Invariance-Like Results for Nonautonomous Switched Systems

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***Abstract*—This paper generalizes the LaSalle–Yoshizawa Theorem to switched nonsmooth systems. The Filippov and Krasovskii regularizations of a switched system are shown to be contained within the convex hull of the Filippov and Krasovskii regularizations of the subsystems, respectively. A common candidate Lyapunov function that has a negative semidefinite generalized time derivative along the trajectories of the subsystems is shown to be sufficient to establish LaSalle–Yoshizawa-like results for the switched system. Of independent interest, are the results on approximate continuity and Filippov regularization of set-valued maps, reduction of differential inclusions using Lipschitz continuous regular functions, and comparative remarks on different generalizations of the time derivative along the trajectories of a nonsmooth system.**

***Index Terms*—Adaptive systems, differential inclusions, nonlinear systems, switched systems.**

I. Introduction

THE focus of this paper is Lyapunov-based stability analysis of switched nonautonomous systems that admit nonstrict candidate Lyapunov functions (cLfs) (i.e., cLfs with time derivatives bounded by a negative semidefinite function of the state). Analysis of adaptive controllers of systems with discontinuities introduced through discontinuous control design and/or dynamics motivates the theoretical development. For example, the neuromuscular electrical stimulation applications such as [1]–[4] involve switching between different muscle groups during different phases of operation to reduce fatigue [1], [4], to compensate for changing muscle geometry [3], or to perform functional tasks that require multilimb coordination [2]. Such applications stand to benefit from adaptive methods where the controller adapts to the uncertain dynamics without strictly relying on robust control methods prone to overstimulation, such as high gain or high frequency feedback.

Switched dynamics are inherent in a variety of modern adaptation strategies. For example, in sparse neural networks [5], the use of different approximation architectures for different regions of the state-space introduces switching via the feedforward part of the controller. In adaptive gain scheduling methods [6], switching is introduced due to changing feedback gains. Switching is also utilized as a tool to improve the transient response of adaptive controllers by selecting between the multiple estimated models of stable plants (see, e.g., [7]–[16]).

The Lyapunov-based stability analysis of switched adaptive systems is challenging because the subsystems under adaptive control typically do not admit strict Lyapunov functions. For each subsystem, convergence of the error signal to the origin is typically established using Barbălat's lemma (e.g., [17, Lemma 8.2]). In traditional methods that utilize multiple cLfs to establish stability of switched systems (e.g., [18, Th. 3.2]), the class of admissible switching signals is restricted based on the rate of decay of the cLfs (cf. [18, eq. (3.10)]). Since Barbălat's lemma provides no information about the rate of decay of a cLf, it alone is insufficient to establish stability of a switched system using multiple Lyapunov functions. While the switched systems can be analyzed using a common strict Lyapunov function, extension to common nonstrict Lyapunov functions is not trivial (cf. [19]–[21] and [18, Example 2.1]).

An adaptive controller for switched nonlinear systems is developed in [22] using a generalization of Barbălat's lemma from [23]. The controller is shown to asymptotically stabilize a switched system with parametric uncertainties in the subsystems. The multiple Lyapunov functions are utilized to analyze the stability of the switched system. However, the generalized Barbălat's Lemma in [23] requires a minimum dwell time, and in general, minimum dwell time cannot be guaranteed when the switching is state-dependent.

Results such as [24]–[27] extend the Barbashin–Krasovskii–LaSalle invariance principle to discontinuous systems. However, these results are for autonomous systems, whereas the development in this paper is focused on nonautonomous systems. An extension of the LaSalle–Yoshizawa Theorem to nonsmooth nonautonomous systems is provided in [28, Th. 2.5]; however, the result requires the cLf to be continuously differentiable,

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whereas the approach developed in this paper uses a more general framework that utilizes locally Lipschitz-continuous cLfs.

This paper generalizes the LaSalle–Yoshizawa Theorem (see, e.g.,[29] and [17, Th. 8.4]) and its nonsmooth extensions (see, e.g.,[28, Th. 2.5] and [30]) to switched nonsmooth systems and nonregular Lyapunov functions. A nonstrict common Lyapunov function is used to establish boundedness of the system state and convergence of a positive semidefinite function of the system state to zero under a mild restriction on the switching signal.

The paper is organized as follows. The notation is defined in Section II. Section III defines the class of systems considered along with the objectives. Sections IV–VII are dedicated to the development of the main results of the paper. Section VIII provides a discussion on the merits of the generalized time derivatives defined in Section V. Section IX presents illustrative examples, and Section X provides concluding remarks. The appendix includes supplementary proofs.

II. Notation

The $n-$ dimensional Euclidean space is denoted by $\mathbb{R}^n$ and μ denotes the Lebesgue measure on $\mathbb{R}^n$. Elements of $\mathbb{R}^n$ are interpreted as column vectors and $(\cdot)^T$ denotes the vector transpose operator. The set of positive integers excluding 0 is denoted by $\mathbb{N}$, and $\mathcal{D}$ denotes an open and connected subset of $\mathbb{R}^n$. For $a \in \mathbb{R}$, the notation $\mathbb{R}_{\geq a}$ denotes the interval $[a, \infty)$ and the notation $\mathbb{R}_{>a}$ denotes the interval (a, ∞). For a relation $(\cdot)$, the notation $\overset{\text{a.e.}}{(\cdot)}$ implies that the relation holds for almost all $t \in \mathcal{I}$, for some interval $\mathcal{I}$. Unless otherwise specified, an interval $\mathcal{I}$ is assumed to be right open, of nonzero length, and $t_0 := \min \mathcal{I}$, where $t_0 \in \mathbb{R}_{\geq 0}$ denotes the initial time. The notation $F : A \rightrightarrows B$ is used to denote a set-valued map from A to the subsets of B. The notations $\operatorname{co} A$ and $\overline{\operatorname{co}} A$ are used to denote the convex hull and the closed convex hull of the set A, respectively and $A \triangle B := (A \setminus B) \cup (B \setminus A)$. If $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ then the notation $[a; b]$ denotes the concatenated vector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathbb{R}^{m+n}$. For $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$ the notations $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$ and $A \times B$ are interchangeably used to denote the set $\{[a; b] \mid a \in A, b \in B\}$. The notations $\overline{\mathrm{B}}(x, r)$ and $\mathrm{B}(x, r)$, for $x \in \mathbb{R}^n$ and $r > 0$, are used to denote the sets $\{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$ and $\{y \in \mathbb{R}^n \mid \|x - y\| < r\}$, respectively. The notation $|(\cdot)|$ denotes the absolute value if $(\cdot) \in \mathbb{R}$ and the cardinality if $(\cdot)$ is a set. The notation $\mathcal{L}_\infty(A, B)$ denotes the set of essentially bounded functions from A to B.

III. Problem Formulation

Consider a switched system of the form[1]

$$\dot{x} = f_{\rho(x,t)}(x, t) \tag{1}$$

where $\rho : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathcal{N}^o$ denotes a state-dependent switching signal, $\mathcal{N}^o \subseteq \mathbb{N}$ is the set of all possible switching indices, and $x \in \mathbb{R}^n$ denotes the system state. Let $f : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n$ denote the function $(x, t) \mapsto f_{\rho(x,t)}(x, t)$. The main objective of this paper is to establish asymptotic properties of the generalized solutions of the system

$$\dot{x} = f(x, t) \tag{2}$$

using asymptotic properties of the generalized solutions of the subsystems

$$\dot{x} = f_\sigma(x, t), \quad \sigma \in \mathcal{N}^o. \tag{3}$$

The advantage of the aforementioned strategy, as opposed to directly analyzing (2), is that the analysis can be made invariant with respect to the switching function over a wide range of admissible (see Assumption 1) switching functions. On the other hand, a direct analysis of (2) is valid only for the specific ρ used to construct (2).

For some classes of switching signals, the switched systems can be modeled and analyzed as hybrid systems (see, e.g., [31, Sec. 1.4.4]). However, when arbitrary state-dependent switching is allowed, the switched systems can have solutions that flow $\forall t \in \mathbb{R}_{\geq t_0}$ with an uncountable set of switching instances (e.g., sliding motion). Since hybrid time domains are not rich enough to describe such solutions while keeping track of the discrete variable, hybrid models are not suitable for the class of systems considered in this paper.

