s} \end{bmatrix} + \begin{bmatrix} f_{d1} & 0 \\ 0 & f_{d2} \end{bmatrix} \begin{bmatrix} \dot{x}_{1\tau_s} \\ \dot{x}_{2\tau_s} \end{bmatrix}$, $x, \dot{x}, \ddot{x} \in \mathbb{R}^2$ denote the state position, velocity, and acceleration, $d(t) \in \mathbb{R}^2$ denotes an additive external disturbance, $u(x, \dot{x}, t - \tau_i(t)) \in \mathbb{R}^2$ denotes the delayed control input and $\tau_s(t), \tau_i(t) \in \mathbb{R}$ denote the unknown non-negative time-varying state delay and the known non-negative time-varying input delay, respectively. Additionally, $p_1 = 3.473$, $p_2 = 0.196$, $p_3 = 0.242$, $p_4 = 0.238$, $p_5 = 0.146$, $f_{d1} = 5.3$, $f_{d2} = 1,1$, and $s_2,\ s_{2\tau_s}$ denote $sin(x_2(t))$ and $sin(x_2(t - \tau_s))$.

Remark 6.2. The system is assumed to have delay-free sensor feedback. This is evident in the dynamic model presented in (6–1), as the state delay only appears within the delayed function $g(x(t - \tau_s(t)), \dot{x}(t - \tau_s(t)), t)$. Scenarios where delays are found in the output are not considered in this work. The dynamics in (6–1) can be transformed into an EL-like system to resemble a class of systems which describes a large number of physical applications (see [129] for additional details on extensions to Euler-Lagrange dynamics).

An additive, non-vanishing, exogenous disturbance was applied as $d_1 = 0.2sin\left(\frac{t}{2}\right)$, and $d_2 = 0.1sin\left(\frac{t}{4}\right)$. The initial conditions for the system were selected as $x_1, x_2 = 0$. The desired trajectories were selected as

$$x_{d1}(t) = (30sin(1.5t) + 20)\left(1 - e^{-0.01t^3}\right),$$
$$x_{d2}(t) = -(20sin(t/2) + 10)\left(1 - e^{-0.01t^3}\right).$$

To illustrate robustness to the delays, several simulations were completed using various time-varying delays. First, to compare the proposed controller to the previous input-delayed work in [123], the controller in (6–12) is simulated with no state delay. Figure 6-1 illustrates the comparative results of the two controllers, assuming $\tau_i = -10 \cdot sin\left(\frac{t}{3}\right) + 30$ and $\tau_s = 0$ (since the result in [123] did not consider state delays). Notably, the the proposed controller achieves better tracking performance compared to the PD-like controller in Chapter 5.

Next, robustness of the proposed controller was examined for the cases when both state and input delays are present in the system. Various time delay combinations were considered and for each case, the RMS errors are shown in Table 6-1. To illustrate the findings, Figure 6-2 depicts the tracking errors, actuation effort and time-varying delays for Case 3. Additionally, Case 5 is provided in Figure 6-3. The results indicate that the performance of the system is relatively less sensitive to the delay frequency and more sensitive to the delay magnitude. This outcome agrees with previous input delay results where the tracking performance reduces as larger delays are applied to the system [31, 181]. In general, simulation results illustrate that the proposed controller is able to achieve better tracking performance as well as handle larger input delays (even with added simultaneous state delays) than the previous time-varying input-delayed work in [181]. Additionally, convergence and performance are achieved in more delay

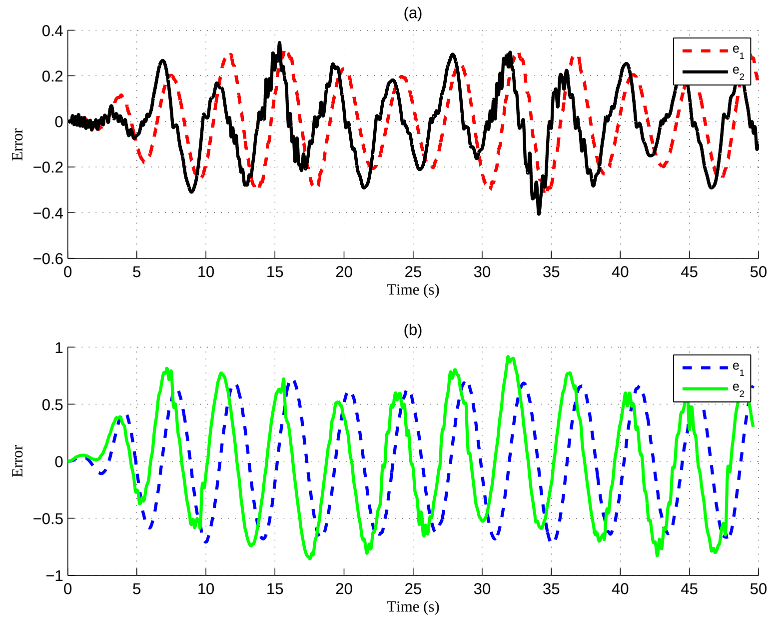

Figure 6-1. Tracking errors for the A) proposed controller and B) the PD-like controller in [181] when considering an input delay of $\tau_i = -10 \cdot sin\left(\frac{t}{3}\right) + 30$ and no state delay.

Table 6-1. RMS errors for time-varying time-delay rates and magnitudes.

	State-Delay	Input-Delay	$\tau_s(t)$ (ms)	$\tau_i(t)$ (ms)	Error x_1	Error x_2
Case 1	Slow, Small	Slow, Large	$5 \cdot sin\left(\frac{t}{8}\right) + 10$	$-50 \cdot sin\left(\frac{t}{10}\right) + 100$	1.46^o	1.47^o
Case 2	Fast, Small	Fast, Small	$10 \cdot sin\left(\frac{t}{2}\right) + 40$	$-10 \cdot sin\left(\frac{t}{3}\right) + 30$	0.29^o	0.30^o
Case 3	Slow, Large	Fast, Large	$10 \cdot sin\left(\frac{t}{10}\right) + 40$	$-50 \cdot sin\left(\frac{t}{2}\right) + 100$	1.32^o	1.39^o
Case 4	Slow, Large	Slow, Large	$10 \cdot sin\left(\frac{t}{10}\right) + 40$	$50 \cdot sin\left(\frac{t}{10}\right) + 100$	1.52^o	2.02^o
Case 5	Fast, Large	Fast, Small	$50 \cdot sin(t) + 800$	$5 \cdot sin\left(\frac{t}{2}\right) + 30$	0.16^o	0.32^o

cases, indicating added robustness to delays in the system. As depicted in the example cases for the given dynamics, the controller is more robust to larger magnitude delays in the state than in the input, as indicated in the stability analysis and is apparent in Case 5. This is not surprising based on the sufficient condition in (6–20).

6.5 Summary

This chapter presents a continuous predictor-based controller for uncertain nonlinear systems which include simultaneous time-varying state and input delays as well as sufficiently smooth additive bounded disturbances. The controller utilizes a DCAL-based design approach to assist in compensation of the unknown state delays coupled with

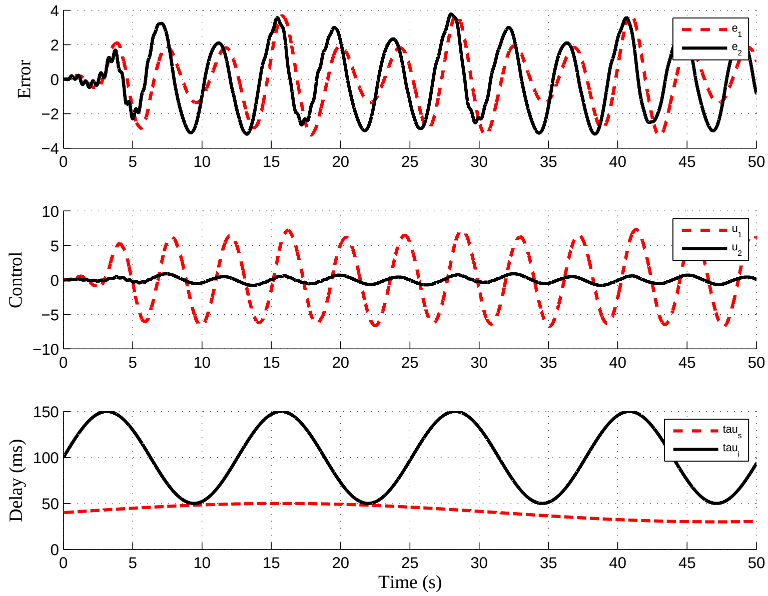

Figure 6-2. Tracking errors, actuation effort and time-varying delays vs time for Case 3.

