Reactive Synthesis for Relay-Explorer Consensus with Intermittent Communication

Runhan Sun, Suda Bharadwaj, Zhe Xu, Ufuk Topcu, Warren E. Dixon

***Abstract*—This paper investigates the planning problem using an optimal reactive synthesis method for multi-agent systems (MAS) to reach approximate consensus with intermittent communication. The reactive synthesis approach can satisfy high-level mission specifications, while the low-level dynamics provide real-time state information for corrections. The MAS control synthesis problem can be cast as a relay-explorer problem, where a relay agent intermittently provides navigational feedback to multiple explorer agents in a pre-defined sub-region. Within each sub-region, there is one relay agent responsible for servicing the corresponding explorer agents. Each time the estimated trajectory of an explorer agent crosses the boundary and enters another sub-region, the neighboring relay agent takes over the servicing responsibility. In this paper, a set of planning strategies corresponding to candidate instantiations (i.e., pre-specified representative information scenarios) is pre-synthesized to dynamically switch among the explorers in real-time. To guarantee the stability of the switched strategies and the approximate consensus of the explorer agents, maximum dwell-time conditions are developed using a Lyapunov-based analysis to allow explorer agents to drift for a pre-defined time period without requiring servicing from relay agents. A simulation study is included to demonstrate the performance of the developed method.**

I. INTRODUCTION

Motivated by advantages of intermittent communication versus requiring continuous communication in multi-agent systems (MAS), recent research has focused on developing event-triggered and self-triggered control. In [1]-[6], the control methods only use sampled data for networked agents when desired stability and performance properties trigger the communication conditions. However, these methods typically assume the network is connected to ensure communication when required.

Recently a class of relay-explorer problems has emerged in [7]-[9] where a relay agent intermittently provides state feedback to a set of explorer agents. To guarantee the stability of the switched systems, stabilizing maximum dwell-time conditions are developed to allow the explorer agents to dead-reckon (i.e., feedback from non-absolute sensors such as wheel encoders) for a pre-defined time period before requiring state feedback from the relay agent. The authors in [8] develop a switched systems approach to enable a distributed MAS to reach consensus at a desired location under intermittent communication. Specifically, a relay agent has full knowledge of state feedback, switches between multiple explorers lacking absolute positional sensors to provide each explorer navigational information intermittently. Similarly, authors in [9] develop a distributed controller to enable formation control and leader tracking for the explorer agents, while a relay agent intermittently provide state feedback to an explorer, enabling a MAS to explore an unknown environment indefinitely. However, the method in [8] relies on one relay agent to service multiple explorer agents, which requires the relay agent to reach certain explorer agent within specified time periods to guarantee system stability. When the number of explorer agents is increased, the relay agent needs to maneuver to the corresponding explorer agent fast enough to ensure stability, which might be impractical in some applications and limits scalability.

Alternatively, the aforementioned stabilizing maximum dwell-time conditions can be encoded by metric temporal logic (MTL) specifications as in [10] and [11]. MTL specifications in [11] express the maximum dwell-time condition and practical constraints for the relay agent such as charging its battery and staying in specific regions of interest. Specifically, the followers' controllers are designed to ensure the stability of the switched system provided the dwell-time conditions are satisfied. The leader's controller is synthesized by using the MTL specifications which encode the dwell-time conditions and the additional practical constraints. The authors solve the mixed-integer linear programming (MILP) problem iteratively to obtain the optimal control inputs for the relay agent. Hence, the relay agent is required to iteratively compute the inputs to ensure the explorer agents can be serviced sufficiently often to reach approximate consensus. However, the computation requirements for the relay agent might not be applicable to agents with limited computation power.

The previous example can be treated as a reactive planning problem, where the MAS has to react to an uncontrolled environment, and guarantee correctness with respect to a given mission specification for all possible behaviors of the environment for all time. Such a planning problem can be solved by using a standard reactive synthesis method such as [12]. Particularly, there is a rich literature focused on synthesis for a fragment of linear temporal logic (LTL), i.e., *Generalized Reactivity 1* (GR(1)) in [13]-[16].

In this paper, we propose a technique to solve the

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approximate consensus problem in MAS by using reactive synthesis in high-level mission specifications, which can encode the stabilizing dwell-time conditions derived from low-level dynamics to ensure system stability. The synthesized planning strategy enables the relay agents to determine the next servicing agent based on the states of real-time execution. The developed approach is scalable to accommodate more explorer agents by incorporating more relay agents and sub-regions. The high-level strategy planning and low-level control design combination enables the MAS to reach approximate consensus under intermittent communication. Additionally, the strategy planning is flexible to adapt to exogenous disturbances, i.e., when an explorer agent leaves a certain region, the relay agents can transfer servicing responsibilities and switch to corresponding strategies. A simulation study is performed to demonstrate the performance of the developed technique.

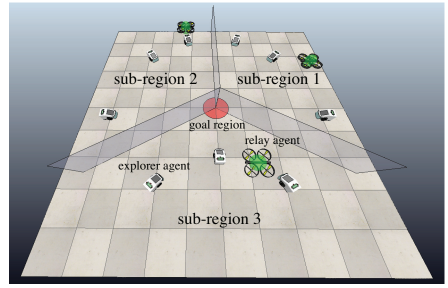


Figure 1: An illustrative example of an MAS consisting of three relay agents (represented by quadcopters) in three different sub-regions (separated by virtual walls) to regulate nine explorer agents (represented by ground robots) to a goal region (represented by a red circle).

II. PRELIMINARIES

Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For $p,q \in \mathbb{Z}_{>0}$, the $p\times q$ zero matrix and the $p\times 1$ zero column vector are denoted by $0_{p\times q}$ and 0_p, respectively. The $p\times p$ identity matrix is denoted by I_p. The maximum singular value of $(\cdot)$ is denoted as $S_{\max}(\cdot)$. The maximum and minimum eigenvalues of a symmetric matrix $G \in \mathbb{R}^{p\times p}$ are denoted by $\lambda_{\max}(G) \in \mathbb{R}$ and $\lambda_{\min}(G) \in \mathbb{R}$, respectively.

III. PROBLEM FORMULATION

A. Problem Statement

Consider a MAS consisting of M relay agents indexed by a set of leaders $L \triangleq \{1,2,...,M\}$ and N explorer agents indexed by a set of followers $F \triangleq \{1,2,...,N\}$ for some $M,N \in \mathbb{Z}_{>0}$, where $M < N$.[1] The MAS is operating within a region denoted by $\mathbb{R}^z$, where $z \in \mathbb{Z}_{>0}$. Within the operating region, the explorer agents lack absolute positional information, while the relay agents have absolute sensing (e.g., GPS). Let $x_i : [0,\infty) \to \mathbb{R}^l$ and $x_j : [0,\infty) \to \mathbb{R}^m$ denote the state of relay agent i and explorer agent j, respectively, where $i \in L$, $j \in F$, and $l,m \in \mathbb{Z}_{>0}$. The objective is to approximately regulate states of the explorer agents within a goal region centered at $g \in \mathbb{R}^z$ with radius $R_g \in \mathbb{R}_{>0}$.

Assumption 1: The entire operating region $\mathbb{R}^z \triangleq \bigcup_{i\in L} S_i$ can be partitioned into M number of sub-regions, and each sub-region is defined by a compact set $S_i \subset \mathbb{R}^z$, where i denotes the index of the corresponding sub-region. The number of sub-regions is equal to the number of relay agents. **Assumption 2:** Each relay agent $i \in L$ is responsible for servicing the explorer agents $j \in F$ within the sub-region S_i for all $t \in [0,\infty)$.

We are interested in *designing a strategy* for the relay agents to service the explorer agents for them to reach approximate consensus. The relay agents cannot control the actions of the explorer agents or the other relay agents. Hence, we represent each relay agent i a *reactive system* in an uncontrolled environment. Formally, we define a finite set $I_i \triangleq \{\mu_{i,1},\ldots,\mu_{i,a}\}$ of *atomic propositions* or Boolean *inputs*, controlled by the environment, and a finite set $O_i \triangleq \{\nu_{i,1},\ldots,\nu_{i,b}\}$ of Boolean *outputs*, controlled by the relay agent i, where $a,b \in \mathbb{Z}_{>0}$. Together, they define the reactive system's input alphabet $\Sigma_{I,i} \triangleq 2^{I_i}$ and the output alphabet $\Sigma_{O,i} \triangleq 2^{O_i}$. We define $\Sigma_i \triangleq \Sigma_{I,i} \times \Sigma_{O,i}$. Informally, we model the status of the environment as observed as agent i's physical sensors by the valuations of the atomic propositions in set I_i. Similarly, we model the actions and state of relay agent i by the valuations of the atomic propositions in set O_i.