In the following, generalized solutions of the systems in (2) and (3), defined using the Filippov and Krasovskii regularization are analyzed. For a Lebesgue measurable function $g : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}$, the Filippov regularization is defined as [32, p. 85]

$$\mathcal{F}[g](x, t) := \bigcap_{\delta > 0} \bigcap_{\mu(N) = 0} \overline{\operatorname{co}} \{g(y, t) \mid y \in \mathrm{B}(x, \delta) \setminus N\} \tag{4}$$

and the Krasovskii regularization is defined as[33, p. 17]

$$\mathcal{K}[g](x, t) := \bigcap_{\delta > 0} \overline{\operatorname{co}} \{g(y, t) \mid y \in \mathrm{B}(x, \delta)\}. \tag{5}$$

The following definition introduces the class of switched systems considered in this paper.

Definition 1: A collection $\{f_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n\}_{\sigma \in \mathcal{N}^o}$ is said to *satisfy the weak basic conditions* if it is locally bounded, uniformly in σ and t,[2] and the functions $t \mapsto f_\sigma(x, t)$ and $t \mapsto \rho(x, t)$ are Lebesgue measurable $\forall x \in \mathbb{R}^n$ and $\forall \sigma \in \mathcal{N}^o$. When a Filippov regularization is considered, the local boundedness requirement on the map $x \mapsto f_\sigma(x, t)$ is relaxed to essential local boundedness and a stronger measurability requirement is imposed so that $(x, t) \mapsto f_\sigma(x, t)$ and $(x, t) \mapsto \rho(x, t)$ are Lebesgue measurable $\forall \sigma \in \mathcal{N}^o$. △

To achieve the aforementioned main objective, the differential inclusion that results from the regularization of the switched system in (2) is proven to be contained within the convex combination of the differential inclusions that result from the regularization of the subsystems in (3), under mild assumptions on the switching signal (Proposition 1, Sec. IV). To facilitate the

[1] For the case where the subsystems are modeled as differential inclusions, see Section VII.

[2] A collection of functions $\{f_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n \mid \sigma \in \mathcal{N}^o\}$ is locally bounded, uniformly in t and σ, if for every compact $K \subset \mathbb{R}^n$, there exists $M > 0$ such that $\|f_\sigma(x, t)\|_2 \leq M$, $\forall (x, t) \in K \times \mathbb{R}_{\geq t_0}$ and $\forall \sigma \in \mathcal{N}^o$.

discussion that follows, the existence of a nonstrict Lyapunov function is shown to be sufficient to infer certain asymptotic properties of solutions of differential inclusions (Th. 1, Sec. V). It is then established that a common nonstrict Lyapunov function for the differential inclusions that result from the regularization of (3) is also a nonstrict Lyapunov function for the differential inclusion that results from the regularization of (2) (Proposition 2, Sec. VI). The main result of the paper then follows, i.e., the conclusions about asymptotic properties of generalized solutions of (2) can be drawn from the asymptotic properties of generalized solutions of (3) (Th. 2, Sec. VI).

The following section develops a relationship between the differential inclusions resulting from the regularization of (2) and (3).

IV. Switching and Regularization

Let $\dot{x} \in \mathcal{F}[f](x,t)$ and $\dot{x} \in \mathcal{F}[f_\sigma](x,t)$ be the Filippov regularizations and $\dot{x} \in \mathcal{K}[f](x,t)$ and $\dot{x} \in \mathcal{K}[f_\sigma](x,t)$ be the Krasovskii regularizations of (2) and (3), respectively. The following assumption imposes a mild restriction on the switching function ρ to establish a relationship between $\mathcal{F}[f]$, $\{\mathcal{F}[f_\sigma]\}$, $\mathcal{K}[f]$, and $\{\mathcal{K}[f_\sigma]\}$.

Assumption 1: For each $(x,t) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0}$, there exists $\delta^* > 0$ such that $|\rho(\mathrm{B}(x,\delta^*),t)| < \infty$. △

Assumption 1 is equivalent to the assumption that ρ is locally bounded in x for each t. Roughly speaking, Assumption 1 restricts infinitely many subsystems from being active in a small neighborhood of the state space. It does not restrict Zeno behavior and arbitrary time-dependent switching, and as such, is not restrictive. For further insight into why Assumption 1 is invoked, see Example 1. The following proposition states that under general conditions, the set-valued maps $\mathcal{F}[f]$ and $\mathcal{K}[f]$ are contained, pointwise, within the convex combination of the collections $\{\mathcal{F}[f_\sigma]\}$ and $\{\mathcal{K}[f_\sigma]\}$, respectively.

Proposition 1: If $\rho : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathcal{N}^o$ satisfies Assumption 1, then the set-valued maps $\mathcal{K}[f]$, $\mathcal{K}[f_\sigma]$, $\mathcal{F}[f]$, and $\mathcal{F}[f_\sigma]$ satisfy

$$\mathcal{K}[f](x,t) \subseteq \overline{\mathrm{co}} \bigcup_{\sigma \in \mathcal{N}^o} \mathcal{K}[f_\sigma](x,t) \tag{6}$$

$$\mathcal{F}[f](x,t) \subseteq \overline{\mathrm{co}} \bigcup_{\sigma \in \mathcal{N}^o} \mathcal{F}[f_\sigma](x,t) \tag{7}$$

$\forall (x,t) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0}$.

Proof for Krasovskii Regularization: Fix $(x,t) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0}$, select $\delta^* > 0$ such that $|\rho(\mathrm{B}(x,\delta^*),t)| < \infty$,[3] and let $\mathcal{N} := \rho(\mathrm{B}(x,\delta^*),t)$. Observe that the containment in (6) is straightforward if the union over σ is placed inside the convex closure operation. That is

$$\bigcap_{\delta>0} \overline{\mathrm{co}}\{f_{\rho(y,t)}(y,t) \mid y \in \mathrm{B}(x,\delta)\} \subseteq \bigcap_{\delta>0} \overline{\mathrm{co}} \bigcup_{\sigma \in \mathcal{N}} \{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\}. \tag{8}$$

[3]Existence of such a δ^* is guaranteed by Assumption 1.

The rest of the proof shows that the right-hand side (RHS) of (8) is contained within the RHS of (6) in two steps. The first step is to show that

$$\bigcap_{\delta>0} \overline{\mathrm{co}} \bigcup_{\sigma \in \mathcal{N}} \{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\} \subseteq \bigcap_{\delta>0} \mathrm{co} \bigcup_{\sigma \in \mathcal{N}} \overline{\mathrm{co}}\{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\}. \tag{9}$$

The second step is to show that

$$\bigcap_{\delta>0} \mathrm{co} \bigcup_{\sigma \in \mathcal{N}} \overline{\mathrm{co}}\{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\} \subseteq \mathrm{co} \bigcup_{\sigma \in \mathcal{N}} \bigcap_{\delta>0} \overline{\mathrm{co}}\{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\}. \tag{10}$$

The result in (6) then follows from (8), (9), and (10).

To prove (9), fix $\delta \in (0,\delta^*]$, let $z \in \overline{\mathrm{co}} \bigcup_{\sigma \in \mathcal{N}} \{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\}$, and let $\{z_i\}_{i \in \mathbb{N}} \in \mathbb{R}^n$ be a sequence such that $z_i \in \mathrm{co} \bigcup_{\sigma \in \mathcal{N}} \{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\}$, $\forall i \in \mathbb{N}$, and $\lim_{i\to\infty} z_i = z$. For each $i \in \mathbb{N}$, there exists a collection of points $\{z_i^1, \ldots, z_i^{|\mathcal{N}|}\} \subset \mathbb{R}^n$ and positive real numbers $\{a_i^1, \ldots, a_i^{|\mathcal{N}|}\}$, for which $\sum_{j=1}^{|\mathcal{N}|} a_i^j = 1$, such that $z_i = \sum_{j=1}^{|\mathcal{N}|} a_i^j z_i^j$ and $z_i^j \in \{f_{\sigma_j}(y,t) \mid y \in \mathrm{B}(x,\delta)\}, \forall j \in \{1,\ldots,|\mathcal{N}|\}$. Hence, $z = \lim_{i\to\infty} \sum_{j=1}^{|\mathcal{N}|} a_i^j z_i^j$, i.e., $z = \lim_{i\to\infty} Z_i A_i$, where $A_i = [a_i^1 \ ;\ldots;\ a_i^{|\mathcal{N}|}]$ and $Z_i = [z_i^1;\ \ldots;\ z_i^{|\mathcal{N}|}]^T$.