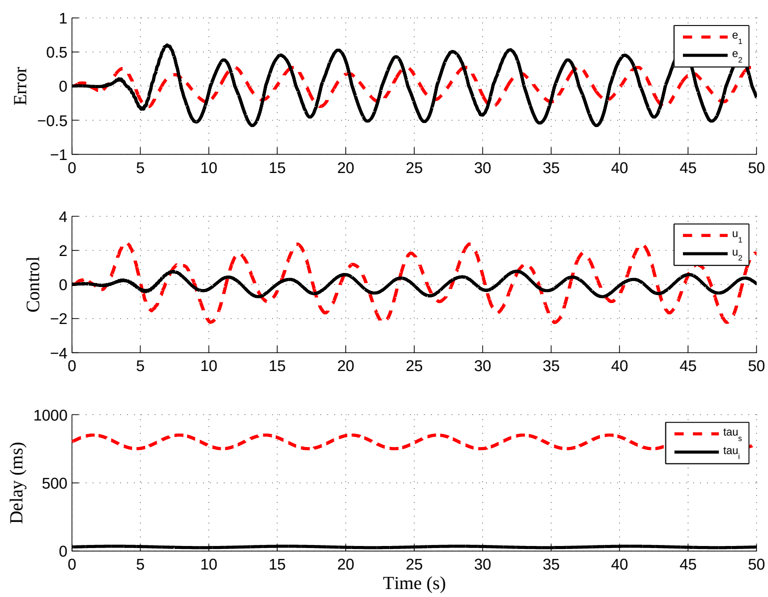

Figure 6-3. Tracking errors, actuation effort and time-varying delays vs time for Case 5.

an error system structure that provides a delay-free open-loop error system. The RISE-based controller and LK functionals guarantee semi-global asymptotic tracking provided the rate of the delays is sufficiently slow, but does not restrict the bound on the delays to be sufficiently small. The control development can be applied when there is uncertainty in the system dynamics and when the state delay is unknown; however, the controller is based on the assumption that the time-varying input delay is known. Numerical simulations compare the result to a previous time-varying input-delay control design and examine the robustness of the method to various combinations of simultaneous input and state delays. The simulation results illustrate robustness to the uncertainty in the magnitude and frequency of the input delays and state delays. These results point to the possibility that different control or analysis methods could be developed to eliminate the assumption that the input delay is known.

CHAPTER 7
SATURATED CONTROL OF AN UNCERTAIN NONLINEAR SYSTEM WITH INPUT DELAY

Leveraging the work of Chapters 3 and 5, this chapter examines saturated control of a general class of uncertain nonlinear systems with time-delayed actuation and additive bounded disturbances. The bound on the control is known a priori and can be adjusted by changing the feedback gains. A Lyapunov-based stability analysis utilizing LK functionals is provided to prove UUB tracking despite uncertainties in the dynamics. The result is extended to general nonlinear systems which can be described by EL dynamics and is illustrated with simulation results to demonstrate the control performance.

7.1 Dynamic Model

Consider a class of nonlinear systems described by

$$\ddot{x} = f(x, \dot{x}, t) + u(t - \tau) + d(t) \tag{7–1}$$

where $x(t), \dot{x}(t) \in \mathbb{R}^n$ are the generalized system states, $u(t-\tau) \in \mathbb{R}^n$ represents the generalized delayed control input vector, where $\tau \in \mathbb{R}^+$ is a constant time delay, $f(x, \dot{x}, t) : \mathbb{R}^{2n} \times [0, \infty) \to \mathbb{R}^n$ is an unknown nonlinear $\mathcal{C}^2$ function, and $d(t) : [0, \infty) \to \mathbb{R}^n$ denotes a sufficiently smooth exogenous disturbance (e.g., unmodeled effects).

The subsequent development is based on the assumption that $x(t)$ and $\dot{x}(t)$ are measurable outputs, the time delay constant, τ, is known, and the control input vector $u(t)$ and its past values (i.e., $u(t-\theta)\,\forall\theta \in [0\ \tau]$) are measurable. Throughout the chapter, a time dependent delayed function is denoted as $\zeta(t-\tau)$ or ζ_τ. Additionally, Assumption 5.3 and the following assumptions are used.

Assumption 7.1. The disturbance term and its first time derivative are bounded by known constants, i.e., $\|d(t)\| \leq c_1, \left\|\dot{d}(t)\right\| \leq c_2$ where $c_1, c_2 \in \mathbb{R}^+$.

Remark 7.1. The nonlinear function $f(x,\dot{x},t)$ is $\mathcal{C}^2$ and the MVT can be used to prove that it satisfies the following inequality

$$\|f(x,\dot{x},t)-f(x_d,\dot{x}_d,t)\|\leq\rho(\|\varphi\|)\|\varphi\|$$

where $\varphi(x,\dot{x},x_d,\dot{x}_d)\in\mathbb{R}^{2n}$ is defined as $\varphi=[x-x_d,\ \dot{x}-\dot{x}_d]^T$. Defining $f(x,\dot{x},t)$ in this way is less restrictive than claiming the function $f(x,\dot{x},t)$ satisfies the global Lipschitz condition (which would yield a linear bound in the states, i.e., $\rho(\|\varphi\|)=\rho$).

As in Chapter 3, to aid the subsequent control design and analysis, the vector $Tanh(\cdot)\in\mathbb{R}^n$ and the matrix $Cosh(\cdot)\in\mathbb{R}^{n\times n}$ are defined as follows

$$Tanh(\xi)\triangleq[tanh(\xi_1),...,tanh(\xi_n)]^T, \tag{7–2}$$

$$Cosh(\xi)\triangleq diag\{cosh(\xi_1),...,cosh(\xi_n)\} \tag{7–3}$$

where $\xi=[\xi_1,...,\xi_n]^T\in\mathbb{R}^n$ and $diag\{\cdot\}$ represents a diagonal matrix. Based on the definitions in (7–2) and (7–3), the following inequalities hold $\forall\xi\in\mathbb{R}^n$ [183]:

$$\|\xi\|^2\geq\sum_{i=1}^{n}ln(cosh(\xi_i))\geq\frac{1}{2}tanh^2(\|\xi\|),\quad\|\xi\|>\|Tanh(\xi)\|,\quad\|Tanh(\xi)\|^2\geq tanh^2(\|\xi\|),$$

$$\xi^TTanh(\xi)\geq Tanh^T(\xi)Tanh(\xi),\quad\frac{\|\xi\|}{tanh(\|\xi\|)}\leq\|\xi\|+1. \tag{7–4}$$

7.2 Control Development

The control objective is to design an amplitude-limited, continuous controller that will ensure the generalized state $x(t)$ of the input-delayed system in (7–1) tracks $x_d(t)$ despite uncertainties and additive bounded disturbances in the dynamic model. To quantify the control objective, a tracking error, denoted by $e(x,t)\in\mathbb{R}^n$, is defined as

$$e\triangleq x_d-x. \tag{7–5}$$

Embedding the control in a bounded trigonometric term (e.g., $tanh(\cdot)$) is an obvious way to limit the control authority below an a priori limit; however, difficulty arises in the closed-loop stability analysis with respect to the delay present in the control. Motivated by these stability analysis complexities and through an iterative analysis procedure, a measurable filtered tracking error is designed which includes additional smooth saturation terms and a finite integral of past control values. Specifically, the filtered tracking error $r(e, \dot{e}, e_f, e_z, t) \in \mathbb{R}^n$ is defined as

$$r \triangleq \dot{e}(\dot{x}, t) + \alpha Tanh(e) + Tanh(e_f) - e_z(t) \tag{7–6}$$

where $\alpha \in \mathbb{R}^+$ is a known adjustable gain constant, $e_f(e, r, t) \in \mathbb{R}^n$ is the solution of the auxiliary error filter dynamics given by

$$\dot{e}_f \triangleq Cosh^2(e_f)(-kr + Tanh(e) - \gamma Tanh(e_f)) \tag{7–7}$$

where $e_f(0) = 0$ and $k, \gamma \in \mathbb{R}^+$ are constant control gains, and $e_z(t) \in \mathbb{R}^n$ denotes the finite integral of past control values, defined as

$$e_z \triangleq \int_{t-\tau}^{t} u(\theta)\, \mathrm{d}\theta. \tag{7–8}$$

From the definition in (7–8), the finite integral can be upper bounded as $\|e_z\| \leq \zeta_z$, where $\zeta_z \in \mathbb{R}^+$ is a known bounding constant provided the control is bounded.

The open-loop error system can be obtained by taking the time derivative of (7–6) and utilizing the expressions in (7–1) and (7–5) to yield

$$\begin{aligned} \dot{r} = {} & \ddot{x}_d(t) - f(x, \dot{x}, t) - u(t) - d(t) \\ & + \alpha Cosh^{-2}(e)\dot{e}(\dot{x}, t) + Cosh^{-2}(e_f)\dot{e}_f(e, e_f, r, t). \end{aligned} \tag{7–9}$$

From (7–9) and the subsequent stability analysis, the control input, $u(e, e_f, t)$, is designed as

$$u \triangleq -kTanh(e_f) + 2Tanh(e) \tag{7–10}$$

where k was introduced in (7–7).[1]

An important feature of the controller given by (5–6) is its applicability to the case where constraints exist on the available actuator commands. Note that the control law is bounded by the adjustable control gain k since $\|u\| \leq (k+2)\sqrt{n}$.