We represent the interaction between relay agent i and the uncontrolled environment as a two-player game. Formally, the game is played on a *game structure* which is a tuple $\mathcal{G}_i = (Q_i, q_0, \Sigma_i, \delta_i)$, where Q_i is a finite set of states and $q_0 \in Q_i$ is the initial state, $\Sigma_i = \Sigma_{I,i} \times \Sigma_{O,i}$ is the alphabet of actions available to the environment and the agent, respectively, and $\delta_i : Q_i \times \Sigma_i \to Q_i$ is a complete transition function, that maps each state, input (environment action) and output (relay agent action) to a successor state.

In every state $q \in Q_i$ (starting with q_0), the environment chooses an input $\sigma_I \in \Sigma_{I,i}$, and then the agent chooses some output $\sigma_O \in \Sigma_{O,i}$. These choices define the next state $q' = \delta(q,(\sigma_I,\sigma_O))$, and so on. The resulting (infinite) sequence $\overline{\pi} = (q_0,\sigma_{I,0},\sigma_{O,0},q_1)(q_1,\sigma_{I,1},\sigma_{O,1},q_2)\ldots$ is called a *play*.

A *strategy for relay agent* i is a function $\rho_{O,i} : [0,\infty) \times \Sigma_{I,i} \to \Sigma_{O,i}$ which maps a prefix (the history of the play so far) and an action of the environment to an action of the relay agent.

We say the game is *winning* for the relay agent if it satisfies a *winning condition*. We consider games in which the agent has a GR(1) winning condition, which are common in a variety of practical applications. In the following, we make use of the LTL operators *next* $\bigcirc$, *always* $\Box$ and *eventually* $\Diamond$ [17].

Given a game structure $\mathcal{G}$ and a GR(1) winning condition φ for the agent, we seek to synthesize a strategy ρ for every relay agent such that for every strategy for the environment it holds that all resulting plays satisfy φ. In such cases we say that ρ *satisfies* φ, denoted $\rho \models \varphi$. The strategy synthesis problem for GR(1) winning conditions was solved in [12].

[1] In this paper, we are interested in the scenario where the number of explorer agents is greater than the number of relay agents.

B. Agent Dynamics

Let $y_i, y_j : [0,\infty) \to \mathbb{R}^z$ denote the position of relay agent i and explorer agent j, respectively. The linear time-invariant dynamics of relay agent i and explorer agent j are

$$\begin{aligned}\dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t), & (1)\\ y_i(t) &= C_i x_i(t), & (2)\\ \dot{x}_j(t) &= A x_j(t) + B u_j(t) + d_j(t), & (3)\\ y_j(t) &= C x_j(t), & (4)\end{aligned}$$

where $A_i \in \mathbb{R}^{l\times l}$, $A \in \mathbb{R}^{m\times m}$, $B_i \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{m\times n}$, $C_i \in \mathbb{R}^{z\times l}$, and $C \in \mathbb{R}^{z\times m}$ are known system matrices, and $n \in \mathbb{Z}_{>0}$. In (1) and (3), $u_i, u_j : [0,\infty) \to \mathbb{R}^n$ denote the control input of relay agent i and explorer agent j, respectively, and $d_j : [0,\infty) \to \mathbb{R}^m$ denotes an exogenous disturbance acting on explorer agent j. The disturbance term in (3) could destabilize the dynamics due to the divergence of the local state feedback from the absolute coordinate information during dead-reckoning periods.

IV. Control Objective

To quantify the objective, let the tracking error $e_j : [0,\infty) \to \mathbb{R}^m$ of explorer agent j be defined as

$$e_j(t) \triangleq x_g - x_j(t), \tag{5}$$

where $x_g \in \mathbb{R}^m$ denotes a predetermined user-selected state. To facilitate the subsequent analysis, define the state estimation error $e_{1,j} : [0,\infty) \to \mathbb{R}^m$ and the estimated tracking error $e_{2,j} : [0,\infty) \to \mathbb{R}^m$ as

$$\begin{aligned}e_{1,j}(t) &\triangleq \hat{x}_j(t) - x_j(t), & (6)\\ e_{2,j}(t) &\triangleq x_g - \hat{x}_j(t), & (7)\end{aligned}$$

respectively, where $\hat{x}_j : [0,\infty) \to \mathbb{R}^m$ denotes the estimate of x_j. Using (6) and (7), (5) can also be expressed as

$$e_j(t) = e_{1,j}(t) + e_{2,j}(t). \tag{8}$$

To facilitate the stability analysis of the relay agents, we define the leader tracking error $e_{3,j} : [0,\infty) \to \mathbb{R}^z$ as

$$e_{3,j}(t) \triangleq C\hat{x}_j(t) - C_i x_i(t). \tag{9}$$

Assumption 3: The state estimate of explorer agent $\hat{x}_j(t)$ is initialized as $\hat{x}_j(0) = x_j(0)$ for all $j \in F$. **Assumption 4:** The initial position of explorer agent $x_j(0)$ is known to the corresponding relay agent $i \in L$ for all $j \in F$. **Assumption 5:** The exogenous disturbance d_j is continuous and bounded, i.e., $\|d_j(t)\| \leq \overline{d}_j$ for all $t \in [0,\infty)$, where $\overline{d}_j \in \mathbb{R}_{>0}$ is a known constant and $\|\cdot\|$ denotes the Euclidean norm. **Assumption 6:** The system matrices B_i and C_i are full-row rank matrices for all $t \in [0,\infty)$, $i \in L$. The right pseudo inverses of B_i and C_i are denoted by B_i^+ and C_i^+, respectively, where $B_i^+ \triangleq B_i^{\mathrm{T}}\left(B_i B_i^{\mathrm{T}}\right)^{-1}$ and $C_i^+ \triangleq C_i^{\mathrm{T}}\left(C_i C_i^{\mathrm{T}}\right)^{-1}$.

Problem 1. Given the system dynamics described in (1)-(4) for a sub-region S_i, the control objective is to design controllers $u_j(t)$ and observers $\hat{x}_j(t)$ for the explorer agents, and design controllers $u_i(t)$ for the relay agents to satisfy the following properties. *Stability*: The error signals $e_{1,j}(t)$, $e_{2,j}(t)$ and $e_{3,j}(t)$ are bounded for each $j \in F$ within the sub-region S_i. *Approximate Consensus:* The states of all the explorer agents within the sub-region S_i reach approximate consensus within the goal region centered at g with radius R_g.

The above problem can be solved by using the combination of the high-level synthesis planning and low-level control design. We first design the controllers and observers for the explorer and relay agents, and derive the corresponding stability conditions required to reach approximate consensus. We incorporate the required stabilizing maximum-dwell time conditions to the synthesis of correct-by-construction strategy planning, which provides the next servicing explorer agents to relay agents.

A. Approximate Consensus

A goal region centered at the position denoted by $g \triangleq Cx_g \in \mathbb{R}^z$ with radius R_g is capable of providing state information to each explorer agent $j \in F$ once $\|Cx_j(t) - Cx_g\| \leq R_g$. The task of the relay agents is to service each explorer agent by providing state, i.e., position and velocity information while the explorer agents navigating to g under the intermittent state feedback provided by the relay agents. Let $R \in \mathbb{R}_{>0}$ denote the communication radius of the relay agents and explorer agents, and relay agent i and explorer agent j can communicate if and only if $\|y_i(t) - y_j(t)\| \leq R$. Let $R_g = R$ for simplicity of exposition and without loss of generality. Given the tracking error in (5), approximate consensus is achieved within the goal region whenever

$$\limsup_{t\to\infty} \|e_j(t)\| \leq \frac{R}{S_{\max}(C)}\ \forall j \in F.$$

Given an integer $K \in \mathbb{Z}_{\geq 0}$, an explorer agent j is in the sub-region S_i at time $t + K$ if its estimated position $C\hat{x}_j \in S_i$ at time $t + K$. We define the function $\eta_i^K : [0,\infty) \to 2^F$ that determines when outputs the subset of explorer agents will be within the sub-region S_i in K time steps for some $K \geq 0$. Put simply, $\eta_i^K(t)$ will output the set of explorer agents $F_i \subseteq F$ whose *estimated state* is in sub-region S_i at time $t + K$. If the estimated trajectory of an explorer agent crosses the boundary of a sub-region in less than $t + K$ steps, the relay agent will communicate with the neighboring relay agent to notify the crossing action, hand-over the servicing responsibility, and transfer the last serviced position of the explorer agent. The parameter K is a user-defined time parameter to allow relay agents to conduct the hand-over without violating the dwell-time condition. This forms an *assume-guarantee* contract between relay agents and we formalize this notion in Section VII. Note that partitioning the region for optimal distribution of relay and explorer agents (such as minimizing boundary crossings and hand-overs) is an active area of current interest. In this paper, we manually partitioned the operating region into three sub-regions for simplicity.