Since the coefficients $a_i^j \geq 0$ are bounded, the sequence $\{A_i\}_{i\in\mathbb{N}}$ is a bounded sequence. Hence, there exists a subsequence $\{A_{i_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} A_{i_k} = A$, for some $A = [a^1 \ ;\cdots;\ a^{|\mathcal{N}|}]$. Furthermore, the continuity of the function $A_i \mapsto \sum_{j=1}^{|\mathcal{N}|} a_i^j$ implies $\sum_{j=1}^{|\mathcal{N}|} a^j = 1$. The boundedness of the set $\bigcup_{\sigma\in\mathcal{N}} \{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\}$ implies that the sequence $\{Z_{i_k}\}_{k\in\mathbb{N}}$ is bounded, and as a result, there exists a subsequence $\{Z_{i_{k_l}}\}_{l\in\mathbb{N}}$ such that $\lim_{l\to\infty} Z_{i_{k_l}} = Z$, elementwise, for some $Z = [z^1 \ ;\cdots;\ z^{|\mathcal{N}|}]^T$. Hence, $z = \lim_{l\to\infty} Z_{i_{k_l}} A_{i_{k_l}} = ZA$, where the columns z^j of the matrix Z are the limits $\lim_{l\to\infty} z^j_{i_{k_l}}$, and $z^j \in \overline{\mathrm{co}}\{f_{\sigma_j}(y,t) \mid y \in \mathrm{B}(x,\delta)\}, \forall j \in \{1,\ldots,|\mathcal{N}|\}$. Therefore, the point z is a convex combination of points from $\overline{\mathrm{co}}\{f_{\sigma_j}(y,t) \mid y \in \mathrm{B}(x,\delta)\}$. That is, $z \in \mathrm{co} \bigcup_{\sigma\in\mathcal{N}} \overline{\mathrm{co}}\{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\}\ \forall \delta \in (0,\delta^*]$, which proves (9).

To establish (10), let $z \in \bigcap_{\delta>0} \mathrm{co} \bigcup_{\sigma\in\mathcal{N}} \overline{\mathrm{co}}\{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\}$. Note that if $0 < \delta_1 \leq \delta_2$, then

$$\mathrm{co} \bigcup_{\sigma \in \mathcal{N}} \overline{\mathrm{co}}\{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta_1)\} \subseteq \mathrm{co} \bigcup_{\sigma \in \mathcal{N}} \overline{\mathrm{co}}\{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta_2)\}.$$

That is, if $z \in \mathrm{co} \bigcup_{\sigma\in\mathcal{N}} \overline{\mathrm{co}}\{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta_1)\}$ for some $0 < \delta_1$, then $z \in \bigcap_{\delta>\delta_1} \mathrm{co} \bigcup_{\sigma\in\mathcal{N}} \overline{\mathrm{co}}\{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\}$. Hence, $\forall k \in \mathbb{N}$, such that $k \geq \frac{1}{\delta^*}$, there exist $\{z_{k1}, \ldots, z_{k|\mathcal{N}|}\} \subset \mathbb{R}^n$, nonnegative real numbers $\{a_{k1}, \ldots, a_{k|\mathcal{N}|}\}$ for which $\sum_{j=1}^{|\mathcal{N}|} a_{kj} = 1$, such that $z_{kj} \in \bigcap_{\delta \geq \frac{1}{k}} \overline{\mathrm{co}}\{f_{\sigma_j}(y,t) \mid y \in \mathrm{B}(x,\delta)\}$ and $z = \sum_{j=1}^{|\mathcal{N}|} a_{kj} z_{kj}$.

That is, $z = Z_k A_k$, where $A_k = [\, a_{k1}\,; \ldots ;\, a_{k|\mathcal{N}|}\,]$ and $Z_k = [\, z_{k1}; \; \cdots \; ; \; z_{k|\mathcal{N}|}\,]^T$. The boundedness of the sequences $\{Z_k\}_{k\in\mathbb{N}}$ and $\{A_k\}_{k\in\mathbb{N}}$ implies the existence of subsequences $\{Z_{k_l}\}_{l\in\mathbb{N}}$ and $\{A_{k_l}\}_{l\in\mathbb{N}}$ and vectors $Z := [\, z_1 \,; \cdots \,; \, z_{|\mathcal{N}|}\,]^T$ and $A := [\, a_1 \,; \cdots \,; \, a_{|\mathcal{N}|}\,]$ such that $A = \lim_{l\to\infty} A_{k_l}$, $\sum_{j=1}^{|\mathcal{N}|} a_j = 1$, and $Z = \lim_{l\to\infty} Z_{k_l}$. Since $z = Z_{k_l} A_{k_l}$, $\forall k_l \in \mathbb{N}$, it can be concluded that $z = ZA$.

It is now claimed that $\forall j \in \{1, \ldots, |\mathcal{N}|\}$, $z_j \in \bigcap_{\delta>0} \overline{\mathrm{co}}\{f_{\sigma_j}(y,t) \mid y \in \mathrm{B}(x,\delta)\}$. To prove the claim by contradiction, assume that $\exists \delta^* > 0$ such that $z_j \notin \overline{\mathrm{co}}\{f_{\sigma_j}(y,t) \mid y \in \mathrm{B}(x,\delta^*)\}$. Since

$$\overline{\mathrm{co}}\big\{f_{\sigma_j}(y,t) \,|\, y\in \mathrm{B}(x,\delta_1)\big\} \subseteq \overline{\mathrm{co}}\big\{f_{\sigma_j}(y,t) \,|\, y\in \mathrm{B}(x,\delta_2)\big\}, \tag{11}$$

$\forall \sigma_j \in \mathcal{N}$ and $\forall \delta_1 \leq \delta_2$, $z_j \notin \overline{\mathrm{co}}\{f_{\sigma_j}(y,t) \mid y \in \mathrm{B}(x,\delta)\}, \forall \delta \geq \delta^*$. That is, $\exists k_l^* \in \mathbb{N}$ such that $z_j \notin \bigcap_{\delta \geq \frac{1}{k_l}} \overline{\mathrm{co}}\{f_{\sigma_j}(y,t) \mid y \in \mathrm{B}(x,\delta)\}$, $\forall k_l \geq k_l^*$. From (11) and the fact that the sets $\bigcap_{\delta \geq \frac{1}{k_l}} \overline{\mathrm{co}}\{f_{\sigma_j}(y,t) \mid y \in \mathrm{B}(x,\delta)\}$ are closed, it can be concluded that there exists $\epsilon > 0$ such that $\forall k_l \geq k_l^*$

$$\mathrm{B}(z_j, \epsilon) \notin \bigcap_{\delta \geq \frac{1}{k_l}} \overline{\mathrm{co}}\big\{f_{\sigma_j}(y,t) \mid y \in \mathrm{B}(x,\delta)\big\}. \tag{12}$$

Since $z_{k_l j} \in \bigcap_{\delta \geq \frac{1}{k_l}} \overline{\mathrm{co}}\{f_{\sigma_j}(y,t) \mid y \in \mathrm{B}(x,\delta)\}, \forall k_l \in \mathbb{N}$, (12) contradicts $z_j = \lim_{l\to\infty} z_{k_l j}$, and hence, the proof of the claim that $\forall j \in \{1, \ldots, |\mathcal{N}|\}$, $z_j \in \bigcap_{\delta>0} \overline{\mathrm{co}}\{f_{\sigma_j}(y,t) \mid y \in \mathrm{B}(x,\delta)\}$ is complete. The claim implies that $z \in \mathrm{co} \bigcup_{\sigma\in\mathcal{N}} \bigcap_{\delta>0} \overline{\mathrm{co}}\{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\}$, which proves (10), and hence, (6).