In review of (7–5)-(7–9), the strategy employed to develop the controller in (7–10) entails several components. One component is the development of the filtered error system in (7–6) and (7–7), which is composed of saturated hyperbolic tangent functions designed from the Lyapunov analysis to cancel cross terms. The filtered error system also includes a predictor term (7–8), which utilizes past values of the control. The motivation for the design of (7–7) stems from the need to inject a $-kr$ signal into the closed-loop error system, since such terms can not be directly injected through the saturated controller, and to cancel cross terms in the analysis. The saturated control structure motivates the need for hyperbolic tangent functions in the Lyapunov analysis to yield $-\left\|Tanh\left(e_f\right)\right\|^2$ terms. The time derivative of the hyperbolic tangent function will yield a $Cosh^{-2}(e_f)$ term. The design of (7–7) is motivated by the desire to cancel the $Cosh^{-2}(e_f)$ term, enabling the remaining terms to provide the desired feedback and cancel nonconstructive terms as dictated by the subsequent stability analysis.

[1] To implement the controller in (7–10), the tracking error $e(\cdot)$ and integral of past control values $e_z(\cdot)$ should be evaluated first. The signal $e_z(\cdot)$ is considered to be 0 until $t = \tau$. The filtered tracking error $r(\cdot)$ can be evaluated using either the initial condition for $e_f(\cdot)$ ($e_f(0) = 0$ as stated after (7–7)) or the computed value after the first iteration. The auxiliary signal $e_f(\cdot)$ can be solved online by evaluating $\dot{e}_f(\cdot)$ at each time step using the computed values for $e(\cdot)$ and $r(\cdot)$ and the previous value for $e_f(\cdot)$. Since each of the terms on the right hand side of (7–7) are measurable, the solution $e_f(t)$ can be found using any of the numerous numerical integration techniques available in literature. Once each of the auxiliary error signals have been computed, (7–10) can be implemented.

The closed-loop error system is obtained by utilizing (7–7), (7–9), and (7–10) to yield

$$\dot{r} = S\left(x_d, \dot{x}_d, \ddot{x}_d, t\right) + \chi\left(e, \dot{e}, e_f, t\right) + kTanh\left(e_f\right) - Tanh\left(e\right) - kr\left(e, \dot{e}, e_f, e_z, t\right) \tag{7–11}$$

where the auxiliary terms $S\left(x_d, \dot{x}_d, \ddot{x}_d, t\right) \in \mathbb{R}^n$ and $\chi\left(e, \dot{e}, e_f, t\right) \in \mathbb{R}^n$ are defined as

$$S \triangleq \ddot{x}_d\left(t\right) - f\left(x_d, \dot{x}_d, t\right) - d\left(t\right), \tag{7–12}$$

$$\chi \triangleq -f\left(x, \dot{x}, t\right) + f\left(x_d, \dot{x}_d, t\right) + \alpha Cosh^{-2}\left(e\right)\dot{e}\left(\dot{x}, t\right) - \gamma Tanh\left(e_f\right). \tag{7–13}$$

The structure of (7–11) is motivated by the desire to segregate terms that can be upper bounded by state-dependent terms and terms that can be upper bounded by constants. Using Assumptions 5.3 and 6.1, the following inequality can be developed based on the expression in (7–12)

$$\|S\| \leq \bar{s} \tag{7–14}$$

where $\bar{s} \in \mathbb{R}^+$ is a known constant. Using the MVT, (7–4) and (7–6), the expression in (7–13) can be upper bounded (see Appendix C for details) as

$$\|\chi\| \leq \bar{\chi}\left(\|z\|\right)\|z\| \tag{7–15}$$

where the bounding function $\bar{\chi} : \mathbb{R}^{4n+1} \to \mathbb{R}$ is a positive, globally invertible, nondecreasing function, and $z\left(e, e_f, r, e_z, P\right) \in \mathbb{R}^{4n+1}$ is defined as

$$z \triangleq \begin{bmatrix} e^T & Tanh^T\left(e_f\right) & r^T & e_z^T & \sqrt{P} \end{bmatrix}^T. \tag{7–16}$$

In (7–16), $P\left(t\right) \in \mathbb{R}^+$ denotes an LK functional defined as

$$P \triangleq \omega \int_{t-\tau}^{t} \left(\int_{s}^{t} \left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta \right) \mathrm{d}s \tag{7–17}$$

where $\omega \in \mathbb{R}^+$ is a known constant.

7.3 Stability Analysis

Theorem 7.1. *Given the dynamics in (7–1), the controller in (7–10) ensures uniformly ultimately bounded tracking provided the adjustable control gains* α, γ, k *are selected according to the following sufficient conditions*

$$\alpha > \frac{\psi^2}{4} + 2\omega\tau(2k+1), \;\; \gamma > k\omega\tau(k+2), \;\; \omega\psi^2 > 2\tau, \tag{7–18}$$

$$4\beta k_2 \geq \bar{\chi}^2\left(\bar{\mu}\right)\left(cosh^{-1}\left(e^{2\bar{\mu}^2}\right)+1\right)^2 \tag{7–19}$$

where $\psi \in \mathbb{R}^+$ *is an known, adjustable, positive constant,* $\bar{\mu} \in \mathbb{R}$ *is defined as* $\bar{\mu} \triangleq max\left\{\bar{d}, \|z(0)\|\right\}$*, and* $\bar{d} \in \mathbb{R}$ *is a subsequently defined, positive constant that defines the radius of a ball containing the position tracking errors.*

Proof. Let $V_L(z,t) : \mathcal{D} \times [0,\ \infty) \rightarrow \mathbb{R}$ be a continuously differentiable, positive-definite functional on a domain $\mathcal{D} \subseteq \mathbb{R}^{4n+1}$, defined as

$$V_L \triangleq \frac{1}{2}r^T r + \sum_{i=1}^{n} ln\left(cosh\left(e_i\right)\right) + \frac{1}{2}Tanh^T\left(e_f\right)Tanh\left(e_f\right) + P, \tag{7–20}$$

which can be bounded using (7–4) as

$$\phi_1\left(\|z\|\right) \leq V_L \leq \phi_2\left(\|z\|\right). \tag{7–21}$$

where the strictly increasing non-negative functions $\phi_1(\cdot), \phi_2(\cdot) : \mathbb{R}^{4n+1} \rightarrow \mathbb{R}$ are defined as

$$\phi_1\left(\|z\|\right) \triangleq \frac{1}{2}ln\left(cosh\left(\|z\|\right)\right), \quad \phi_2\left(\|z\|\right) \triangleq \|z\|^2. \tag{7–22}$$

After utilizing (7–6), (7–7), (7–11) and by canceling similar terms, the time derivative of (7–20) can be expressed as

$$\begin{aligned}\dot{V}_L = {} & r^T\chi + r^T S - kr^T r - \alpha Tanh^T(e)\,Tanh(e) - \gamma Tanh^T\left(e_f\right)Tanh\left(e_f\right) \\ & + Tanh^T(e)\,e_z + \omega\tau\|u\|^2 - \omega\int_{t-\tau}^{t}\|u(\theta)\|^2\,\mathrm{d}\theta\end{aligned} \tag{7–23}$$

where the Leibniz Rule was applied to determine the time derivative of (7–17). Using (7–4), (7–10), (7–14), and (7–15), (7–23) can be upper bounded by

$$\begin{aligned}\dot{V}_L \leq & -k\left\|r\right\|^2-\alpha\left\|Tanh\left(e\right)\right\|^2-\gamma\left\|Tanh\left(e_f\right)\right\|^2+\left\|r\right\|\bar{\chi}\left(\left\|z\right\|\right)\left\|z\right\|+\left\|r\right\|\bar{s}\\ & +\left\|Tanh\left(e\right)\right\|\left\|e_z\right\|+k^2\omega\tau\left\|Tanh\left(e_f\right)\right\|^2+4\omega\tau\left\|Tanh\left(e\right)\right\|^2\\ & +4k\omega\tau\left\|Tanh\left(e_f\right)\right\|\left\|Tanh\left(e\right)\right\|-\omega\int_{t-\tau}^{t}\left\|u\left(\theta\right)\right\|^2\mathrm{d}\theta.\end{aligned} \tag{7–24}$$

Young's Inequality can be used to upper bound select terms in (7–24) as

$$\begin{aligned}\|Tanh(e)\|\|e_z\| & \leq \frac{\psi^2}{4}\|Tanh(e)\|^2+\frac{1}{\psi^2}\|e_z\|^2,\\ \|Tanh(e_f)\|\|Tanh(e)\| & \leq \frac{1}{2}\|Tanh(e_f)\|^2+\frac{1}{2}\|Tanh(e)\|^2\end{aligned} \tag{7–25}$$

where ψ is a known constant. Utilizing the Cauchy-Schwarz Inequality, the last integral in (7–23) can be upper bounded as

$$-\omega\int_{t-\tau}^{t}\left\|u\left(\theta\right)\right\|^2 d\theta \leq -\frac{\omega}{2\tau}\left\|e_z\right\|^2-\frac{\omega}{2}\int_{t-\tau}^{t}\left\|u\left(\theta\right)\right\|^2 d\theta. \tag{7–26}$$