Let $\zeta_i : [0,\infty) \to F$ be a piece-wise constant switching signal that determines which explorer the relay agent

i is to service within the sub-region S_i. At $t = 0$, relay agent i will compute the servicing time of each explorer agent j as denoted by t_s^j, where s indicates the s^{th} servicing instance. Immediately after $t = 0$, relay agent i will maneuver towards explorer agent j, where j is dictated by ζ_i and provide state information once $\|y_i(t) - y_j(t)\| \leq R$. Hence, the $(s+1)^{\text{th}}$ servicing time for explorer agent j is defined as $t_{s+1}^j \triangleq \inf\left\{t \geq t_s^j : (\|y_i(t) - y_j(t)\| \leq R) \wedge (\zeta_i(t) = j)\right\}$, where $\wedge$ denotes the conjunction logical connective.[2] Let $\left\{t_s^j\right\}_{s=0}^{\infty} \subset \mathbb{R}$ be an increasing sequence of servicing times determined by the subsequently defined maximum dwell-time condition (see Theorem 1) for explorer agent j. The servicing time in t_{s+1}^j defines the necessary conditions to enable communication between the relay agent i and explorer agent j. Nonetheless, the maximum dwell-time condition provides an upper bound on the servicing time based on the need to ensure stability as subsequently shown.

At time t_s^j, $\left\|y_i\left(t_s^j\right) - y_j\left(t_s^j\right)\right\| \leq R$, where the relay agent i will service explorer agent j and compute the future servicing time t_{s+1}^j. Immediately after t_s^j, the relay agent i will leave explorer agent j to go service other explorers. Let t_r^j denote the time the relay agent i begins maneuvering towards explorer agent j, where $t_r^j \triangleq \inf\left\{t \geq t_s^j : (\|y_i(t) - y_j(t)\| > R) \wedge (\zeta_i(t) = j)\right\}$. Proper design of ζ_i requires $t_r^j < t_{s+1}^j$ for the relay agent i to satisfy the maximum dwell-time condition. Let $\left\{t_r^j\right\}_{r=0}^{\infty} \subset \mathbb{R}$ be an increasing sequence of return times for explorer agent j. Note that one of the contributions of this work is to provide a *scalable* and *provably correct* method to compute $\zeta_i(t)$ for all relay agents $i \in L$. We detail this process in Section VII.

V. OBSERVER AND CONTROLLER DEVELOPMENT

The state estimate of explorer agent $j \in F$ is obtained from the following model-based observer

$$\dot{\hat{x}}_j(t) \triangleq -Ae_{2,j}(t) + Bu_j(t), \ t \in \left[t_s^j, t_{s+1}^j\right), \quad (10)$$

$$\hat{x}_j\left(t_s^j\right) \triangleq x_j\left(t_s^j\right), \quad (11)$$

where the position estimate $\hat{y}_j : [0, \infty) \to \mathbb{R}^z$ of explorer agent j can be modeled as

$$\hat{y}_j(t) \triangleq C\hat{x}_j(t). \quad (12)$$

The control input of explorer agent j is designed as

$$u_j(t) \triangleq B^{\mathrm{T}} P e_{2,j}(t), \quad (13)$$

where $P \in \mathbb{R}^{m \times m}$ is the positive definite solution to the Algebraic Riccati Equation (ARE) given by

$$A^{\mathrm{T}}P + PA - 2PBB^{\mathrm{T}}P + k_{ARE}I_m = 0_{m \times m} \quad (14)$$

such that $k_{ARE} > 0$ is a user-defined parameter. The control input of relay agent i is designed as

$$\begin{aligned} u_i(t) \ \triangleq \ & B_i^+ C_i^+ \left(-C_i A_i x_i(t) + k_i(t) e_{3,j}(t)\right) \\ & + B_i^+ C_i^+ C\left(-Ae_{2,j}(t) + Bu_j(t)\right), \end{aligned} \quad (15)$$

[2]For $s = 0$, t_0^j is taken to be the initial time, e.g., $t_0^j = 0$.

where $k_i : [0, \infty) \to \mathbb{R}_{>0}$ is a subsequently defined piecewise constant parameter. Substituting (3), (6), (7), (10), and (11) into the time derivative of (6) yields

$$\dot{e}_{1,j}(t) = Ae_{1,j}(t) - Ax_g - d_j(t), \ t \in \left[t_s^j, t_{s+1}^j\right), \quad (16)$$

$$e_{1,j}\left(t_s^j\right) = 0_m. \quad (17)$$

Substituting (10), (11), and (13) into the time derivative of (7) yields

$$\dot{e}_{2,j}(t) = \left(A - BB^{\mathrm{T}}P\right) e_{2,j}(t), \ t \in \left[t_s^j, t_{s+1}^j\right), \quad (18)$$

$$e_{2,j}\left(t_s^j\right) = x_g - x_j\left(t_s^j\right). \quad (19)$$

Substituting (3), (8), and (13) into the time derivative of (5) yields

$$\begin{aligned} \dot{e}_j(t) \ = \ & \left(A - BB^{\mathrm{T}}P\right) e_j(t) + BB^{\mathrm{T}} P e_{1,j}(t) \\ & - Ax_g - d_j(t). \end{aligned} \quad (20)$$

Substituting (1), (10), (15) into the time derivative of (9) yields

$$\begin{aligned} \dot{e}_{3,j}(t) = \ & C\left(-Ae_{2,j}(t) + Bu_j(t)\right) \\ & - C_i\left(A_i x_i(t) + B_i u_i(t)\right), \ t \in \left[t_s^j, t_r^j\right) \end{aligned} \quad (21)$$

$$e_{3,j}\left(t_s^j\right) = Cx_j\left(t_s^j\right) - C_i x_i\left(t_s^j\right) \quad (22)$$

and

$$\dot{e}_{3,j}(t) = -k_i(t) e_{3,j}(t), \ t \in \left[t_r^j, t_{s+1}^j\right) \quad (23)$$

$$e_{3,j}\left(t_r^j\right) = C\hat{x}_j\left(t_r^j\right) - C_i x_i\left(t_r^j\right). \quad (24)$$

VI. STABILITY CONDITIONS

In this section, we provide conditions that generate a stable switched system for each sub-region, and then prove approximate consensus for the corresponding explorer agents within the sub-region. When explorer agents cross boundaries, the synthesized strategies are changed for the relay agents to adapt to the different number of explorer agents within sub-regions. Eventually, all the explorer agents in the entire operating region reach approximate consensus within the goal region centered at g with radius R_g. Specifically, Theorem 1 presents the maximum dwell-time condition the relay agent i has to satisfy to ensure the state estimation error $e_{1,j}(t)$ is bounded for all $t \in \left[t_s^j, t_{s+1}^j\right]$. Theorem 2 shows the observer in (10) and controller in (13) ensure the estimated tracking error $e_{2,j}(t)$ is exponentially regulated for all $t \in \left[t_s^j, t_{s+1}^j\right)$ when the ARE in (14) is satisfied. Theorem 3 indicates the observer in (10) and controller in (13) ensure the tracking error $e_j(t)$ is uniformly ultimately bounded (UUB) provided the relay agent i satisfies the maximum dwell-time condition in (25) and $e_{1,j}\left(t_0^j\right) = 0_m$. Theorem 4 provides a sufficient gain condition to enable timely servicing by the relay agent i, and shows the leader tracking error $e_{3,j}(t)$ is bounded for $t \in \left[t_s^j, t_{s+1}^j\right]$. Theorem 5 shows when the GR(1) specifications for relay agents described in (54) are satisfied, the observer in (10), the controllers in (13) and (15) enable the explorer agents reach approximate consensus within the goal region.