The proof for the Filippov regularization involves the technical details related to exclusion of measure-zero sets that are provided in the appendix. ■

The following example demonstrates that Assumption 1 is not vacuous.[4]

Example 1: Let $\mathcal{N}^o = \mathbb{N}$ and for $\sigma \in \mathcal{N}^o$, let f_σ be defined as

$$f_\sigma(x) := \begin{cases} 0 & |x| < {}^1\!/_2{}^\sigma \\ 1 & |x| \geq {}^1\!/_2{}^\sigma \end{cases}$$

so that $\mathcal{K}[f_\sigma](0) = \mathcal{F}[f_\sigma](0) = \{0\}, \forall \sigma \in \mathcal{N}^o$. Let

$$\rho(x) = \begin{cases} \sigma & x \in \left(-\dfrac{1}{2^{\sigma-1}}, -\dfrac{1}{2^\sigma}\right] \cup \left[\dfrac{1}{2^\sigma}, \dfrac{1}{2^{\sigma-1}}\right) \\ 1 & \text{otherwise} \end{cases}.$$

Clearly, ρ violates Assumption 1 at $x = 0$. In this case, $f(x) = \{\begin{smallmatrix} 1 & x\neq 0 \\ 0 & x=0 \end{smallmatrix}$, $\mathcal{K}[f](0) = [0,1]$, and $\mathcal{F}[f](0) = \{1\}$, that is, the conclusion of Proposition 1 does not hold without the switching restriction in Assumption 1. △

To facilitate the analysis of $\mathcal{F}[f]$ and $\mathcal{K}[f]$ based on the analysis of $\mathcal{F}[f_\sigma]$ and $\mathcal{K}[f_\sigma]$, respectively, a stability result for differential inclusions that relies on the nonstrict Lyapunov functions is developed in the following section. While the results developed in this section are specific to differential inclusions that arise from the Filippov and Krasovskii regularization of differential equations with discontinuous RHSs, the results developed in the following sections are more general in the sense that they apply to generic set-valued maps, not necessarily resulting from the Filippov or Krasovskii regularization.

V. Nonstrict Lyapunov Functions for Differential Inclusions

Let $F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n$ be a set-valued map. Consider a differential inclusion of the form

$$\dot{x} \in F(x,t). \tag{13}$$

A locally absolutely continuous function $x : \mathcal{I} \to \mathbb{R}^n$ is called a solution of (13) over the closed interval $\mathcal{I}$ provided

$$\dot{x}(t) \in F(x(t), t) \tag{14}$$

for almost all $t \in \mathcal{I}$ [32, p. 50]. The following analysis focuses on the Lyapunov-based analysis of maximal solutions (see [24, Definition 2.1]) of set-valued maps that admit local solutions.[5]

Definition 2: The set-valued map $F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n$ is said to *admit local solutions* over $\mathcal{D} \times \mathcal{J}$, where $\mathcal{J}$ is an interval, if $\forall (y,t) \in \mathcal{D} \times \mathcal{J}$, $\exists T > t$ such that a solution $x : \mathcal{I} \to \mathbb{R}^n$ of (13), starting from $x(t) = y$ exists over $\mathcal{I} := [t, T)$. △

To facilitate the analysis, generalized time derivatives and the nonstrict Lyapunov functions are defined as follows.

Definition 3: Let $F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n$ have nonempty and compact values. The *generalized time derivative* of a locally Lipschitz-continuous function $V : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}$ with respect to F is the function $\dot{\overline{V}}_F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}$ defined as (cf. [34])

$$\dot{\overline{V}}_F(x,t) := \max_{p\in\partial V(x,t)} \max_{q\in F(x,t)} p^T [q; 1] \tag{15}$$

where ∂V denotes the Clarke gradient of V [35, p. 39]. △

For a detailed comparison of Definition 3 with more popular set-valued notions of generalized time derivatives (i.e., [36, eq. 13] and [37, p. 364]), see Section VIII.

Definition 4: Let $\Omega := \mathcal{D} \times \mathcal{I}$ for some interval $\mathcal{I}$. Let $F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n$ have nonempty and compact values over Ω. Let $V : \Omega \to \mathbb{R}$ be a locally Lipschitz-continuous positive definite function. Let $\overline{W}$, $\underline{W} : \mathcal{D} \to \mathbb{R}$ be continuous positive definite functions and let $W : \mathcal{D} \to \mathbb{R}$ be a continuous positive semidefinite function. If

$$\underline{W}(x) \leq V(x,t) \leq \overline{W}(x), \quad \forall (x,t) \in \Omega \tag{16}$$

and

$$\dot{\overline{V}}_F(x,t) \leq -W(x) \tag{17}$$

$\forall x \in \mathcal{D}$ and for almost all $t \in \mathbb{R}_{\geq t_0}$, then V is called a *nonstrict Lyapunov function* for F over Ω with the bounds $\underline{W}$, $\overline{W}$, and W. △

The following theorem establishes the fact that the existence of a nonstrict Lyapunov function implies that $t \mapsto W(x(t))$ asymptotically decays to zero.

[4]The authors thank the anonymous reviewer who suggested this example.

[5]Sufficient conditions for existence of local solutions can be found in, e.g., [32, p. 83, Th. 5].

Theorem 1: Let $0 \in \mathcal{D}, r > 0$ be selected such that $\overline{\mathrm{B}}(0, r) \subset \mathcal{D}$, and $\Omega := \mathcal{D} \times \mathbb{R}_{\geq t_0}$. Let $F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n$ be a map that admits local solutions over Ω and is locally bounded, uniformly in t, over Ω.[6] If $V : \Omega \to \mathbb{R}$ is a nonstrict Lyapunov function for F over Ω with the bounds $\underline{W} : \mathcal{D} \to \mathbb{R}$, $\overline{W} : \mathcal{D} \to \mathbb{R}$, and $W : \mathcal{D} \to \mathbb{R}$, then all maximal solutions of (13) with $x(t_0) \in \{x \in \overline{\mathrm{B}}(0, r) \mid \overline{W}(x) \leq c\}$, for some $c \in (0, \min_{\|x\|_2 = r} \underline{W}(x))$, are complete, bounded, and satisfy $\lim_{t\to\infty} W(x(t)) = 0$. In addition, if $\mathcal{D} = \mathbb{R}^n$ and the sets $\{x \in \mathbb{R}^n \mid \underline{W}(x) \leq c\}$ are compact for all $c \in \mathbb{R}_{>0}$, then all maximal solutions of (13), regardless of the initial condition, are complete, bounded, and satisfy $\lim_{t\to\infty} W(x(t)) = 0$. Furthermore, if the nonstrict Lyapunov function is regular [35, Definition 2.3.4], then (17) can be relaxed to $\underline{\dot{V}}_F(x, t) \leq -W(x)$, where

$$\underline{\dot{V}}_F(x, t) := \min_{p \in \partial V(x,t)} \max_{q \in F(x,t)} p^T [q; 1]. \tag{18}$$

■

Proof: See the appendix.

The following section utilizes the results of Sections IV and V to develop the main results of this paper.

VI. Invariance-Like Results for Switched Systems

The following proposition states that a common nonstrict Lyapunov function for a family of differential inclusions is also a nonstrict Lyapunov function for the closure of their convex combination.[7]

Proposition 2: Let $\Omega := \mathcal{D} \times \mathcal{I}$ for some interval $\mathcal{I}$. Let $\{F_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n \mid \sigma \in \mathcal{N}^o\}$ be a family of set-valued maps with compact and nonempty values that is locally bounded, uniformly in σ, over $\Omega \times \mathcal{N}^o$.[8] If $V : \Omega \to \mathbb{R}$ is a common nonstrict Lyapunov function for the family $\{F_\sigma\}$ over Ω with the bounds $\underline{W} : \mathcal{D} \to \mathbb{R}$, $\overline{W} : \mathcal{D} \to \mathbb{R}$, and $W : \mathcal{D} \to \mathbb{R}$ (i.e., V is a nonstrict Lyapunov function for F_σ for each $\sigma \in \mathcal{N}^o$ and the bounds W, $\overline{W}$, and $\underline{W}$ in (16) are independent of σ), then V is also a nonstrict Lyapunov function for $\overline{\mathrm{co}} \bigcup_{\sigma \in \mathcal{N}^o} F_\sigma(x, t)$ over Ω with the bounds $\underline{W}$, $\overline{W}$, and W.