Using (7–25) and (7–26), (7–24) can be upper bounded as

$$\begin{aligned}\dot{V}_L \leq & -k_1\left\|r\right\|^2-\left(\alpha-\frac{\psi^2}{4}-4\omega\tau\left(\frac{k}{2}+1\right)\right)\left\|Tanh\left(e\right)\right\|^2\\ & -\left(\gamma-2k\omega\tau-k^2\omega\tau\right)\left\|Tanh\left(e_f\right)\right\|^2-\left(\frac{\omega}{2\tau}-\frac{1}{\psi^2}\right)\left\|e_z\right\|^2\\ & -k_2\left\|r\right\|^2+\bar{\chi}\left(\left\|z\right\|\right)\left\|z\right\|\left\|r\right\|-k_3\left\|r\right\|^2+\bar{s}\left\|r\right\|-\frac{\omega}{2}\int_{t-\tau}^{t}\left\|u\left(\theta\right)\right\|^2\mathrm{d}\theta\end{aligned} \tag{7–27}$$

where k, introduced in (7–7) and (7–10), is split into adjustable constants $k_1, k_2, k_3 \in \mathbb{R}^+$ as $k \triangleq k_1+k_2+k_3$. After completing the squares, the expression in (7–27) can be upper bounded as

$$\begin{aligned}\dot{V}_L \leq & -k_1\left\|r\right\|^2-\left(\alpha-\frac{\psi^2}{4}-4\omega\tau\left(\frac{k}{2}+1\right)\right)\left\|Tanh\left(e\right)\right\|^2-\left(\gamma-2k\omega\tau-k^2\omega\tau\right)\left\|Tanh\left(e_f\right)\right\|^2\\ & -\left(\frac{\omega}{2\tau}-\frac{1}{\psi^2}\right)\left\|e_z\right\|^2+\frac{\bar{\chi}^2\left(\left\|z\right\|\right)}{4k_2}\left\|z\right\|^2-\frac{\omega}{2}\int_{t-\tau}^{t}\left\|u\left(\theta\right)\right\|^2\mathrm{d}\theta+\frac{\bar{s}^2}{4k_3}.\end{aligned} \tag{7–28}$$

The inequality

$$\int_{t-\tau}^{t}\left(\int_{s}^{t}\left\|u\left(\theta\right)\right\|^{2}\mathrm{d}\theta\right)\mathrm{d}s\leq\tau\sup_{s\in[t,\,t-\tau]}\left[\int_{s}^{t}\left\|u\left(\theta\right)\right\|^{2}\mathrm{d}\theta\right]=\tau\int_{t-\tau}^{t}\left\|u\left(\theta\right)\right\|^{2}\mathrm{d}\theta$$

can be used to upper bound (7–28) as

$$\dot{V}_{L}\leq-k_{1}\left\|r\right\|^{2}-\left(\alpha-\frac{\psi^{2}}{4}-4\omega\tau\left(\frac{k}{2}+1\right)\right)\left\|Tanh\left(e\right)\right\|^{2}-\left(\gamma-2k\omega\tau-k^{2}\omega\tau\right)\left\|Tanh\left(e_{f}\right)\right\|^{2}$$
$$-\left(\frac{\omega}{2\tau}-\frac{1}{\psi^{2}}\right)\left\|e_{z}\right\|^{2}+\frac{\bar{\chi}^{2}\left(\left\|z\right\|\right)}{4k_{2}}\left\|z\right\|^{2}-\frac{\omega}{2\tau}\int_{t-\tau}^{t}\left(\int_{s}^{t}\left\|u\left(\theta\right)\right\|^{2}\mathrm{d}\theta\right)\mathrm{d}s+\frac{\bar{s}^{2}}{4k_{3}}. \qquad (7–29)$$

Let $y\left(e,e_{f},e_{z},r,P\right)\in\mathbb{R}^{4n+1}$ be defined as

$$y\triangleq\left[Tanh^{T}\left(e\right)\;\;Tanh^{T}\left(e_{f}\right)\;\;e_{z}^{T}\;\;r^{T}\;\;\sqrt{P}\right]^{T}. \qquad (7–30)$$

By using (7–16) and (7–30), (7–29) can be upper bounded as

$$\dot{V}_{L}\leq-\beta\left\|y\right\|^{2}+\frac{\bar{\chi}^{2}\left(\left\|z\right\|\right)}{4k_{2}}\left\|z\right\|^{2}+\frac{\bar{s}^{2}}{4k_{3}} \qquad (7–31)$$

where the auxiliary constant $\beta\in\mathbb{R}^{+}$ is defined as

$$\beta\triangleq min\left\{k_{1},\;\alpha-\frac{\psi^{2}}{4}-4\omega\tau\left(\frac{k}{2}+1\right),\;\gamma-2k\omega\tau-k^{2}\omega\tau,\;\frac{\omega}{2\tau}-\frac{1}{\psi^{2}},\;\frac{1}{2\tau}\right\}. \qquad (7–32)$$

If the sufficient conditions in (7–18) are satisfied, then $\beta>0$. Provided the following inequality is satisfied

$$\frac{\bar{\chi}^{2}\left(\left\|z\right\|\right)}{4k_{2}}\left\|z\right\|^{2}-\beta\left\|y\right\|^{2}\leq0, \qquad (7–33)$$

(7–31) can be expressed as

$$\dot{V}_{L}\leq-\beta_{2}\left\|y\right\|^{2}+\frac{\bar{s}^{2}}{4k_{3}} \qquad (7–34)$$

where $\beta_{2}\in\mathbb{R}^{+}$ is some constant. From the definitions in (7–16) and (7–30) and utilizing the fact that $\left\|y\right\|^{2}\geq tanh^{2}\left(\left\|z\right\|\right)$ from (7–4), the expression in (7–33) is satisfied if

$$\left(\frac{\left\|z\right\|}{tanh\left(\left\|z\right\|\right)}\right)^{2}\leq\frac{4\beta k_{2}}{\bar{\chi}^{2}\left(\left\|z\right\|\right)}. \qquad (7–35)$$

Using the properties in (7–4), a sufficient condition for (7–35) is

$$(\|z\|+1)^2 \leq \frac{4\beta k_2}{\bar{\chi}^2(\|z\|)}. \tag{7–36}$$

The first lower bound on $V_L(z,t)$ from (7–21) can be used to state that

$$\|z\| \leq cosh^{-1}(exp(2V_L)); \tag{7–37}$$

hence, a sufficient condition for (7–36) can be obtained as

$$\bar{\chi}^2\left(cosh^{-1}(exp(2V_L))+1\right)^2 \leq 4\beta k_2. \tag{7–38}$$

If the condition (7–38) is satisfied, then from (7–4), the expression in (7–34) can be rewritten as

$$\dot{V}_L \leq -\phi_3(\|z\|)+\frac{\bar{s}^2}{4k_3} \tag{7–39}$$

where the strictly increasing non-negative function $\phi_3 : \mathbb{R}^{4n+1} \to \mathbb{R}$ is defined as $\phi_3(\|z\|) \triangleq \beta_2 tanh^2(\|z\|)$. Given (7–21), and (7–39), $z(\cdot)$ (as well as $e(\cdot)$ and $r(\cdot)$ via the definition in (7–16) and standard linear analysis) is UUB [184] in the sense that

$$\|e(t)\| \leq \|z(t)\| < \bar{d}, \quad \forall t \geq T\left(\bar{d},\ \|z(0)\|\right) \tag{7–40}$$

provided the sufficient conditions in (7–18) and the inequality in (7–38) are satisfied. In (7–40), $\bar{d} \in \mathbb{R}$ is a positive constant that defines the radius of a ball containing the position tracking errors, selected according to [184]

$$\bar{d} > \left(\phi_1^{-1}\circ\phi_2\right)\left(\phi_3^{-1}\left(\frac{\bar{s}^2}{4k_3}\right)\right), \tag{7–41}$$

and $T\big(\bar{d},\ \|z(0)\|\big) \in \mathbb{R}$ is a positive constant that denotes the ultimate time to reach the ball [184]

$$T \triangleq \begin{cases} 0 & \|z(0)\| \leq \left(\phi_2^{-1}\circ\phi_1\right)\left(\bar{d}\right) \\ \frac{\phi_2(\|z(0)\|)-\phi_1\left(\left(\phi_2^{-1}\circ\phi_1\right)\left(\bar{d}\right)\right)}{\phi_3\left(\phi_2^{-1}\circ\phi_1\right)\left(\bar{d}\right)-\frac{\bar{s}^2}{4k_3}} & \|z(0)\| > \left(\phi_2^{-1}\circ\phi_1\right)\left(\bar{d}\right). \end{cases}$$

From (7–21) and (7–40), a final sufficient condition for (7–38), given in (7–19), can be expressed in terms of either the initial conditions of the system or the ultimate bound. □

Remark 7.2. Based on (7–39), the size of the ultimate bound in (7–41) can be made smaller by selecting k_3 larger. For arbitrarily large delays or arbitrarily large initial conditions, the control gains required to satisfy the sufficient gain conditions in (7–19) may demand torque that is not physically deliverable by the system (i.e., the gain k may be required to be larger than the saturation limit of the actuator). The gain condition in (7–19) is directly influenced by the bound given in (7–15), which results from the bounds derived in Remark 7.1. For example, if f is globally Lipschitz, then the upper bound on χ reduces to a constant times the state and a local condition on the state z can be determined as $\|z(0)\| \leq \sqrt{4\beta k_2/\bar{\chi}} - 1$, which can be enlarged by increasing k_2 (up to a point based on the actuator constraints). Given the current, more general bound for χ in Remark 7.1, a simplified closed-form initial condition bound can not be derived. However, given an upper bound on the disturbance, an upper bound on the time delay, and the initial conditions, (7–19) and (7–32) can be used to determine the sufficient gain βk_2, if possible, based on the actuator limit. This result does not satisfy the standard semi-global result because under the consideration of input constraints, k cannot be arbitrarily increased and consequently cannot satisfy all initial conditions. This outcome is not surprising from a physical perspective in the sense that such demands may yield cases where the actuation is insufficient to stabilize the system.