A. Explorer Agent Analysis

To demonstrate the tracking error $e_j(t)$ is bounded for the explorer agent j, we provide three theorems. The following theorem proves the state estimation error $e_{1,j}(t)$ is bounded for all $t \in \left[t_s^j, t_{s+1}^j\right]$.

Theorem 1. *When the relay agent i satisfies the maximum dwell-time condition given by*

$$T_j \triangleq t_{s+1}^j - t_s^j \leq \frac{1}{S_{\max}(A)} \ln\left(\frac{V_T S_{\max}(A)}{\kappa_j} + 1\right), \quad (25)$$

where $T_j \in \mathbb{R}_{>0}$ denotes the maximum dwell-time for explorer agent j, $V_T \in \left(0, \frac{R}{S_{\max}(C)}\right)$ is a user-defined parameter, $\kappa_j \triangleq S_{\max}(A)\,\overline{x}_g + \overline{d}_j \in \mathbb{R}_{>0}$, $\overline{x}_g \in \mathbb{R}_{>0}$ is a bounding constant such that $\|x_g\| \leq \overline{x}_g$, then $\|e_{1,j}(t)\| \leq V_T$ for all $t \in \left[t_s^j, t_{s+1}^j\right]$.

Proof: Let $t \geq t_s^j$, and suppose $\left\|e_{1,j}\left(t_s^j\right)\right\| = 0$.[3] Consider the common Lyapunov-like functional candidate $V_{1,j} : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ defined as

$$V_{1,j}(e_{1,j}(t)) \triangleq \frac{1}{2} e_{1,j}^{\mathrm{T}}(t)\, e_{1,j}(t). \quad (26)$$

Substituting the closed-loop error system (16) into the time derivative of (26) yields

$$\dot{V}_{1,j}(e_{1,j}(t)) = e_{1,j}^{\mathrm{T}}(t)\left(A e_{1,j}(t) - A x_g - d_j(t)\right). \quad (27)$$

Using the definition of κ_j in (25), (27) can be upper bounded by

$$\dot{V}_{1,j}(e_{1,j}(t)) \leq S_{\max}(A)\,\|e_{1,j}(t)\|^2 + \kappa_j \|e_{1,j}(t)\|. \quad (28)$$

Substituting (26) into (28) yields

$$\begin{aligned}\dot{V}_{1,j}(e_{1,j}(t)) \quad &\leq \quad 2 S_{\max}(A)\, V_{1,j}(e_{1,j}(t)) \\ &\quad + \kappa_j \sqrt{2 V_{1,j}(e_{1,j}(t))}. \end{aligned} \quad (29)$$

Invoking the Comparison Lemma in [18, Lemma 3.4] on (29) over $\left[t_s^j, t_{s+1}^j\right)$ yields

$$V_{1,j}(e_{1,j}(t)) \leq \frac{\kappa_j^2}{2 S_{\max}^2(A)} \left(\exp\left(S_{\max}(A)\left(t - t_s^j\right)\right) - 1\right)^2. \quad (30)$$

Substituting (26) into (30) yields

$$\|e_{1,j}(t)\| \leq \frac{\kappa_j}{S_{\max}(A)} \left(\exp\left(S_{\max}(A)\left(t - t_s^j\right)\right) - 1\right). \quad (31)$$

Define $\Phi_j : \left[t_s^j, t_{s+1}^j\right) \to \mathbb{R}$ as

$$\Phi_j(t) \triangleq \frac{\kappa_j}{S_{\max}(A)} \left(\exp\left(S_{\max}(A)\left(t - t_s^j\right)\right) - 1\right). \quad (32)$$

Since $\|e_{1,j}(t)\| \leq \frac{\kappa_j}{S_{\max}(A)} \left(\exp\left(S_{\max}(A)\left(t - t_s^j\right)\right) - 1\right)$ for all $t \in \left[t_s^j, t_{s+1}^j\right)$ and $\left\|e_{1,j}\left(t_{s+1}^j\right)\right\| = 0$, where $t_{s+1}^j > t_s^j$ and $\Phi_j\left(t_{s+1}^j\right) > 0$, therefore $\|e_{1,j}(t)\| \leq \Phi_j(t)$ for all $t \in \left[t_s^j, t_{s+1}^j\right]$. If $\Phi_j\left(t_{s+1}^j\right) \leq V_T$, then $\|e_{1,j}(t)\| \leq V_T$ for all $t \in \left[t_s^j, t_{s+1}^j\right]$. In addition, $\Phi_j\left(t_{s+1}^j\right) \leq V_T$ yields the maximum dwell-time condition in (25). Therefore, $\|e_{1,j}(t)\| \leq V_T$ for all $t \in \left[t_s^j, t_{s+1}^j\right]$ provided $\left\|e_{1,j}\left(t_s^j\right)\right\| = 0$ and (25) hold. ∎

[3] $\left\|e_{1,j}\left(t_s^j\right)\right\| = 0$ because relay agent i serviced explorer agent j at time t_s^j.

Next, we show the estimated tracking error $e_{2,j}(t)$ is exponentially regulated for all $t \in \left[t_s^j, t_{s+1}^j\right)$.

Theorem 2. *If the ARE in (14) is satisfied, then the observer in (10) and controller in (13) ensure the estimated tracking error in (7) is exponentially regulated in the sense that*

$$\|e_{2,j}(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|e_{2,j}(t_s^j)\| \exp\left(-\frac{k_{ARE}}{2\lambda_{\max}(P)}\left(t - t_s^j\right)\right) \quad (33)$$

for all $t \in \left[t_s^j, t_{s+1}^j\right)$ and each servicing instance $s \in \mathbb{Z}$.

Proof: Consider the common Lyapunov functional $V_{2,j} : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ defined as

$$V_{2,j}(e_{2,j}(t)) \triangleq e_{2,j}^{\mathrm{T}}(t)\, P e_{2,j}(t). \quad (34)$$

By the Rayleigh quotient, (34) can be bounded as

$$\lambda_{\min}(P)\|e_{2,j}(t)\|^2 \leq V_{2,j}(e_{2,j}(t)) \leq \lambda_{\max}(P)\|e_{2,j}(t)\|^2. \quad (35)$$

Substituting the closed-loop error system (18) into the time derivative of (34) yields

$$\dot{V}_{2,j}(e_{2,j}(t)) = e_{2,j}^{\mathrm{T}}(t)\left(A^{\mathrm{T}}P + PA - 2PBB^{\mathrm{T}}P\right) e_{2,j}(t). \quad (36)$$

Using (14), (36) can be obtained as

$$\dot{V}_{2,j}(e_{2,j}(t)) = -k_{ARE}\|e_{2,j}(t)\|^2. \quad (37)$$

Using (35) in (37) yields

$$\dot{V}_{2,j}(e_{2,j}(t)) \leq -\frac{k_{ARE}}{\lambda_{\max}(P)} V_{2,j}(e_{2,j}(t)). \quad (38)$$

Invoking the Comparison Lemma in [18, Lemma 3.4] on (38) over $\left[t_s^j, t_{s+1}^j\right)$ and substituting in (35) yields (33). ∎

Using the relationship described in (8), and results from Theorems 1 and 2, the following theorem shows the tracking error $e_j(t)$ is bounded.

Theorem 3. *If the relay agent i satisfies the maximum dwell-time condition in (25) for each $s \in \mathbb{Z}$ and $e_{1,j}\left(t_0^j\right) = 0_m$, then the observer in (10) and controller in (13) ensure the tracking error in (5) is UUB in the sense that*

$$\begin{aligned}\|e_j(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|e_j(0)\| \exp\left(-\frac{k_{ARE}}{2\lambda_{\max}(P)} t\right) + \\ \frac{c}{k_{ARE}} \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \left(1 - \exp\left(-\frac{k_{ARE}}{2\lambda_{\max}(P)} t\right)\right), \end{aligned} \quad (39)$$

where $c \triangleq 2V_T S_{\max}\left(PBB^T P\right) + 2\overline{x}_g S_{\max}(PA) + 2\overline{d}_j S_{\max}(P) \in \mathbb{R}_{>0}$ is a known constant.