Proof: Since the maps $\{F_\sigma\}$ are locally bounded, uniformly in σ, over $\Omega \times \mathcal{N}^o$, $\overline{\mathrm{co}} \bigcup_{\sigma \in \mathcal{N}^o} F_\sigma(x, t)$ is nonempty and compact $\forall (x, t) \in \Omega$. Since V is a common nonstrict Lyapunov function, $\max_{p \in \partial V(x,t)} \max_{q \in F_\sigma(x,t)} p^T [q; 1] \leq -W(x), \forall \sigma \in \mathcal{N}^o$. Let $q^* \in F(x, t) := \overline{\mathrm{co}} \bigcup_{\sigma \in \mathcal{N}^o} F_\sigma(x, t)$. There exists a sequence $\{q_j\}_{j \in \mathbb{N}}$ such that $\lim_{j\to\infty} q_j = q^*$ and $q_j \in \mathrm{co} \bigcup_{\sigma \in \mathcal{N}^o} F_\sigma(x, t)$. By Carathéodory's theorem [38, p. 103], $q_j = \sum_{i=1}^m a_i^j z_i^j$, where $m \leq n + 1$, $\sum_{i=1}^m a_i^j = 1$, $a_i^j \geq 0$, and $z_i^j \in F_{\sigma_i^j}(x, t)$, $\forall i \in \{1, \ldots, m\}$.

For any fixed $p \in \partial V(x, t)$,

$$p^T [z_i^j; 1] \leq \max_{q \in F_{\sigma_i^j}(x,t)} p^T [q; 1],$$

$\forall i \in \{1, \ldots, m\}$ and $\forall j \in \mathbb{N}$. Hence,

$$\max_{p \in \partial V(x,t)} p^T \left[z_i^j; 1\right] \leq \max_{p \in \partial V(x,t)} \max_{q \in F_{\sigma_i^j}(x,t)} p^T [q; 1] \leq -W(x).$$

$\forall i \in \{1, \ldots, m\}$ and $\forall j \in \mathbb{N}$. Since

$$\sum_{i=1}^m a_i^j = 1, \max_{p \in \partial V(x,t)} p^T [q_j; 1] \leq -W(x),$$

$\forall j \in \mathbb{N}$. Now, since $(p, q) \mapsto p^T [q; 1]$ is continuous, and $\partial V(x, t)$ and $\overline{\mathrm{co}} \bigcup_{\sigma \in \mathcal{N}^o} F_\sigma(x, t)$ are compact, the function $q \mapsto \max\{p^T [q; 1] \mid p \in \partial V(x, t)\}$ is continuous on $\overline{\mathrm{co}} \bigcup_{\sigma \in \mathcal{N}^o} F_\sigma(x, t)$. Hence, $\max_{p \in \partial V(x,t)} p^T [q; 1] \leq -W(x)$, $\forall q \in \overline{\mathrm{co}} \bigcup_{\sigma \in \mathcal{N}^o} F_\sigma(x, t)$. ■

The following corollary demonstrates that if V is regular and the set-valued maps $\{F_\sigma\}$ are continuous, then the bound (17) in Proposition 2 can be relaxed to utilize $\underline{\dot{V}}_F$ instead of $\dot{\overline{V}}_F$.

Corollary 1: Let the family of set-valued maps $\{F_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n \mid \sigma \in \mathcal{N}^o\}$ satisfy the conditions of Proposition 2. If a regular [35, Definition 2.3.4] function $V : \Omega \to \mathbb{R}$ is a common nonstrict Lyapunov function for the family $\{F_\sigma\}$, over Ω, with the bounds $\underline{W} : \mathcal{D} \to \mathbb{R}$, $\overline{W} : \mathcal{D} \to \mathbb{R}$, and $W : \mathcal{D} \to \mathbb{R}$, and with (17) in Definition 4 relaxed to $\underline{\dot{V}}_{F_\sigma}(x, t) \leq -W(x)$, $\forall (x, \sigma) \in \mathbb{R}^n \times \mathcal{N}^o$ and for almost all $t \in \mathbb{R}_{\geq t_0}$, then $\underline{\dot{V}}_{\overline{\mathrm{co}} \bigcup_{\sigma \in \mathcal{N}^o} F_\sigma}(x, t) \leq -W(x)$, $\forall (x, \sigma) \in \mathbb{R}^n \times \mathcal{N}^o$ and for almost all $t \in \mathbb{R}_{\geq t_0}$, provided the set-valued maps $\{F_\sigma\}$ are continuous (in the sense of [39, Definition 1.4.3]) and convex valued.[9]

Proof: See the appendix. ■

The main result of the paper can now be summarized in the following theorem.

Theorem 2: Let $0 \in \mathcal{D}$, $\Omega := \mathcal{D} \times \mathbb{R}_{\geq t_0}$, and let $r > 0$ be selected such that $\overline{\mathrm{B}}(0, r) \subset \mathcal{D}$. Let $\{f_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n\}_{\sigma \in \mathcal{N}^o}$ be a collection that satisfies the weak basic conditions in Definition 1. If Assumption 1 holds and the (Filippov) Krasovskii regularizations of the subsystems in (3) admit a common nonstrict Lyapunov function $V : \Omega \to \mathbb{R}$, over Ω, with the bounds $\underline{W} : \mathcal{D} \to \mathbb{R}$, $\overline{W} : \mathcal{D} \to \mathbb{R}$, and $W : \mathcal{D} \to \mathbb{R}$, then every maximal solution of the (Filippov) Krasovskii regularization of the switched system in (2) such that $x(t_0) \in \{x \in \overline{\mathrm{B}}(0, r) \mid \overline{W}(x) \leq c\}$, for some $c \in (0, \min_{\|x\|_2 = r} \underline{W}(x))$, is complete, bounded, and satisfies $\lim_{t\to\infty} W(x(t)) = 0$. In addition, if $\mathcal{D} = \mathbb{R}^n$ and the sets $\{x \in \mathbb{R}^n \mid \underline{W}(x) \leq c\}$ are compact for all $c \in \mathbb{R}_{>0}$, then every maximal solution of the (Filippov) Krasovskii regularization of the switched system in (2), regardless of the initial condition, is complete, bounded, and satisfies $\lim_{t\to\infty} W(x(t)) = 0$.

[6] A set valued map $F : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^n$ is locally bounded, uniformly in t, over Ω, if for every compact $K \subset \mathcal{D}$, there exists $M > 0$ such that $\forall (x, t, y)$ such that $(x, t) \in K \times \mathbb{R}_{\geq t_0}$, and $y \in F(x, t)$, $\|y\|_2 \leq M$.

[7] The observation that a common (strong) continuously differentiable Lyapunov function for a family of finitely many differential inclusions is also a Lyapunov function for the closure of their convex combination is stated in [19, Proposition 1]. In this paper, it is proved and extended to families of countably infinite differential inclusions and semidefinite locally Lipschitz-continuous Lyapunov functions.

[8] A collection of set valued maps $\{F_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n \mid \sigma \in \mathcal{N}^o\}$ is locally bounded, uniformly in σ, over $\Omega \times \mathcal{N}^o$, if for every compact $K \subset \Omega$, there exists $M > 0$ such that $\forall (x, t, \sigma, y)$ such that $(x, t, \sigma) \in K \times \mathcal{N}^o$ and $y \in F_\sigma(x, t)$, $\|y\|_2 \leq M$.

[9] Example 2 demonstrates that there are collections of upper semicontinuous set-valued maps for which Corollary 1 fails to hold, i.e., the continuity condition in Corollary 1 is not vacuous.