7.4 Euler-Lagrange Extension

Similar to the development in Section 5.4 of Chapter 5, the controller presented in (7–10) can be extended to nonlinear EL systems of the form

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + G(q) + F(\dot{q}) + d(t) = u(t-\tau) \tag{7–42}$$

where $M(q) \in \mathbb{R}^{n\times n}$ denotes the generalized inertia, $V_m(q,\dot{q}) \in \mathbb{R}^{n\times n}$ denotes the generalized centrifugal and Coriolis forces, $G(q) \in \mathbb{R}^n$ denotes the generalized gravity, $F(\dot{q}) \in \mathbb{R}^n$ denotes the generalized friction and $q(t), \dot{q}(t), \ddot{q}(t) \in \mathbb{R}^n$ denote the generalized states. Utilizing standard properties of the inertia and centrifugal/Coriolis matrices, the control development can be extended to achieve the same result as in Section 7.3.

The design of the error systems and controller follow similarly to the method presented previously. Specifically, (7–6) can be modified slightly to accommodate the uncertain inertia effects in the dynamics and (7–7), (7–8) and (7–10) can be designed as in Section 7.2.

As in the development of (7–6), the filtered tracking error, $r(e, e_f, e_z, t)$ is redefined as

$$r \triangleq \dot{e} + \alpha Tanh(e) + Tanh(e_f) - Be_z$$

where e_f is defined as in (7–7) and $B \in \mathbb{R}^{n\times n}$ is a symmetric, positive-definite constant gain matrix that satisfies the following inequality $\|B\|_\infty \leq \bar{b}$ where $\bar{b} \in \mathbb{R}^+$ is a known constant. The error between B and $M^{-1}(q)$ is denoted by $\eta(q) \in \mathbb{R}^{n\times n}$ and is defined as

$$\eta \triangleq B - M^{-1}$$

and satisfies the following inequality $\|\eta\|_\infty \leq \bar{\eta}$ where $\bar{\eta} \in \mathbb{R}^+$ is a known constant.

Due to the inclusion of B, the open-loop error system will contain an additive time-delayed term: $M\eta(u - u_\tau)$. Motivated by the need to cancel this term in the stability analysis, based on the structure of (7–10) and inspired by the work in [180], two additional LK functionals are added to the Lyapunov functional candidate, $V_L(z,t) \in \mathbb{R}$, defined as

$$V_L \triangleq \frac{1}{2}r^T Mr + \sum_{i=1}^{n} ln(cosh(e_i)) + \frac{1}{2}Tanh^T(e_f)\,Tanh(e_f) + P + Q + R$$

where $P(t)$ is defined as in (7–17) and $Q(e_f), R(e) \in \mathbb{R}$ denote LK functionals defined as

$$Q \triangleq \frac{k\bar{m}\bar{\eta}}{2}\int_{t-\tau}^{t}\|Tanh(e_f)\|^2 \,\mathrm{d}\theta, \qquad R \triangleq \frac{\bar{m}\bar{\eta}}{2}\int_{t-\tau}^{t}\|Tanh(e)\|^2 \,\mathrm{d}\theta.$$

A Lyapunov-based stability analysis similar to the one presented for the general second-order nonlinear system in Section 7.3 is used to derive sufficient conditions for UUB tracking. The EL system in (7–42) requires a sufficient condition on B (in addition to gain conditions similar to those given in (7–18) and (7–19)), given by

$$k_1 > \frac{(k_2 + k_3)\, 2\bar{m}\bar{\eta}}{1 - \frac{2\bar{m}\bar{\eta}}{\underline{m}}}$$

where $\bar{m}, \underline{m} \in \mathbb{R}$ are the known constant upper and lower bound on the inertia matrix. The sufficient gain conditions indicate that k_1 can be selected sufficiently large provided $1 - \frac{2\bar{m}\bar{\eta}}{\underline{m}} > 0$. This condition indicates that the constant approximation matrix B must be chosen sufficiently close to $M^{-1}(q)$ so that $\|B - M^{-1}(q)\|_\infty < \frac{\bar{m}}{2\underline{m}}$. Additional details regarding the EL extension of this chapter can be found in [185].

7.5 Simulation Results

Utilizing the extension from Section 7.4, the controller was simulated for a two-link planar manipulator. The EL dynamics of the manipulator are given as

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} p_1 + 2p_3c_2 & p_2 + p_3c_2 \\ p_2 + p_3c_2 & p_2 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -p_3s_2\dot{q}_2 & -p_3s_2(\dot{q}_1 + \dot{q}_2) \\ p_3s_2\dot{q}_1 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$
$$+ \begin{bmatrix} f_{d1} & 0 \\ 0 & f_{d2} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} \tau_{d1} \\ \tau_{d2} \end{bmatrix}$$

where $p_1 = 3.473\ kg \cdot m^2$, $p_2 = 0.196\ kg \cdot m^2$, $p_3 = 0.242\ kg \cdot m^2$, $f_{d1} = 5.3\ Nm\ sec$, $f_{d2} = 1,1\ Nm\ sec$, c_2 denotes $cos(q_2)$, and s_2 denotes $sin(q_2)$. The disturbance terms were selected as $\tau_{d1} = 0.5sin\left(\frac{t}{5}\right)$, and $\tau_{d2} = 0.1sin\left(\frac{t}{5}\right)$. The desired trajectories for links

1 and 2 for all simulations were selected as

$$q_{d1}(t) = 1.5sin(t/2)\ rad, \qquad q_{d2}(t) = 0.5sin(t/4)\ rad.$$

The initial conditions for the manipulator were selected as stationary with a significant offset from the initial conditions of the desired trajectory as $\begin{bmatrix} q_1 & q_2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \end{bmatrix}^T rad.$ For comparison, the simulation was completed using various values of input delay, ranging from 100 ms to 1 s. For each case it is desired for the actuation torque to be limited to $\tau_1 \leq 20\ N,\ \ \tau_2 \leq 10\ N.$ Because the controller assumes that the inertia matrix is unknown, a best guess estimate of the constant matrix B is selected as

$$B = \begin{bmatrix} 4.0 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}.$$

Additional results show that the performance/robustness of the developed saturated controller with respect to the mismatch between B and $M^{-1}(q)$ indicating an insignificant amount of variation in the performance even when each element of $M^{-1}(q)$ is overestimated by as much as 300%, an improvement over the result shown in [31].

Figure (7-1) illustrates the tracking errors associated with each of the input delay cases. As the delay magnitude is increased, the performance degrades and the tracking error bound increases. Figure (7-2) shows that even with a large input delay in the system, the proposed controller is able to ensure that the control torque does not exceed the actuator limits (as specified by the controller gains) while ensuring the boundedness of the tracking error.

7.6 Summary

A continuous saturated controller is developed for uncertain nonlinear systems which include input delays and sufficiently smooth additive bounded disturbances. The bound on the control is known a priori and can be adjusted by changing the feedback gains. The saturated controller is shown to guarantee UUB tracking provided the delay

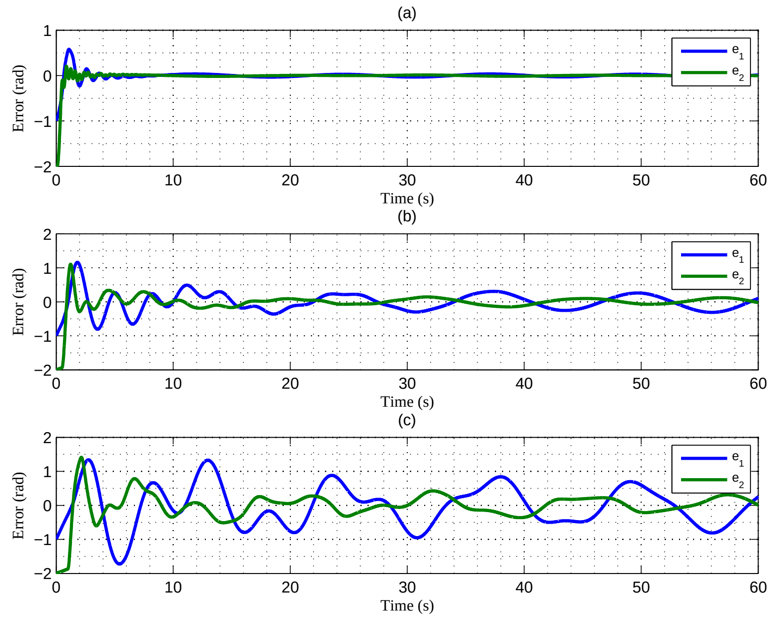

Figure 7-1. Tracking error vs. time for proposed controller in (7–10). A) 100 ms input delay, B) 500 ms input delay, C) 1 s input delay.