Proof: Suppose the relay agent i satisfies the maximum dwell-time condition in (25) for each $s \in \mathbb{Z}$ and $e_{1,j}\left(t_0^j\right) = 0_m$. Consider the common Lyapunov functional $V_j : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ defined as

$$V_j\left(e_j\left(t\right)\right) \triangleq e_j^{\mathrm{T}}\left(t\right) P e_j\left(t\right). \tag{40}$$

By the Rayleigh quotient, (40) can be bounded as

$$\lambda_{\min}\left(P\right)\left\|e_j\left(t\right)\right\|^2 \leq V_j\left(e_j\left(t\right)\right) \leq \lambda_{\max}\left(P\right)\left\|e_j\left(t\right)\right\|^2. \tag{41}$$

Substituting the closed-loop error system (20) into the time derivative of (40) yields

$$\begin{aligned}\dot{V}_j(e_j(t)) = {} & 2e_j^{\mathrm{T}}(t)P\big(BB^{\mathrm{T}}Pe_{1,j}(t) - Ax_g - d_j(t)\big) \\ & + e_j^{\mathrm{T}}(t)\big(A^{\mathrm{T}}P + PA - 2PBB^{\mathrm{T}}P\big)e_j(t).\end{aligned} \tag{42}$$

Using (14), (42) can be upper bounded as

$$\begin{aligned}\dot{V}_j\left(e_j\left(t\right)\right) \leq {} & -k_{ARE}\left\|e_j\left(t\right)\right\|^2 + 2S_{\max}\left(P\right)\left\|e_j\left(t\right)\right\|\overline{d}_j \\ & + 2S_{\max}\left(PBB^{\mathrm{T}}P\right)\left\|e_j\left(t\right)\right\|\left\|e_{1,j}\left(t\right)\right\| \\ & + 2S_{\max}\left(PA\right)\left\|e_j\left(t\right)\right\|\overline{x}_g.\end{aligned} \tag{43}$$

Since the relay agent i satisfies the maximum dwell-time condition in (25) for each $s \in \mathbb{Z}$, $\left\|e_{1,j}\left(t\right)\right\| \leq V_T$ for all $t \in [0, \infty)$ by Theorem 1. Using the definition for c, (43) can be upper bounded as

$$\dot{V}_j\left(e_j\left(t\right)\right) \leq -k_{ARE}\left\|e_j\left(t\right)\right\|^2 + c\left\|e_j\left(t\right)\right\|. \tag{44}$$

Invoking the Comparison Lemma in [18, Lemma 3.4] on (44) over $[0, \infty)$ and substituting in (41) yields (39). Note that (39) implies $e_j\left(t\right) \in \mathcal{L}_\infty$. Since $e_j\left(t\right) \in \mathcal{L}_\infty$ and $e_{1,j}\left(t\right) \in \mathcal{L}_\infty$ given the relay agent i satisfies the maximum dwell-time condition in (25) for each $s \in \mathbb{Z}$, (5) and (8) imply $x_j\left(t\right), e_{2,j}\left(t\right) \in \mathcal{L}_\infty$. Since $x_j\left(t\right), e_{1,j}\left(t\right), e_{2,j}\left(t\right) \in \mathcal{L}_\infty$, (6) and (13) imply $\hat{x}_j\left(t\right), u_j\left(t\right) \in \mathcal{L}_\infty$ provided B and P are constant matrices. Hence, $\dot{x}_j\left(t\right), y_j\left(t\right) \in \mathcal{L}_\infty$ by (3) and (4). Since $e_{1,j}\left(t\right), e_{2,j}\left(t\right), e_j\left(t\right), u_j\left(t\right), \hat{x}_j\left(t\right) \in \mathcal{L}_\infty$, (10), (12), (16), (18), (20) imply $\dot{\hat{x}}_j\left(t\right), \hat{y}_j\left(t\right), \dot{e}_{1,j}\left(t\right), \dot{e}_{2,j}\left(t\right), \dot{e}_j\left(t\right) \in \mathcal{L}_\infty$. ■

Remark 1. From Theorem 3, note that

$$\limsup_{t\to\infty}\left\|e_j\left(t\right)\right\| \leq \frac{c}{k_{ARE}}\frac{\lambda_{\max}\left(P\right)}{\lambda_{\min}\left(P\right)} \triangleq \gamma\left(c\right), \tag{45}$$

where $\gamma\left(c\right)$ can be made arbitrarily small by selecting a small c, i.e., selecting a small V_T and setting the desired state as the origin.

B. Relay Agent Analysis

To prove the leader tracking error $e_{3,j}\left(t\right)$ is bounded for $t \in \left[t_s^j, t_{s+1}^j\right]$, we provide the following theorem.

Theorem 4. *If $\left\|y_i\left(t_r^j\right) - y_j\left(t_r^j\right)\right\| > R$, then the controller of the relay agent i in (15) can satisfy the maximum dwell-time condition in (25) for explorer agent j provided*

$$k_i\left(t\right) \geq \frac{1}{\left(t_{s+1}^j - t_r^j\right)}\ln\left(\frac{\left\|e_{3,j}\left(t_r^j\right)\right\|}{R - S_{\max}\left(C\right)V_T}\right) \tag{46}$$

for all $t \in \left[t_r^j, t_{s+1}^j\right)$, where $k_i\left(t\right)$ is a piece-wise constant. In addition, the leader tracking error in (9) is bounded for $t \in \left[t_s^j, t_{s+1}^j\right]$.

Proof: Consider the common Lyapunov functional candidate $V_{3,j} : \mathbb{R}^z \to \mathbb{R}_{\geq 0}$ defined as

$$V_{3,j}\left(e_{3,j}\left(t\right)\right) \triangleq \frac{1}{2}e_{3,j}^{\mathrm{T}}\left(t\right)e_{3,j}\left(t\right). \tag{47}$$

Substituting the closed-loop error system (23) when $t \in \left[t_r^j, t_{s+1}^j\right)$ into the time derivative of (47) yields

$$\dot{V}_{3,j}\left(e_{3,j}\left(t\right)\right) = -k_i\left(t\right)e_{3,j}^{\mathrm{T}}\left(t\right)e_{3,j}\left(t\right), \tag{48}$$

where $k_i\left(t\right)$ is constant over $\left[t_r^j, t_{s+1}^j\right)$. Substituting (47) into (48) yields

$$\dot{V}_{3,j}\left(e_{3,j}\left(t\right)\right) = -2k_i\left(t\right)V_{3,j}\left(e_{3,j}\left(t\right)\right). \tag{49}$$

Invoking the Comparison Lemma in [18, Lemma 3.4] on (49) over $\left[t_r^j, t_{s+1}^j\right)$ and substituting in (47) yields

$$\left\|e_{3,j}\left(t\right)\right\| = \left\|e_{3,j}\left(t_r^j\right)\right\|\exp\left(-k_i\left(t\right)\left(t - t_r^j\right)\right). \tag{50}$$

Consider $t \in \left[t_r^j, t_{s+1}^j\right)$, the jump discontinuity of $e_{3,j}\left(t\right)$ at t_{s+1}^j is given by $\Psi_j\left(t_{s+1}^j\right) \triangleq e_{3,j}\left(t_{s+1}^j\right) - \lim_{t\to\left(t_{s+1}^j\right)^-} e_{3,j}\left(t\right) \in \mathbb{R}^z$, where $e_{3,j}\left(t_{s+1}^j\right)$ is defined by (22) and $\lim_{t\to\left(t_{s+1}^j\right)^-} e_{3,j}\left(t\right)$ denotes the limit of $e_{3,j}\left(t\right)$ as $t \to t_{s+1}^j$ from the left. Since $\Psi_j\left(t_{s+1}^j\right) = -\lim_{t\to\left(t_{s+1}^j\right)^-} Ce_{1,j}\left(t\right)$, then by Theorem 1 $\left\|\Psi_j\left(t_{s+1}^j\right)\right\| \leq S_{\max}\left(C\right)V_T$. It then follows that the magnitude of the jump discontinuity is bounded by

$$\left|\left\|e_{3,j}\left(t_{s+1}^j\right)\right\| - \lim_{t\to\left(t_{s+1}^j\right)^-}\left\|e_{3,j}\left(t\right)\right\|\right| \leq S_{\max}\left(C\right)V_T. \tag{51}$$