Proof: The first step is to show that under the weak basic conditions in Definition 1, the maps $\mathcal{K}[f]$, $\mathcal{K}[f_\sigma]$, $\mathcal{F}[f]$, and $\mathcal{F}[f_\sigma]$ admit local solutions $\forall\sigma\in\mathcal{N}^o$. Since the collection $\{f_\sigma\}_{\sigma\in\mathcal{N}^o}$ is locally bounded, uniformly in σ and t, the function f is locally bounded, uniformly in t. To establish Lebesgue measurability of f, consider the representation $f(x,t)=\sum_{\sigma\in\mathcal{N}^o} I_\sigma(\rho(x,t))f_\sigma(x,t)$, where

$$I_\sigma(i) := \begin{cases} 1, & i=\sigma,\\ 0, & i\neq\sigma.\end{cases}$$

Since $I_\sigma:\mathbb{N}\to\mathbb{R}$ is continuous $\forall\sigma\in\mathcal{N}^o$, $t\mapsto I_\sigma(\rho(x,t))$ is Lebesgue measurable $\forall(\sigma,x)\in\mathcal{N}^o\times\mathbb{R}^n$ (and $(x,t)\mapsto I_\sigma(\rho(x,t))$ is Lebesgue measurable $\forall\sigma\in\mathcal{N}^o$ in the Filippov case). The Lebesgue measurability of $t\mapsto f(x,t)$, $\forall x\in\mathbb{R}^n$ (and of $(x,t)\mapsto f(x,t)$ in the Filippov case) then follows from that of $t\mapsto f_\sigma(x,t), \forall(\sigma,x)\in\mathcal{N}^o\times\mathbb{R}^n$ (and of $(x,t)\mapsto f_\sigma(x,t)$, $\forall\sigma\in\mathcal{N}^o$ in the Filippov case). Since f is locally bounded, uniformly in t, $\mathcal{F}[f]$ and $\mathcal{K}[f]$ are also locally bounded, uniformly in t. In the Krasovskii case, since the functions $(x,t)\mapsto f_\sigma(x,t)$ and $(x,t)\mapsto f(x,t)$ are locally bounded and the functions $t\mapsto f_\sigma(x,t)$ and $t\mapsto f(x,t)$ are Lebesgue measurable, the maps $\mathcal{K}[f]$ and $\mathcal{K}[f_\sigma]$ admit local solutions [40, p. 101]. In the Filippov case, since the functions $(x,t)\mapsto f_\sigma(x,t)$ and $(x,t)\mapsto f(x,t)$ are essentially locally bounded and the functions $(x,t)\mapsto f_\sigma(x,t)$ and $(x,t)\mapsto f(x,t)$ are Lebesgue measurable, the maps $\mathcal{F}[f]$ and $\mathcal{F}[f_\sigma]$ admit local solutions [32, p. 85].

Since the collection $\{f_\sigma \mid \sigma\in\mathcal{N}^o\}$ is locally bounded, uniformly in t and σ, over $\Omega\times\mathcal{N}^o$, the collections $\{\mathcal{F}[f_\sigma]\mid\sigma\in\mathcal{N}^o\}$ and $\{\mathcal{K}[f_\sigma]\mid\sigma\in\mathcal{N}^o\}$ are also locally bounded, uniformly in t and σ, over $\Omega\times\mathcal{N}^o$. Hence, by Proposition 2, V is also a nonstrict Lyapunov function for the set-valued maps $(x,t)\mapsto\overline{\text{co}}\bigcup_{\sigma\in\mathcal{N}^o}\mathcal{F}[f_\sigma](x,t)$ and $(x,t)\mapsto\overline{\text{co}}\bigcup_{\sigma\in\mathcal{N}^o}\mathcal{K}[f_\sigma](x,t)$, over Ω, with the bounds $\underline{W}$, $\overline{W}$, and W. From Proposition 1, $\mathcal{F}[f](x,t)\subseteq\overline{\text{co}}\bigcup_{\sigma\in\mathcal{N}^o}\mathcal{F}[f_\sigma](x,t)$ and $\mathcal{K}[f](x,t)\subseteq\overline{\text{co}}\bigcup_{\sigma\in\mathcal{N}^o}\mathcal{K}[f_\sigma](x,t)$. Hence, V is also a nonstrict Lyapunov function for $\mathcal{F}[f]$ and $\mathcal{K}[f]$, over Ω, with the bounds $\underline{W}$, $\overline{W}$, and W. The conclusion then follows by Theorem 1. ■

Remark 1: The geometric condition in (17) can be relaxed to the following trajectory-based condition. For all the generalized solutions $x_\sigma:\mathcal{I}\to\mathbb{R}^n$ to (3), if the subsystems in (3) satisfy $\dot{\tilde{V}}_{F_\sigma}(x_\sigma(t),t)\leq -W(x_\sigma(t))$, $\forall\sigma\in\mathcal{N}^o$ and for almost all $t\in\mathcal{I}$, and for a specific maximal generalized solution $x^*:\mathcal{I}\to\mathbb{R}^n$ of (2), if the set $\{t\in\mathcal{I}\mid\rho(x^*(\cdot),\cdot)\text{ is discontinuous at }t\}$ is countable for every $\mathcal{I}\subseteq\mathbb{R}_{\geq t_0}$, then weak versions of Theorem 1 and Proposition 2 that establish the convergence of $W(x^*(t))$ to the origin as $t\to\infty$ can be proven using techniques similar to [30, Corollary 1].

Remark 2: If the subsystems are autonomous, and if they admit a common nonstrict Lyapunov function that is regular, then by applying the invariance principle (e.g., [37, Th. 3]) to the differential inclusions $\dot{x}\in\overline{\text{co}}\bigcup_{\sigma\in\mathcal{N}^o}\mathcal{F}[f_\sigma](x)$ and $\dot{x}\in\overline{\text{co}}\bigcup_{\sigma\in\mathcal{N}^o}\mathcal{K}[f_\sigma](x)$, it can be shown that all maximal generalized solutions of (2) that start in the set $\overline{C_l}$ converge to the largest weakly forward invariant set contained within $\overline{C_l}\cap\overline{E}$, where $E:=\{x\in\mathcal{D}\mid W(x)=0\}$ and C_l is a bounded connected component of the level set $\{x\in\mathcal{D}\mid V(x)\leq l\}$. Hence, Propositions 1 and 2 also generalize results such as [26] to switched nonsmooth systems. A similar result can also be obtained for the case where the subsystems are periodic.

VII. Switching Between Differential Inclusions

The results in Section IV, and hence, those in Section VI can be generalized to switched systems of the form

$$\dot{x}\in F_{\rho(x,t)}(x,t). \tag{19}$$

Let $F:\mathbb{R}^n\times\mathbb{R}_{\geq t_0}\rightrightarrows\mathbb{R}^n$ denotes the set valued map $(x,t)\mapsto F_{\rho(x,t)}(x,t)$. The asymptotic properties of the generalized solutions of the system

$$\dot{x}\in F(x,t) \tag{20}$$

can then be inferred using the asymptotic properties of the generalized solutions of the subsystems

$$\dot{x}\in F_\sigma(x,t) \tag{21}$$

where generalized solutions of a system of the form $\dot{x}\in F(x,t)$ are defined as the solutions of the differential inclusion $\dot{x}\in\mathcal{K}[F](x,t)$ in the Krasovskii case and $\dot{x}\in\mathcal{F}[F](x,t)$ in the Filippov case. The operators $\mathcal{F}$ and $\mathcal{K}$ are defined as in (4) and (5), respectively, where for a set $A\in\mathbb{R}^n$, the notation $\overline{\text{co}}\{F(y,t)\mid y\in A\}$ denotes the set $\overline{\text{co}}\cup_{y\in A}F(y,t)$.

Definition 5: The collection $\{F_\sigma:\mathbb{R}^n\times\mathbb{R}_{\geq t_0}\rightrightarrows\mathbb{R}^n\}_{\sigma\in\mathcal{N}^o}$ is said to *satisfy the weak basic conditions* if:

1) it is locally bounded in the Krasovskii case and essentially locally bounded in the Filippov case, uniformly in σ and t and
2) the maps $t\mapsto F_\sigma(x,t)$ and the functions $t\mapsto\rho(x,t)$ are Lebesgue measurable $\forall(x,\sigma)\in\mathbb{R}^n\times\mathcal{N}^o$ in the Krasovskii case and the maps $(x,t)\mapsto F_\sigma(x,t)$ and $(x,t)\mapsto\rho(x,t)$ are Lebesgue measurable $\forall\sigma\in\mathcal{N}^o$ in the Filippov case.

△

The following theorem generalizes Theorem 2 to switched differential inclusions.