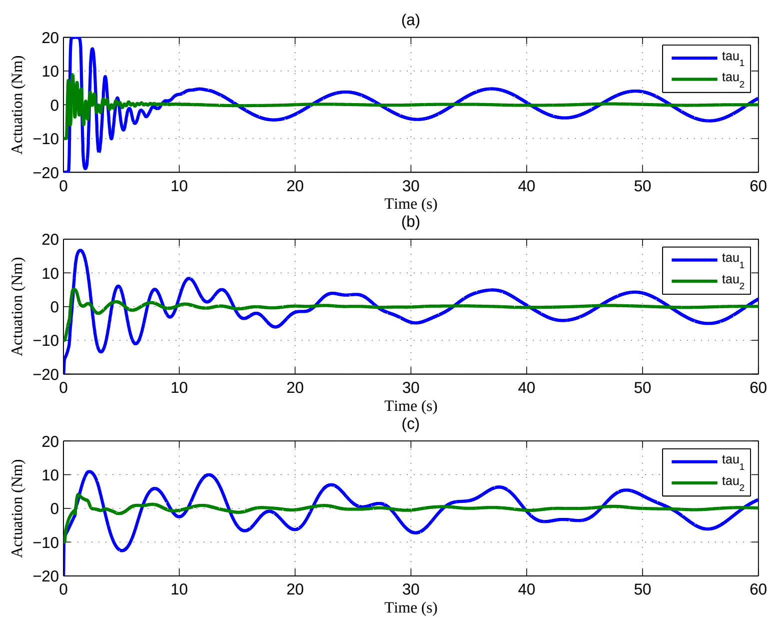

Figure 7-2. Control torque vs. time for proposed controller in (7–10). A) 100 ms input delay, B) 500 ms input delay, C) 1 s input delay.

is sufficiently small. The result is extended to general EL systems and simulation results are performed on a two-link robotic manipulator to demonstrate the effectiveness of the control design.

CHAPTER 8
CONCLUSION AND FUTURE WORK

8.1 Dissertation Summary

The focus of this work is to develop control methods for uncertain nonlinear systems with real world considerations including time-delays and actuator saturation. The work covers a wide variety of systems with practical considerations and has facilitated the introduction of an important Lyapunov-based stability Corollary, suitable for use in numerous applications as well as many theoretical studies in nonlinear control design and analysis. Because real world systems are affected by nonlinear behaviors that are often not considered, the work in this dissertation aims to compensate for these phenomena with practical control designs that can be implemented in relevant engineered systems.

Chapter 2 focuses on introducing the mechanics required to utilize nonsmooth analysis in Lyapunov-based control design and extending the LYT to differential systems with a discontinuous RHS using generalized solutions in the sense of Filippov. The result presents theoretical tools applicable to nonlinear systems with discontinuities in the plant dynamics or in the control structure. Generalized Lyapunov-based analysis methods are developed using differential inclusions to achieve asymptotic convergence of the state when the Lyapunov derivative is upper bounded by a negative semi-definite function. Semi-global sliding mode control and RISE control examples illustrate the use of the Corollaries in control design and analysis.

RISE-based control techniques have been shown to effectively suppress additive bounded disturbances and parametric uncertainties in nonlinear systems; however, the technique's high-gain nature often limits its applicability in systems where actuator limitations exist since standard RISE techniques can potentially demand large actuation efforts when large initial offsets or disturbances are present. In Chapter 3, a continuous saturated controller is developed for a class of uncertain nonlinear systems which includes time-varying and non-LP functions and additive bounded disturbances, achieving semi-global asymptotic tracking. The bound on the control is known a priori and can be adjusted by changing the feedback gains. Embedding the $sgn\left(\cdot\right)$ of an error signal inside and integral term allows for a continuous control design without the risk of infinite actuation demand or chatter, as commonly found in standard sliding mode control designs. However, with the inclusion of an $sgn\left(\cdot\right)$ function and because most Lyapunov-based stability analyses are based on existence of solutions when continuous differential equations are utilized, the work in Chapter 2 is motivated.

The next four chapters focus on nonlinear systems with time delays. Chapter 4 extends our RISE-based control techniques to systems with time-varying state delays. While the generalized state delay problem for nonlinear systems has been studied rigorously, our continuous controller, which achieves asymptotic tracking in the presence of parametric uncertainties and bounded disturbances, is one of the first of its kind. The development involves a DCAL-based approach to separate delayed and non-delayed terms and a neural network to compensate for non-LP uncertainties.

The more challenging open time delay problems in literature focus on the input delay problem for nonlinear systems: the focus of Chapters 5-7. Chapter 5 begins this work by developing a continuous predictor-based controller for uncertain nonlinear systems which include time-varying input delays and sufficiently smooth additive bounded disturbances, guaranteeing UUB tracking, provided the delay is sufficiently small and slowly varying. The error system development includes a novel predictor-based integral of past control values, facilitating the use of a delay-free control signal which can be designed. Chapter 6 goes a step further by combining the input delay problem from Chapter 5 and the state delay problem from Chapter 6 to develop a RISE-based control method capable of handling both time-varying state and input delays and the inclusion of plant uncertainties and bounded disturbances. A new error system development which introduces auxiliary filtered signals allows additional flexibility in control design over the previous UUB approach. Now, the predictor-based controller and LK functional design guarantee semi-global asymptotic tracking provided the rate of the delays is sufficiently slow, but does not restrict the bound on the delays to be sufficiently small.

The final chapter circles back to analyze the effect of actuator saturation for input-delayed systems. Since errors can build over the actuator dead-time, this work is highly motivated. Specifically, a continuous saturated controller is developed for uncertain nonlinear systems which include constant, known input delays and sufficiently smooth additive bounded disturbances. The bound on the control is known a priori and can be adjusted by changing the feedback gains while the controller is shown to guarantee UUB tracking provided the delay is sufficiently small.

8.2 Limitations and Future Work

The work in this dissertation opens new doors for both RISE-based control methods and control designs for time-delayed nonlinear systems. In this section, open problems related to the work in this dissertation are discussed.

From Chapter 2:

1. The LYC utilizes existence of solutions in the sense of Filippov. However, similar results in literature that develop stability techniques utilizing differential inclusions have also been shown for other types of solution definitions. One such example are Krasovskii solutions, as utilized in results such as [153]. Extending the work of Chapter 2 to general solution definitions (not restricted to Filippov) is a likely achievable goal. In general, utilizing Krasovskii solutions in place of Filippov solutions facilitates stronger overall stability results at the cost of requiring more restrictive system assumptions.

From Chapter 3:

1. The stability result achieved in this chapter is semi-global, meaning that the control gains must be selected according to the initial conditions of the system. However,

in practice (both simulation and experiments), the controller exhibits global-like performance. Utilizing techniques similar to [174], it may be possible to exploit the $Tanh(\cdot)$ properties in the control development to manipulate the bound on χ to achieve a global stability result.

2. In the EL extension, the inertia matrix $M(q)$ is required to be known. This is due to the fact that $M^{-1}(q)$ pre-multiplies the control input throughout the open and closed loop analysis. It may be possible to modifying the control design or the error system development to eliminate this assumption.

From Chapter 4:

1. All LK functional-based analysis techniques for time-varying-delayed systems introduce a restriction on the derivative of the delay. This is intuitively due to the fact that the derivative of the LK functional results in a $(1-\dot{\tau})$. Alternative stability techniques such as Razumikhin methods may be capable of eliminating the assumption on the rate of the delay, but future efforts have yet to confirm this.

From Chapter 5:

1. As in Chapter 4, the use of LK functionals introduces restrictive sufficient conditions on the rate of the delay.
2. Due to the fact that the stability result depends on a sufficient condition related to the initial conditions of the system, it is difficult to predict the admissible values of delay that the system can tolerate.
3. The delay is assumed to be sufficiently small and slowly varying. This limits the number of applications for which the controller can be applied. Relaxing these assumptions to a) uncertain delays, b) arbitrarily large delays, and c) arbitrarily fast delays are future goals of this work, in addition to reducing the steady state ultimately bounded tracking.
4. The condition on B in the EL extension that requires B to be selected sufficiently close to $M^{-1}(q)$ is rather restrictive if the inertia matrix is entirely unknown. While

these parameters are often measurable offline, incorporating a more robust method of handling the inertia uncertainties is motivated.

From Chapter 6:

1. The modified error system development allows us to integrate a RISE-based control law in the closed-loop system, allowing us to achieve asymptotic tracking. However, the controller is still restricted by knowledge of the input delay (since it is used in the error system) and the bound on the delay rates. It may be possible to extend the result to include uncertain input delays (the state delays are already assumed uncertain) and to relax the assumption on the delay rates.
2. As in 3, the the Euler-Lagrange extension of this result requires knowledge of the inertia matrix.
3. The second order dynamics presented in (6–1) do not facilitate a trivial extension to Euler-Lagrange dynamics. This extension would introduce signals that have both state and input delays present. Considerations for these composite delay terms require additional focus.