Communication between the relay agent i and explorer agent j occurs when $\left\|y_i\left(t\right) - y_j\left(t\right)\right\| \leq R$ where $\left\|y_i\left(t\right) - y_j\left(t\right)\right\| \leq S_{\max}\left(C\right)\left\|e_{1,j}\left(t\right)\right\| + \left\|e_{3,j}\left(t\right)\right\|$. Therefore, $\left\|y_i\left(t_{s+1}^j\right) - y_j\left(t_{s+1}^j\right)\right\| \leq R$ can be ensured provided $S_{\max}\left(C\right)\left\|e_{1,j}\left(t_{s+1}^j\right)\right\| + \left\|e_{3,j}\left(t_{s+1}^j\right)\right\| \leq R$. From Theorem 1, $\left\|e_{1,j}\left(t_{s+1}^j\right)\right\| \leq V_T$. Using (50) and (51), it follows that $S_{\max}\left(C\right)\left\|e_{1,j}\left(t_{s+1}^j\right)\right\| + \left\|e_{3,j}\left(t_{s+1}^j\right)\right\| \leq S_{\max}\left(C\right)V_T + \left\|e_{3,j}\left(t_r^j\right)\right\|\exp\left(-k_i\left(t\right)\left(t_{s+1}^j - t_r^j\right)\right) \leq R$ provided (46) holds. To ensure $k_i\left(t\right)$ for $t \in \left[t_r^j, t_{s+1}^j\right)$ is well-defined, V_T must be selected such that $\left\|e_{3,j}\left(t_r^j\right)\right\| > R - S_{\max}\left(C\right)V_T > 0$. Note that if $0 < \left\|e_{3,j}\left(t_r^j\right)\right\| \leq R - S_{\max}\left(C\right)V_T$, then $S_{\max}\left(C\right)\left\|e_{1,j}\left(t_r^j\right)\right\| + \left\|e_{3,j}\left(t_r^j\right)\right\| \leq S_{\max}\left(C\right)V_T + R - S_{\max}\left(C\right)V_T \leq R$ provided $V_T \in \left(0, \frac{R}{S_{\max}(C)}\right)$

and communication between the relay agent i and explorer agent j is possible without the need to maneuver the relay agent i towards explorer agent j. By (47) and (50), the leader tracking error in (9) is bounded. Since $e_{3,j}(t) \in \mathcal{L}_\infty$ and $\hat{x}_j(t) \in \mathcal{L}_\infty$ by Theorem 3, then $x_i(t) \in \mathcal{L}_\infty$. Since $x_i(t), e_{3,j}(t) \in \mathcal{L}_\infty$ and $e_{2,j}(t), u_j(t) \in \mathcal{L}_\infty$ by Theorem 3, the controller $u_i(t) \in \mathcal{L}_\infty$ by (15). Substituting (21) when $t \in \left[t_s^j, t_r^j\right)$ into the time derivative of (47) yields $\dot{V}_{3,j}(e_{3,j}(t)) = e_{3,j}^{\mathrm{T}}(t)\left(C\left(-Ae_{2,j}(t) + Bu_j(t)\right) - C_i\left(A_i x_i(t) + B_i u_i(t)\right)\right)$. From Theorem 2, $e_{2,j}(t) \in \mathcal{L}_\infty$ for $t \in \left[t_s^j, t_{s+1}^j\right)$. Since $t_r^j < t_{s+1}^j$ by design, $e_{2,j}(t) \in \mathcal{L}_\infty$, i.e., $\|e_{2,j}(t)\| \leq \overline{e}_{2,j}$ for $t \in \left[t_s^j, t_r^j\right)$, where $\overline{e}_{2,j} \in \mathbb{R}_{>0}$. Using (7), since $\|x_g\| \leq \overline{x}_g$ and $e_{2,j}(t) \in \mathcal{L}_\infty$, $\hat{x}_j(t) \in \mathcal{L}_\infty$, i.e., $\|\hat{x}_j(t)\| \leq \overline{\hat{x}}_j$ for $t \in \left[t_s^j, t_r^j\right)$, where $\overline{\hat{x}}_j \in \mathbb{R}_{>0}$. Since $u_i(t), u_j(t) \in \mathcal{L}_\infty$, then there exist $\overline{U}_i, \overline{U}_j \in \mathbb{R}_{>0}$ such that $\|u_i(t)\| \leq \overline{U}_i$ and $\|u_j(t)\| \leq \overline{U}_j$ for all t.[4] Therefore, $\dot{V}_{3,j}(e_{3,j}(t))$ can be upper bounded as

$$\dot{V}_{3,j}(e_{3,j}(t)) = S_{\max}(A_i)\|e_{3,j}(t)\|^2 + \epsilon\|e_{3,j}(t)\|, \quad (52)$$

where $\epsilon \triangleq S_{\max}(CA)\overline{e}_{2,j} + S_{\max}(CB)\overline{U}_j + S_{\max}(C_iB_i)\overline{U}_i + S_{\max}(A_i)S_{\max}(C)\overline{\hat{x}}_j \in \mathbb{R}_{>0}$ is a bounding constant. Invoking the Comparison Lemma in [18, Lemma 3.4] on (52) over $\left[t_s^j, t_r^j\right)$ and substituting in (47) yields

$$\begin{aligned}\|e_{3,j}(t)\| \leq & \frac{\epsilon}{S_{\max}(A_i)}\left(\exp\left(S_{\max}(A_i)\left(t-t_s^j\right)\right)-1\right) \\ & + \left\|e_{3,j}\left(t_s^j\right)\right\|\exp\left(S_{\max}(A_i)\left(t-t_s^j\right)\right). \quad (53)\end{aligned}$$

By (51) and (53), $e_{3,j}(t) \in \mathcal{L}_\infty$ for $t \in \left[t_s^j, t_r^j\right)$. Since $e_{3,j}(t) \in \mathcal{L}_\infty$ for $t \in \left[t_r^j, t_{s+1}^j\right)$, the leader tracking error in (9) is bounded for $t \in \left[t_s^j, t_{s+1}^j\right]$. ■

VII. SYNTHESIS

A. Strategy Synthesis

Recall that the goal of the synthesized strategy is to compute switching signal $\zeta_i(t)$ for all relay agents $i \in L$. We approach the problem using reactive synthesis as it is a natural formulation to capture any potential unknowns in the environment (such as travel time between explorer agents) as environmental inputs and still provide theoretical guarantees of correctness that the maximum dwell-time condition given in Theorem 1 for all explorer agents is satisfied. In this section, we highlight how we can use *contract-based* synthesis to decentralize the reactive synthesis problem amongst the relay agents. In other words, our method enables each relay agent to compute their own $\zeta_i(t)$ independently and in parallel.

We decentralize the problem by enforcing each relay agent to only be responsible for servicing explorer agents in its region. Each relay agent thus needs to keep track of which explorer agents it is responsible for, as well as how much time has elapsed since that agent had last been serviced. To this end, we introduce two sets of atomic propositions. First, for a relay agent i, we define a set of *service propositions* $Y_i = \{y_i^1, \ldots, y_i^N\}$ that corresponds to the explorer agents that relay agent i is currently responsible for servicing, i.e., $y_i^j = \top$ if explorer agent j is in S_i. We additionally define $service_i : [0, \infty) \to 2^{Y_i}$ which maps the history of the play so far to the set of explorer agents in the corresponding sub-region S_i. In practice, the function $\eta_i^K(t)$ outputs the set of explorer agents $F_i \subseteq F$, and $service_i$ converts F_i into valuations of the service propositions Y_i.

Second, we define the discrete time set $\mathbb{T}_d \triangleq \{t[0], t[1], \ldots\}$, where $t[h] = hT_s$ for $h \in \mathbb{I}$, $\mathbb{I} \triangleq \{0, 1, \ldots\}$ is the time index set, and $T_s \in \mathbb{R}_{>0}$ is the sampling period. Then we define the set of *timing propositions* $\mathcal{T}_i^j = \{\tau_0, \tau_1, \ldots, \tau_{T_j}\}$, where T_j denotes the maximum dwell-time defined in Theorem 1, and $\mathcal{T}_i^j$ encodes how much time explorer agent j has to be serviced before violating the dwell-time condition, i.e., $\tau_h = \top$ if explorer j has to be serviced in at most $t[h]$ time steps for the maximum dwell-time condition to be satisfied.