Theorem 3: Let $0\in\mathcal{D}$, $r>0$ be selected such that $\overline{\text{B}}(0,r)\subset\mathcal{D}$, and let $\Omega:=\mathcal{D}\times\mathbb{R}_{\geq t_0}$. Let $\{F_\sigma:\mathbb{R}^n\times\mathbb{R}_{\geq t_0}\rightrightarrows\mathbb{R}^n\}_{\sigma\in\mathcal{N}^o}$ be a collection that satisfies the weak basic conditions in Definition 5. If Assumption 1 holds and the (Filippov) Krasovskii regularizations of the subsystems in (21) admit a common nonstrict Lyapunov function $V:\Omega\to\mathbb{R}$, over Ω, with the bounds $\underline{W}:\mathcal{D}\to\mathbb{R}$, $\overline{W}:\mathcal{D}\to\mathbb{R}$, and $W:\mathcal{D}\to\mathbb{R}$, then every maximal solution of the (Filippov) Krasovskii regularization of the switched system in (20) with $x(t_0)\in\{x\in\overline{\text{B}}(0,r)\mid\overline{W}(x)\leq c\}$, for some $c\in(0,\min_{\|x\|_2=r}\underline{W}(x))$, is complete, bounded, and satisfies $\lim_{t\to\infty}W(x(t))=0$. In addition, if $\mathcal{D}=\mathbb{R}^n$ and the sets $\{x\in\mathbb{R}^n\mid\underline{W}(x)\leq c\}$ are compact for all $c\in\mathbb{R}_{>0}$, then every maximal solution of the (Filippov) Krasovskii regularization of the switched system in (20), regardless of the initial condition, is complete, bounded, and satisfies $\lim_{t\to\infty}W(x(t))=0$.

Proof: The first step is to show that under the weak basic conditions, the maps $\mathcal{K}[F_\sigma]$, $\mathcal{K}[F]$, $\mathcal{F}[F_\sigma]$, and $\mathcal{F}[F]$ admit local solutions $\forall \sigma \in \mathcal{N}^o$. Since the collection $\{F_\sigma\}_{\sigma\in\mathcal{N}^o}$ is (essentially) locally bounded, uniformly in σ and t, the map F is (essentially) locally bounded, uniformly in t. To establish measurability of F, consider the representation $F(x,t) = \cup_{\sigma\in\mathcal{N}^o} I_\sigma(\rho(x,t)) \cap F_\sigma(x,t)$, where

$$I_\sigma(i) := \begin{cases} \mathbb{R}^n, & i = \sigma, \\ \emptyset, & i \neq \sigma. \end{cases}$$

Since $I_\sigma : \mathbb{N} \to \mathbb{R}$ is a step mapping [41, p. 643] $\forall \sigma \in \mathcal{N}^o$, $t \mapsto I_\sigma(\rho(x,t))$ is Lebesgue measurable $\forall(\sigma,x) \in \mathcal{N}^o \times \mathbb{R}^n$. Using [41, Proposition 14.11], it can be concluded that $t \mapsto F(x,t)$ is Lebesgue measurable $\forall x \in \mathbb{R}^n$ if $t \mapsto F_\sigma(x,t)$ is Lebesgue measurable $\forall(\sigma,x) \in \mathcal{N}^o \times \mathbb{R}^n$ and $(x,t) \mapsto F(x,t)$ is Lebesgue measurable if $(x,t) \mapsto F_\sigma(x,t)$ is Lebesgue measurable $\forall \sigma \in \mathcal{N}^o$.

In the Krasovskii case, it is clear that the maps $\mathcal{K}[F]$ and $\mathcal{K}[F_\sigma]$ are locally bounded, upper semicontinuous in x for each t, and have compact, nonempty, and convex values. Since $F(x,t) \subseteq \mathcal{K}[F](x,t)$ and $F_\sigma(x,t) \subseteq \mathcal{K}[F_\sigma](x,t)$, $\forall(x,t) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0}$, and since F and F_σ are Lebesgue measurable in t for all x, the maps $\mathcal{K}[F]$ and $\mathcal{K}[F_\sigma]$ admit selections that are Lebesgue measurable in t for all x [41, Corollary 14.6], and as a result, admit local solutions [32, Th. 5, p. 83].

In the Filippov case, using Rockafeller and Wets' generalization of Lusin's theorem [41, Th. 14.10], the proof of [40, Proposition 1, p. 102] can be extended to show that the maps $\mathcal{F}[F]$ and $\mathcal{F}[F_\sigma]$ are locally bounded, upper semicontinuous in x for all t, have compact, nonempty, and convex values, and for each fixed $t \in \mathbb{R}_{\geq t_0}$, $F(x,t) \subseteq \mathcal{F}[F](x,t)$ and $F_\sigma(x,t) \subseteq \mathcal{F}[F_\sigma](x,t)$ for almost all $x \in \mathbb{R}^n$. Since F and F_σ admit Lebesgue measurable selections [41, Corollary 14.6], there exist Lebesgue measurable functions $g : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n$ and $g_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n$ such that $g(x,t) \in \mathcal{F}[F](x,t)$ and $g_\sigma(x,t) \in \mathcal{F}[F_\sigma](x,t)$ almost everywhere. Therefore, $\mathcal{F}[F]$ and $\mathcal{F}[F_\sigma]$ admit Lebesgue measurable selections, and as a result, admit local solutions.

Since the maps the maps $\mathcal{K}[F]$, $\mathcal{K}[F_\sigma]$, $\mathcal{F}[F]$, and $\mathcal{F}[F_\sigma]$ satisfy all the conditions of Theorem 1, Proposition 2, and Corollary 1, similar arguments as the proof of Theorem 2 can be used to prove Theorem 3 if Proposition 1 can be generalized to Filippov and Krasovskii regularization of set-valued maps. The proof of Proposition 1 in the Krasovskii case does not rely on any properties of f and $\{f_\sigma\}$ other then local boundedness, uniformly in σ. Therefore, it can be trivially generalized to include Krasovskii regularization of set-valued maps.

In the Filippov case, the proof of Proposition 1 relies on Lemma 1 from [42]. Using Rockafeller and Wets' generalization of Lusin's theorem [41, Th. 14.10], Lemma 1 from [42] can be extended to include Lebesgue measurable set-valued maps (see Th. 4 in the appendix), and hence, Proposition 1 can be generalized to include the Filippov regularization of set-valued maps. ■

VIII. Comments on the Generalized Time Derivative

If V is regular then the generalized time derivative obtained using Definition 3 is generally more conservative than (i.e., greater than or equal to) the maximal element of the more popular set-valued generalized derivatives defined in [36] and [37]. The motivation behind the use of the seemingly restrictive definition is that the invariance-like results in Section VI do not hold if the time derivative of the cLf is interpreted in the set-valued sense (see Example 2). Furthermore, through a reduction of the admissible directions in F using locally Lipschitz-continuous regular functions, a generalized time derivative that is less conservative than the set-valued derivatives in [36] and [37] can be obtained (see Lemma 1 and Corollary 2).

Lemma 1: Let $\Omega := \mathcal{D} \times \mathbb{R}_{\geq t_0}$, $V : \Omega \to \mathbb{R}$ be a locally Lipschitz continuous function, and $\mathcal{V} := \{V_i : \Omega \to \mathbb{R}\}_{i\in\mathcal{M}\subseteq\mathbb{N}}$ be a countable collection of locally Lipschitz-continuous regular [35, Definition 2.3.4] functions. Let $F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n$ be a map that admits local solutions over Ω and let $G, G_i, \tilde{F} : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n$ be defined as

$$G_i(x,t) := \left\{q \in F(x,t) \,|\, \exists a_f \,|\, p^T[q;1] = a_f \quad \forall p \in \partial V_i(x,t)\right\},$$
$$\tilde{F}(x,t) := F(x,t) \cap \left(\cap_{i=1}^{\infty} G_i(x,t)\right) \qquad \forall (x,t) \in \Omega.$$

If

$$\underline{\dot{V}}_{\tilde{F}}(x,t) \leq -W(x) \qquad \forall (x,t) \in \Omega \tag{22}$$

where $\underline{\dot{V}}_{(\cdot)}$ is introduced in (18) and $\underline{\dot{V}}_{\tilde{F}}(x,t)$ is understood to be $-\infty$ when $\tilde{F}(x,t)$ is empty, then each solution of (13), such that $x(t_0) \in \mathcal{D}$, satisfies $\dot{V}(x(t),t) \leq -W(x(t))$, for almost all $t \in [t_0, T)$, where $T := \min(\sup \mathcal{I}, \inf\{t \in \mathcal{I} \mid x(t) \notin \mathcal{D}\})$.