From Chapter 7:

1. The stability analysis in this work is also dependent on a sufficient condition related to the initial conditions of the system. Accurate prediction of the admissible values of the delay for a given system are obtained, however; they are limited by the assumption that the delay is constant and known.
2. Utilizing the work in Chapter 3 and 6, design of a saturated controller for uncertain nonlinear systems with asymptotic tracking may now be possible using RISE techniques.

APPENDIX A
PROOF OF P (CH 3)

Lemma A.1. *Given the differential equation in (3–22), $P(e_2, t) \geq 0$ if β satisfies*

$$\beta\gamma_1 > \zeta_{N_{d1}} + \frac{\zeta_{N_{d2}}}{\alpha_3}. \tag{A–1}$$

Proof. For sake of notation, define an auxiliary signal $\sigma(e_2, r, t) \in \mathbb{R}^n$ as the integral of terms found in $\dot{P}(e_2, t)$ in (3–22)

$$\sigma(e_2, r, t) = \int_0^t r^T(\tau)\left(N_d(\tau) - \beta\gamma_1 sgn(e_2(\tau))\right) \mathrm{d}\tau.$$

By using (3–7), integrating by parts, and regrouping yields

$$\begin{aligned}
\sigma(e_2, r, t) \;=\; & \int_{t_0}^t \alpha_2 Tanh^T(e_2(\tau))\left[N_d(\tau) - \beta\gamma_1 sgn(e_2(\tau))\right] d\tau \\
& + \int_{t_0}^t \alpha_3 e_2^T(\tau)\left[N_d(\tau) - \beta\gamma_1 sgn(e_2(\tau))\right] d\tau \\
& - \int_{t_0}^t \alpha_3 e_2^T(\tau)\left[\frac{1}{\alpha_3}\frac{\partial N_d(\tau)}{\partial\tau}\right] d\tau + e_2^T N_d(t) \\
& - e_2^T(t_0) N_d(t_0) - \beta\gamma_1 \sum_{i=1}^n |e_{2i}(t)| + \beta\gamma_1 \sum_{i=1}^n |e_{2i}(t_0)|.
\end{aligned} \tag{A–2}$$

From (3–8) and (3–18), the expression in (A–2) can be upper bounded by

$$\begin{aligned}
\sigma(e_2, r, t) \;\leq\; & \int_{t_0}^t \alpha_2 \left\|Tanh(e_2(\tau))\right\| \left[\zeta_{N_{d1}} - \beta\gamma_1\right] d\tau \\
& + \int_{t_0}^t \alpha_3 \left\|e_2(\tau)\right\| \left[\zeta_{N_{d1}} + \frac{\zeta_{N_{d2}}}{\alpha_3} - \beta\gamma_1\right] d\tau \\
& + \left\|e_2(t)\right\| \left[\zeta_{N_{d1}} - \beta\gamma_1\right] + \beta\gamma_1 \sum_{i=1}^n |e_{2i}(t_0)| \\
& - e_2^T(t_0) N_d(t_0).
\end{aligned} \tag{A–3}$$

Thus, from (A–3), if β satisfies (A–1), then

$$\sigma(e_2, r, t) \leq \beta\gamma_1 \sum_{i=1}^n |e_{2i}(t_0)| - e_2^T(t_0) N_d(t_0) = P(e_2(t_0), t_0). \tag{A–4}$$

Integrating both sides of (3–22) yields

$$P\left(e_2, t\right) = P\left(e_2\left(t_0\right), t_0\right) - \sigma\left(e_2, r, t\right),$$

which indicates $P\left(e_2, t\right) \geq 0$ from (A–4). $\square$

APPENDIX B
PROOF OF P (CH 6)

Lemma B.1. *Given the differential equation in (6–22),* $P(\eta, t) \geq 0$ *if* β *satisfies*

$$\beta > \zeta_{N_{d1}} + \frac{\zeta_{N_{d2}}}{\alpha_2}. \tag{B–1}$$

Proof. For sake of notation brevity, define an auxiliary signal $\sigma(r, \eta, t) \in \mathbb{R}^n$ as the integral of the terms found in $\dot{P}(r, \eta, t)$ in (6–22) as

$$\sigma(r, \eta, t) = \int_{t_0}^{t} r^T(\xi)\left(N_d(\xi) - \beta sgn(\eta)\right) d\xi. \tag{B–2}$$

Utilizing the expression for (6–9) in terms of (6–8), (B–3) can be expanded as

$$\begin{aligned} \sigma(r, \eta, t) \;\; = \;\; & \int_{t_0}^{t} \alpha_2 \eta^T(\xi)\left[N_d(\xi) - \beta sgn(\eta)\right] d\xi \\ & + \int_{t_0}^{t} \frac{\partial \eta^T(\xi)}{\partial \xi} N_d(\xi)\, d\xi - \int_{t_0}^{t} \frac{\partial \eta^T(\xi)}{\partial \xi} \beta sgn(\eta)\, d\xi. \end{aligned} \tag{B–3}$$

Integrating the last two integrals in (B–4) by parts yields

$$\begin{aligned} \sigma(r, \eta, t) \;\; = \;\; & \int_{t_0}^{t} \alpha_2 \eta^T(\xi)\left[N_d(\xi) - \beta sgn(\eta)\right] d\xi \\ & + \eta^T(t) N_d(t) - \eta^T(t_0) N_d(t_0) - \int_{t_0}^{t} \alpha_2 \eta^T(\xi)\left[\frac{1}{\alpha_2}\frac{\partial N_d(\xi)}{\partial \xi}\right] d\xi \\ & - \beta \sum_{i=1}^{n} \left|\eta_i(t)\right| + \beta \sum_{i=1}^{n} \left|\eta_i(t_0)\right|. \end{aligned} \tag{B–4}$$

Based on (6–17), the expression in (B–4) can be upper bounded as

$$\begin{aligned} \sigma(r, \eta, t) \;\; \leq \;\; & \int_{t_0}^{t} \alpha_2 \left\|\eta(\xi)\right\| \left[\zeta_{N_{d1}} + \frac{\zeta_{N_{d1}}}{\alpha_2} - \beta\right] d\xi \\ & + \left\|\eta^T(t)\right\| \left[\zeta_{N_{d1}} - \beta\right] + \beta \sum_{i=1}^{n} \left|\eta_i(t_0)\right| - \eta^T(t_0) N_d(t_0). \end{aligned} \tag{B–5}$$

From (B–5), if β satisfies the sufficient condition in (B–1), then

$$\sigma(r, \eta, t) \leq \beta \sum_{i=1}^{n} \left|\eta_i(t_0)\right| - \eta^T(t_0) N_d(t_0) = P(\eta(t_0), t_0). \tag{B–6}$$

Integrating both sides of $P\left(\eta\left(t_0\right), t_0\right)$ in (6–22) yields

$$P\left(\eta, t\right) = P\left(\eta\left(t_0\right), t_0\right) - \sigma\left(r, \eta, t\right)$$

which indicates that $P\left(\eta, t\right) \geq 0$ from (B–6). $\square$

APPENDIX C
PROOF OF χ BOUND (CH 7)

Lemma C.1. *The MVT can be used to develop the upper bound in (7–15)*

$$\|\chi\| \leq \bar{\chi}\left(\|z\|\right)\|z\|$$

where the bounding function $\bar{\chi}:\mathbb{R}^{4n+1}\rightarrow\mathbb{R}$ *is a positive, globally invertible, nondecreasing function, and* $z\left(e, e_f, r, e_z, P\right)\in\mathbb{R}^{4n+1}$ *is defined as*

$$z \triangleq \left[\begin{array}{ccccc} e^T & Tanh^T\left(e_f\right) & r^T & e_z^T & \sqrt{P}\end{array}\right]^T. \tag{C–1}$$

Proof. The proof of Lemma C.1 follows from that of [164, App A]. The auxiliary error $\bar{\chi}\left(\cdot\right)$ in (7–13) can be written as the sum of errors pertaining to each of its arguments as follows:

$$\begin{aligned}
\chi\left(\cdot\right) &= \chi\left(x,\dot{x},e,r,Tanh\left(e_f\right),e_z\right)-\chi\left(x_d,\dot{x}_d,0,0,0,0\right)\\
&= \chi\left(x,\dot{x}_d,0,0,0,0\right)-\chi\left(x_d,\dot{x}_d,0,0,0,0\right)\\
&\quad +\chi\left(x,\dot{x},0,0,0,0\right)-\chi\left(x,\dot{x}_d,0,0,0,0\right)\\
&\quad +\chi\left(x,\dot{x},e,0,0,0\right)-\chi\left(x,\dot{x},0,0,0,0\right)\\
&\quad +\chi\left(x,\dot{x},e,r,0,0\right)-\chi\left(x,\dot{x},e,0,0,0\right)\\
&\quad +\chi\left(x,\dot{x},e,r,Tanh\left(e_f\right),0\right)-\chi\left(x,\dot{x},e,r,0,0\right)\\
&\quad +\chi\left(x,\dot{x},e,r,Tanh\left(e_f\right),e_z\right)-\chi\left(x,\dot{x},e,r,Tanh\left(e_f\right),0\right).
\end{aligned}$$