Formally, each relay agent i will have environment atomic propositions $E_i = Y_i \cup \left(\bigcup_{j=1}^N \mathcal{T}_i^j\right)$. The GR(1) requirements that each relay agent must satisfy are $\varphi_i = \bigwedge_{j=1}^N \left(\square\left(y_j \to \neg\tau_0\right)\right)$, where the valuation y_j is set by $service_i$. Informally, φ_i states that if explorer agent j is in S_i, then it must be serviced by relay agent i before the time left to service reaches 0 as denoted by $\tau_0 = \top$.

Each relay agent is unaware of the specification and implementation details of the other relay agents. To ensure that relay agents coordinate to satisfy their specifications, every controller must additionally satisfy *contract specifications*. These contract specifications take the form of *assume-guarantee* contracts. Informally, a relay agent gives a *guarantee* of satisfying a contract specification with all other relay agents. This guarantee is used as an assumption for the synthesis of the other relay agents' controllers and vice-versa. In this paper, we focus on providing a framework to conduct the assume-guarantee synthesis. However, in practice, the contract specifications are domain and environment-specific. We provide an example of a contract specification used to coordinate *hand-offs* used in the implementation in section VIII. Since explorer agents can enter and leave sub-regions, the currently responsible relay agent must ensure there is sufficient time for the next relay agent to service the incoming explorer agent. We denote this contract specification as ϕ_i and define it as $\phi_i = \bigwedge_{j=1}^N \left(\square\left(\left(y_j \wedge \neg\bigcirc y_j\right) \to \neg\left(\bigwedge_{h=0}^K \tau_0\right)\right)\right)$ for some user-provided integer $K \leq T_j$. This contract specification states that if explorer agent j is leaving region S_i in the next time step, it must have at least K time steps before it needs to be serviced again. This contract gives the next relay agent some buffer time to service explorer agent j when it enters the next region.

The full GR(1) specifications for relay agent i to satisfy

[4]The relay agent i executes (15) by cycling through all $j \in F$ for all t, which was shown to be bounded for each $j \in F$.

are

$$\Phi_i = \Box\Diamond\left(\bigwedge_{\alpha=1,\alpha\neq i}^{M}\phi_\alpha\right) \to \bigwedge_{j=1}^{N}\left(\Box\left(y_j \to \neg\tau_0\right)\wedge\phi_i\right). \tag{54}$$

By construction, if $\rho_i \models \Phi_i$ for all $i \in L$ then the maximum dwell-time condition for all explorer agents are satisfied and consensus is achieved.

Last, we present Theorem 5, which provides theoretical guarantees for achieving stability and approximate consensus (in Problem 1) by satisfying the full GR(1) specifications described in (54).

Theorem 5. *With the observer in (10), controllers in (13) for explorer agents, controllers in (15) for relay agents, the parameters are selected such that* $k_i(t) \geq \frac{1}{\left(t_{s+1}^j - t_r^j\right)}\ln\left(\frac{\left\|e_{3,j}\left(t_r^j\right)\right\|}{R - S_{\max}(C)V_T}\right)$, $V_T \in \left(0, \frac{R}{S_{\max}(C)}\right)$, $\gamma(c)\,S_{\max}(C) \leq R$, *Assumptions 1-6 and the GR(1) specifications for relay agents described in (54) are satisfied, then the explorer agents reach approximate consensus within the goal region in the sense that*

$$\lim_{t\to\infty}\sup\left\|e_j(t)\right\| \leq \gamma(c^*), \tag{55}$$

where $c^* = 2\overline{x}_g S_{\max}(PA) + 2\overline{d}_j S_{\max}(P)$.

Proof: From results of Theorems 1-3, the tracking error $e_j(t)$ is UUB provided the relay agent i satisfies the maximum dwell-time condition described in (25) for all $t \in \left[t_s^j, t_{s+1}^j\right]$. By satisfying the GR(1) specifications for relay agent i described in (54) for all $i \in L$, then the maximum dwell-time condition for all the explorer agents are satisfied. According to (45), $\left\|e_j(t)\right\| \leq \gamma(c)$. By satisfying $\gamma(c)\,S_{\max}(C) \leq R$, then $e_{1,j}(t) = 0_m$, and $\gamma(c)$ can be reduced to $\gamma(c^*)$. Therefore, we obtain (55). ■

VIII. SIMULATION

Two simulation examples demonstrate that the developed technique of combining the high-level strategy planning and low-level control design can regulate the explorer agents to reach approximate consensus. Specifically, Section VIII-A shows nine explorer agents originated in three different pre-defined sub-regions (divided by functions $X = 0$, $\sqrt{3}X - 3Y = 0$ and $\sqrt{3}X + 3Y = 0$ in the Cartesian coordinate system) that are serviced by three relay agents for state corrections. Each of the three relay agents is responsible for servicing the corresponding three explorer agents within its sub-region, and the nine explorer agents reach a goal region centered at $g \triangleq [0, 0] \in \mathbb{R}^2$ with radius R. To demonstrate the developed method requires less control effort and can be used in a distributed fashion, we provide the following two baseline methods for comparison. We use the *round-robin* scheduler for the relay agents to service certain explorer agents while satisfying the maximum dwell-time conditions. We also conduct a centralized reactive synthesis planning to compare to the developed distributed strategy planning.

To further demonstrate that the applicability of the developed method, Section VIII-B showcases that eight explorer agents reach approximate consensus even when an explorer agent's trajectory crosses sub-regions. The servicing responsibilities among relay agents can be transferred to account for boundary crossing between sub-regions, and the corresponding planning strategies can accommodate the changing number of explorer agents within a sub-region.

A. Local Maneuvering

We adopt the dynamics of the relay and explorer agents in (1)-(4), where $A_i = B_i = C_i = A = B = C \triangleq I_2$, and $i = 1, 2, 3$. The disturbances for the explorer agents are modeled as $d_j(t) \triangleq d_j^*\left[\sin(t), \cos(t)\right]^{\mathrm{T}}$, where $j = 1, 2, 3, ..., 9$.[5] The initial positions of explorer agents 1-9 and relay agents 1-3 are shown in Figure 3, and the simulation parameters are selected as shown in Table I. We use the tool Slugs [13] for the strategy synthesis.

As shown in the following figures, Figure 2(a) depicts the norm of the state estimation error $e_{1,j}(t)$ throughout the simulation, showing the errors are bounded. Figure 2(b) depicts the norm of the estimated tracking error $e_{2,j}(t)$ is regulated to zero. Figure 2(c) shows the leader tracking error $e_{3,j}(t)$ for each explorer agent with respect to its corresponding servicing relay agent. Figure 3 depicts the true and estimated trajectories for the explorer agents, and the trajectories for the relay agents. As shown in Figures 2 and 3, the errors are bounded and the states of nine explorer agents are regulated towards the origin.

To illustrate the developed method requires less control effort than the other standard scheduler methods, we provide a comparison using round-robin scheduler. Specifically, we set the target servicing sequence to be 1-2-3 in a loop for the relay agent within the sub-region while the round-robin scheduler also satisfies the corresponding maximum dwell-time conditions. Since the round-robin scheduler can not achieve the objective while using the same initial control gains for the relay agents and exogenous disturbances for the explorer agents, we select the initial gains for the relay agents to be $k_1(0) = 4$, $k_2(0) = 3.8$, and $k_3(0) = 4$ as shown in Table I. As shown in Figure 4, the round-robin scheduler in Figure 4(b) requires 101.4% more control effort to complete the objective compare to the control effort needed for the developed method in Figure 4(a). The synthesized strategies enable the relay agents to service the explorer agents who need the state corrections the most, based on their previous servicing times and the corresponding maximum dwell-time conditions before the state estimation errors exceed the user-defined threshold, i.e., $\left\|e_{1,j}(t)\right\| \leq V_T$. As shown in Figure 3, the relay agent in the top-right sub-region services explorer agents 1 (initialized at $[100, -10]^{\mathrm{T}}$) and 2 (initialized at $[70, 70]^{\mathrm{T}}$) more often than servicing explorer agent 3 (initialized at $[30, 100]^{\mathrm{T}}$). Because the explorer agents experienced different exogenous disturbances with the same user-defined state estimation error bound, the corresponding maximum dwell-time conditions are different, i.e., (25), which leads to some

[5] For the specific values used in the simulation, we refer the reader to Table I.

Table I: Simulation parameters

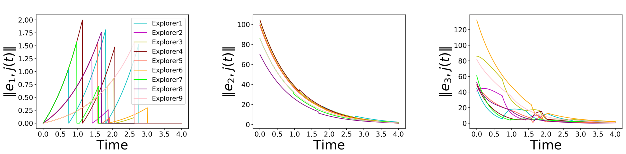
Local Maneuvering			Round-robin			Global Maneuvering		
$d_1^* = 1$	$d_2^* = 0.45$	$d_3^* = 0.15$	$d_1^* = 1$	$d_2^* = 0.45$	$d_3^* = 0.15$	$d_1^* = 1$	$d_2^* = 0.15$	$d_3^* = 0.45$
$d_4^* = 1$	$d_5^* = 0.45$	$d_6^* = 0.15$	$d_4^* = 1$	$d_5^* = 0.45$	$d_6^* = 0.15$	$d_4^* = 1$	$d_5^* = 0.45$	$d_6^* = 1$
$d_7^* = 1$	$d_8^* = 0.45$	$d_9^* = 0.15$	$d_7^* = 1$	$d_8^* = 0.45$	$d_9^* = 0.15$	$d_7^* = 0.45$	$d_8^* = 0.15$	
$R_g, R = 5$	$V_T = 3$	$k_{ARE} = 0.005$	$R_g, R = 5$	$V_T = 3$	$k_{ARE} = 0.005$	$R_g, R = 5$	$V_T = 3$	$k_{ARE} = 0.005$
$k_1(0) = 2.8$	$k_2(0) = 2.1$	$k_3(0) = 2.8$	$k_1(0) = 4$	$k_2(0) = 3.8$	$k_3(0) = 4$	$k_1(0) = 7$	$k_2(0) = 6$	$k_3(0) = 3$

Table II: Computation time for generating the synthesized strategies

	$M=2, N=3$	$M=2, N=5$	$M=2, N=7$	$M=3, N=6$	$M=3, N=9$	$M=3, N=12$
distributed	0.028s	3.05s	6.91s	0.084s	9.06s	11.67s
centralized	109.43s	TO	TO	TO	TO	TO

explorer agents needing more service than others. Because the round-robin scheduler sets a specific servicing sequence, some explorer agents got redundant services while ensuring the maximum dwell-time condition for each explorer agent is satisfied. Therefore, the developed method requires less control effort to achieve the objective.

A centralized strategy planning approach is also compared to our distributed method. The centralized strategy refers to a method where more than one relay agent is pre-synthesized in the planning to service all the explorer agents at the same time. For example, a distributed strategy can incorporate two relay agents, and each relay agent is responsible for servicing three explorer agents. While the centralized strategy will have these two relay agents servicing all six explorer agents together. As shown in Table II, the centralized strategies scale badly in computation time as the number of relay and explorer agents increased, which impedes applicability.[6]



(a) (b) (c)

Figure 2: Norm for (a) state estimation error, (b) estimated tracking error, and (c) leader tracking error for the nine explorer agents without crossing the sub-region boundaries, i.e., local maneuvering.

B. Global Maneuvering

By further demonstrating the developed method is applicable, we now consider eight explorer agents and three relay agents initialized in three different pre-defined sub-regions. Throughout the simulation, an explorer agent (i.e., explorer

[6]When generating the synthesized strategies, the maximum dwell-time for each explorer agent is selected as 5 time units. The times listed in Table II are generated using a Linux Ubuntu 20.04 operating system, Intel i7-4820K CPU @ 3.70GHz x 8 processor, and 32 GB memory computer.

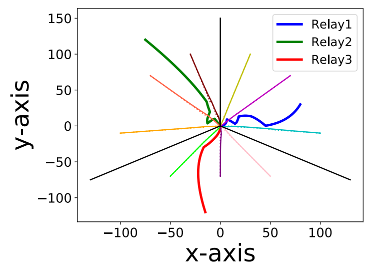
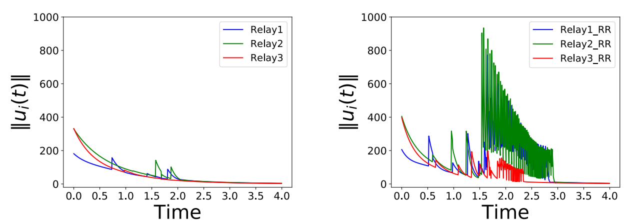


Figure 3: Agent trajectories for the nine explorer agents without crossing the sub-region boundaries, i.e., local maneuvering. The blue, green and red lines denote the three relay agents, and the other lines denote the nine explorer agents.

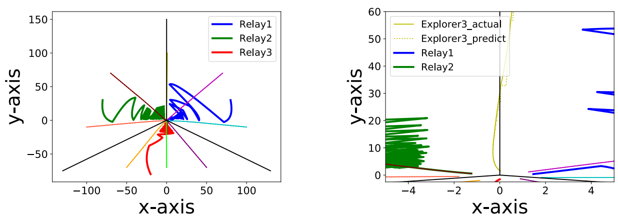


(a) (b)

Figure 4: Control effort of the relay agents using (a) the developed approach, and (b) the round-robin scheduler.

agent 3 initialized at $[1, 100]^{\mathrm{T}}$) leaves top-right sub-region and enters top-left sub-region as depicted in Figure 5. While the trajectory of explorer agent 3 crosses the boundaries, servicing responsibilities between the relay agents in top-right and top-left sub-regions are transferred, and the relay agents only need to service the explorer agents in their own sub-regions. The dynamics and system matrices used in this simulation example are the same as those in Section VIII-A, and the disturbances for the explorer agents are modeled as $d_j(t) \triangleq d_j^* [-\sin(t), \cos(t)]^{\mathrm{T}}$, where $j = 1, 2, 3, ..., 8$. The initial positions of explorer agents 1-8 and relay agents 1-3 are shown in Figure 5, and the simulation parameters are selected as shown in Table I.

Similar to Section VIII-A, Figure 5 shows that the states of the explorer agents reach approximate consensus at the origin. Note that explorer agent 3 leaves top-right sub-region and enters the top-left sub-region during the simulation, and the relay agent in top-left sub-region needs



(a) (b)

Figure 5: (a) Agent trajectories for the eight explorer agents with one explorer agent crossing the sub-region boundaries, i.e., global maneuvering. The blue, green and red lines denote the three relay agents, and the other lines denote the eight explorer agents. (b) Zoomed in plot of the trajectories, where the explorer agent 3 crosses the boundaries because of exogenous disturbance, and the relay agent in the right sub-region transfers the servicing responsibility to the relay agent in the left sub-region after the crossing.

to start servicing explorer agent 3 after crossing, and the relay agent in top-right sub-region does not need to service explorer agent 3 after crossing. As shown in Figure 5(b), explorer agent 3 crosses the boundary (denoted by $X = 0$ in the Cartesian coordinate system) because of the exogenous disturbance. After explorer agent 3 crosses the boundary, the relay agent in top-right sub-region only services explorer agents 1 (initialized at $[100, -10]^{\mathrm{T}}$) and 2 (initialized at $[70, 70]^{\mathrm{T}}$). The relay agent in top-left sub-region services explorer agents 3 (initialized at $[1, 100]^{\mathrm{T}}$), 4 (initialized at $[-70, 70]^{\mathrm{T}}$) and 5 (initialized at $[-100, -10]^{\mathrm{T}}$). This simulation example shows the developed method can accommodate for transferring of servicing responsibilities in between relay agents.

IX. CONCLUSION

By using the reactive synthesis approach to satisfy the high-level mission specifications and the low-level control design to provide performance guarantees, we show the distributed MAS can reach approximate consensus while relay agents switch among explorer agents to provide state information. Future work will focus on extending the current approach to satisfy more complicated mission specifications.

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