Applying the MVT to further describe $\chi\left(\cdot\right)$,

$$\begin{aligned}
\chi\left(\cdot\right) = \; & \frac{\partial\chi\left(\sigma_1,\dot{x}_d,0,0,0,0\right)}{\partial\sigma_1}|_{\sigma_1=v_1}\left(x-x_d\right)+\frac{\partial\chi\left(x,\sigma_2,0,0,0,0\right)}{\partial\sigma_2}|_{\sigma_2=v_2}\left(\dot{x}-\dot{x}_d\right)\\
& +\frac{\partial\chi\left(x,\dot{x},\sigma_3,0,0,0\right)}{\partial\sigma_3}|_{\sigma_3=v_3}\left(e-0\right)+\frac{\partial\chi\left(x,\dot{x},e,\sigma_4,0,0\right)}{\partial\sigma_4}|_{\sigma_4=v_4}\left(r-0\right)\\
& \frac{\partial\chi\left(x,\dot{x},e,r,\sigma_4,0\right)}{\partial\sigma_5}|_{\sigma_5=v_5}\left(Tanh\left(e_f\right)-0\right)\\
& +\frac{\partial\chi\left(x,\dot{x},e,r,Tanh\left(e_f\right),\sigma_5\right)}{\partial\sigma_6}|_{\sigma_6=v_6}\left(e_z-0\right)
\end{aligned} \tag{C–2}$$

where

$$v_1 = x - c_1 (x - x_d), \quad v_2 = \dot{x} - c_2 (\dot{x} - \dot{x}_d), \quad v_3 = e (1 - c_3),$$

$$v_4 = r (1 - c_4), \quad v_5 = e_f (1 - c_5), \quad v_6 = e_z (1 - c_6)$$

and $c_i \in (0,1) \in \mathbb{R},\ i = [1,6]$ are unknown constants. From (C–2), $\chi(\cdot)$ can be upper bounded as

$$\begin{aligned} \|\chi(\cdot)\| \;=\; & \left\| \frac{\partial \chi(\sigma_1, \dot{x}_d, 0, 0, 0)}{\partial \sigma_1} |_{\sigma_1 = v_1} \right\| \|e\| \\ & + \left\| \frac{\partial \chi(x, \sigma_2, 0, 0, 0)}{\partial \sigma_2} |_{\sigma_2 = v_2} \right\| \|r - \alpha Tanh(e) + Tanh(e_f) - e_z\| \\ & + \left\| \frac{\partial \chi(x, \dot{x}, \sigma_3, 0, 0)}{\partial \sigma_3} |_{\sigma_3 = v_3} \right\| \|e\| + \left\| \frac{\partial \chi(x, \dot{x}, e, \sigma_4, 0, 0)}{\partial \sigma_4} |_{\sigma_4 = v_4} \right\| \|r\| \\ & + \left\| \frac{\partial \chi(x, \dot{x}, e, r, \sigma_4, 0)}{\partial \sigma_5} |_{\sigma_5 = v_5} \right\| \|Tanh(e_f)\| \\ & + \left\| \frac{\partial \chi(x, \dot{x}, e, r, Tanh(e_f), \sigma_5)}{\partial \sigma_6} |_{\sigma_6 = v_6} \right\| \|e_z\|. \end{aligned} \tag{C–3}$$

We can upper bound the partial derivatives as

$$\begin{aligned} \left\| \frac{\partial \chi(\sigma_1, \dot{x}_d, 0, 0, 0)}{\partial \sigma_1} |_{\sigma_1 = v_1} \right\| &\leq \rho_1(e) \\ \left\| \frac{\partial \chi(x, \sigma_2, 0, 0, 0)}{\partial \sigma_2} |_{\sigma_2 = v_2} \right\| &\leq \rho_2(r, e, Tanh(e_f), e_z) \\ \left\| \frac{\partial \chi(x, \dot{x}, \sigma_3, 0, 0)}{\partial \sigma_3} |_{\sigma_3 = v_3} \right\| &\leq \rho_3(e) \\ \left\| \frac{\partial \chi(x, \dot{x}, e, \sigma_4, 0, 0)}{\partial \sigma_4} |_{\sigma_4 = v_4} \right\| &\leq \rho_4(r) \\ \left\| \frac{\partial \chi(x, \dot{x}, e, r, \sigma_4, 0)}{\partial \sigma_5} |_{\sigma_5 = v_5} \right\| &\leq \rho_5(Tanh(e_f)) \\ \left\| \frac{\partial \chi(x, \dot{x}, e, r, e_f, \sigma_5)}{\partial \sigma_6} |_{\sigma_6 = v_6} \right\| &\leq \rho_6(e_z) \end{aligned}$$

where $\rho_i(\cdot) \in \mathbb{R},\ i = [1,6]$ are positive, nondecreasing functions. The bound on $\chi(\cdot)$ can be further simplified

$$\begin{aligned} \|\chi(\cdot)\| \;\leq\; & \rho_1(e)\|e\| + \rho_2(r, e, Tanh(e_f), e_z)\|r - \alpha Tanh(e) + Tanh(e_f) - e_z\| \\ & + \rho_3(e)\|e\| + \rho_4(r)\|r\| + \rho_5(Tanh(e_f))\|Tanh(e_f)\| + \rho_6(e_z)\|e_z\|. \end{aligned}$$

Since $\|Tanh\,(e)\| \leq \|e\|$, the upper bound

$$\|r - \alpha Tanh\,(e) + Tanh\,(e_f) - e_z\| \leq \|r\| + \alpha\,\|e\| + \|Tanh\,(e_f)\| + \|e_z\|\,,$$

can be used to bound $\chi\,(\cdot)$ as

$$\begin{aligned}
\|\chi\,(\cdot)\| \;\; \leq \;\; & (\rho_1\,(e) + \rho_3\,(e) + \alpha\rho_2\,(r, e, Tanh\,(e_f)\,, e_z))\,\|e\| \\
& + (\rho_2\,(r, e, Tanh\,(e_f)\,, e_z) + \rho_4\,(r))\,\|r\| \\
& + (\rho_2\,(r, e, Tanh\,(e_f)\,, e_z) + \rho_5\,(Tanh\,(e_f)))\,\|Tanh\,(e_f)\| \\
& + (\rho_2\,(r, e, Tanh\,(e_f)\,, e_z) + \rho_6\,(e_z))\,\|e_z\|\,.
\end{aligned}$$

Using the definition from (C–1), $\chi\,(\cdot)$ can be expressed in terms of $z\,(\cdot)$ as

$$\begin{aligned}
\|\chi\,(\cdot)\| \;\; \leq \;\; & (\rho_1\,(e) + \rho_3\,(e) + \alpha\rho_2\,(r, e, Tanh\,(e_f)\,, e_z))\,\|z\| \\
& + (\rho_2\,(r, e, Tanh\,(e_f)\,, e_z) + \rho_4\,(r))\,\|z\| \\
& + (\rho_2\,(r, e, Tanh\,(e_f)\,, e_z) + \rho_5\,(Tanh\,(e_f)))\,\|z\| \\
& + (\rho_2\,(r, e, Tanh\,(e_f)\,, e_z) + \rho_6\,(e_z))\,\|z\|\,,
\end{aligned}$$

$$\|\chi\,(\cdot)\| \leq (\rho_1 + (\alpha + 3)\,\rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6)\,\|z\|\,.$$

Therefore,

$$\|\chi\,(\cdot)\| \leq \hat{\chi}\,(\|z\|)\,\|z\|$$

where $\hat{\chi}\,(\|z\|)$ is some positive, nondecreasing function. Any positive nondecreasing function can be upper bounded by a positive strictly increasing function. Thus, the conditions for global invertibility hold. Finally,

$$\|\chi\,(\cdot)\| \leq \hat{\chi}\,(\|z\|)\,\|z\| \leq \bar{\chi}\,(\|z\|)\,\|z\|$$

where $\hat{\chi}\,(\|z\|)$ is a positive globally invertible, nondecreasing function. Note that since $\sqrt{P}$ is positive by definition, its conservative matter in the bounding of χ does not play a factor. $\square$

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BIOGRAPHICAL SKETCH

Nic Fischer was born in St. Petersburg, Florida. He received a Bachelor of Science degree in mechanical engineering from the University of Florida in 2008. He joined the Nonlinear Controls and Robotics (NCR) research group in 2006 with the hopes of continuing his research into graduate school. After competing his bachelor's degree, Nic decided to pursue doctoral research in under the advisement of Dr. Warren Dixon at the University of Florida. Focusing on nonlinear control theory and applications, Nic earned a Master of Science degree in December of 2010 and completed his Ph.D. in December of 2012, both in mechanical engineering. As a graduate researcher, Nic was awarded the Outstanding Graduate Research Award for the Department of Mechanical and Aerospace Engineering for his work in nonlinear control theory in 2012. He also received the Ph.D. Gator Engineering Attribute of Professional Excellence award in 2012 from the UF College of Engineering. Nic has led several student projects including UF's SubjuGator autonomous underwater vehicle. Additionally, Nic worked as a mechanical engineering intern at Honeywell International in 2006.