Proof: See the appendix. ■

Instead of maximizing over $\tilde{F}$, the upper bound of the generalized time derivative $\dot{\tilde{V}}^{(F)}$, introduced in [37, p. 364], is computed using maximization over the set $G(x,t) := \{q \in F(x,t) \,|\, \exists a_f \,|\, p^T[q;1] = a_f, \forall p \in \partial V(x,t)\}$, i.e.,[10]

$$\max \dot{\tilde{V}}^{(F)}(x,t) = \min_{p\in\partial V(x,t)} \max_{q\in G(x,t)} p^T[q;1].$$

Note that if $V \in \mathcal{V}$ then $\underline{\dot{V}}_{\tilde{F}} = \dot{\tilde{V}}_{\tilde{F}}$, $\tilde{F} \subseteq G$, and hence, $\underline{\dot{V}}_{\tilde{F}}(x,t) = \dot{\tilde{V}}_{\tilde{F}}(x,t) \leq \max \dot{\tilde{V}}^{(F)}(x,t), \forall(x,t) \in \Omega$. Thus, depending on the functions $\mathcal{V}$ selected to reduce the inclusions, the notions of the generalized time derivative introduced here can be less conservative than the set-valued derivative in [37] (and hence, the set-valued derivative in [36]). Naturally, if $\mathcal{V} = \{V\}$ then the notions introduced here are equivalent to [37].

A function V that satisfies the conditions of Lemma 1 is hereafter called a *$\mathcal{V}-$nonstrict Lyapunov function* for $F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n$ over Ω with the bounds $\overline{W}$, $\underline{W}$, and W. The following corollary is a straightforward consequence of Theorem 1 and Lemma 1.

Corollary 2: Let $0 \in \mathcal{D}$ and $\Omega := \mathcal{D} \times \mathbb{R}_{\geq t_0}$. Assume that the differential inclusion in (13) admits a $\mathcal{V}-$ nonstrict Lyapunov function over Ω with the bounds $\overline{W} : \mathcal{D} \to \mathbb{R}$, $\underline{W} : \mathcal{D} \to \mathbb{R}$, and $W : \mathcal{D} \to \mathbb{R}$. If $F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n$ admits local solutions over Ω and is locally bounded, uniformly in t, over Ω, then every maximal solution of (13) with $x(t_0) \in \{x \in \overline{\mathrm{B}}(0,r) \mid \overline{W}(x) \leq$

[10]The minimization here serves to maintain consistency of notation, but is in fact, redundant.

$c\}$, for some $c \in (0, \min_{\|x\|_2 = r} \underline{W}(x))$, is complete, bounded, and satisfies $\lim_{t\to\infty} W(x(t)) = 0$. △

At this juncture, it would be natural to ask whether the result in Theorem 2 can be established using the set-valued derivatives in [36] and [37] or a common $\mathcal{V}-$ nonstrict Lyapunov function. The following example demonstrates that a common $\mathcal{V}-$ nonstrict Lyapunov function is not sufficient to establish the results in Section VI and neither are the set-valued derivatives in [36] or [37]. Furthermore, the example also demonstrates that the continuity assumption in Corollary 1 is not vacuous.

Example 2: Let $g_1, g_2, g_3 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $g_1(x) := [x_1; 0]$, $g_2(x) := [0; x_2]$, and $g_3(x) := [-x_1; -x_2]$. Let the subsystems be defined by $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$f_1(x) = \begin{cases} g_1(x) & |x_1| < |x_2| \\ g_3(x) & |x_1| \geq |x_2| \end{cases}, \quad f_2(x) = \begin{cases} g_2(x) & |x_1| < |x_2| \\ g_3(x) & |x_1| \geq |x_2|. \end{cases}$$

The subsystems have identical Krasovskii and Filippov regularizations, given by

$$F_1(x) = \begin{cases} \overline{\text{co}}\{g_1(x), g_3(x)\} & |x_1| = |x_2| \\ f_1(x) & \text{otherwise,} \end{cases}$$

$$F_2(x) = \begin{cases} \overline{\text{co}}\{g_2(x), g_3(x)\} & |x_1| = |x_2| \\ f_2(x) & \text{otherwise.} \end{cases}$$

The function $V : \mathbb{R}^2 \to \mathbb{R}$, defined as $V(x) := \max(|x_1|, |x_2|)$, is a locally Lipschitz-continuous regular function[11] that satisfies (16) and

$$\partial V(x) = \begin{cases} v_1(x) & |x_1| < |x_2| \\ v_2(x) & |x_1| > |x_2| \\ \overline{\text{co}}\{v_1(x), v_2(x)\} & |x_1| = |x_2| \end{cases}$$

where $v_1(x) = [\text{sgn}(x_1); 0]$ and $v_2(x) = [0; \text{sgn}(x_2)]$. Hence, with $\mathcal{V} = \{V\}$,

$$\tilde{F}_i(x) = \begin{cases} \{0\} & |x_1| = |x_2| \\ F_i(x) & \text{otherwise} \end{cases},$$

for $i = 1, 2$.

In this case, $(v_1(x))^T f_2(x) = (v_2(x))^T f_1(x) = 0$, $(v_1(x))^T f_3(x) = -|x_1|$, and $(v_2(x))^T f_3(x) = -|x_2|$. It follows that $\underline{\dot{V}}_{F_i}(x) \leq 0$ and $\underline{\dot{V}}_{\tilde{F}_i}(x) = \dot{\overline{V}}_{\tilde{F}}(x) \leq 0$, $\forall x \in \mathbb{R}^2$ and $i = 1, 2$. It is also easy to see that $\max \dot{\bar{V}}^{(F_i)}(x) \leq 0$ and $\max \dot{\tilde{V}}^{(F_i)}(x) \leq 0$, $\forall x \in \mathbb{R}^2$ and $i = 1, 2$, where $\dot{\tilde{V}}^{(F_i)}$ is defined in [36, eq. (13)]. Thus, V is a common nonstrict Lyapunov function for the subsystems according to all the notions of the generalized time derivatives discussed above.

Let $F := x \mapsto \overline{\text{co}}(F_1(x) \cup F_2(x))$. For any $x \in \mathbb{R}^2$ such that $|x_1| = |x_2|$, $q := \frac{1}{2}[x_1; x_2] \in \overline{\text{co}}\{g_1(x), g_2(x), g_3(x)\} = F(x)$. Thus, whenever $|x_1| = |x_2| = V(x) > 0$, $\min_{p \in \partial V(x)} p^T q = 0.5V(x) > 0$, i.e., Proposition 2 does not hold. Furthermore, a solution of $\dot{x} \in F(x)$, starting at $x = [1; 1]$, is $x(t) = \mathrm{e}^{0.5t}[1; 1]$, i.e., Theorem 2 does not hold.

[11]Pointwise maxima of locally Lipschitz-continuous regular functions is locally Lipschitz-continuous and regular.

Thus, Proposition 2 and Theorem 2 may not hold if the generalized time derivative is understood in the sense of Lemma 1, $\dot{\bar{V}}^{(\cdot)}$ in [37], or $\dot{\tilde{V}}^{(\cdot)}$ in [36]. Furthermore, if $\underline{\dot{V}}_F$ is used as the generalized time derivative instead of $\dot{\overline{V}}_F$ then Corollary 1 may not hold if the set-valued maps $\{F_\sigma\}$ are not continuous. △

IX. Design Examples

Many of the applications discussed in the opening paragraphs of Section I can be represented by the following example problems. The first example demonstrates the utility of the developed technique on an adaptive control problem where only the regession matrices are discontinuous. In the second example, an adaptive controller for a switched system that exhibits arbitrary switching between the subsystems with different parameters and disturbances is analyzed.

Example 3: Consider the nonlinear dynamical system

$$\dot{x} = Y_{\rho(x,t)}(x)\theta + u + d(t) \tag{23}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}^n$ denotes the control input, $d : \mathbb{R}_{\geq t_0} \to \mathbb{R}^n$ denotes an unknown disturbance, $\rho : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{N}$ denotes a switching signal that satisfies Assumption 1, $Y_\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times L}$, for each $\sigma \in \mathbb{N}$, is a known continuous function, and $\theta \in \mathbb{R}^L$ is the vector of constant unknown parameters. The control objective is to regulate the system state to the origin. The disturbance is assumed to be bounded, with a known bound $\overline{d}$ such that $\|d(t)\|_\infty \leq \overline{d}$, for almost all $t \in \mathbb{R}_{\geq t_0}$. Furthermore, $t \mapsto d(t)$ is assumed to be Lebesgue measurable.

One example of an adaptive controller designed to satisfy the control objective is $u = -kx - Y_{\rho(x,t)}(x)\hat{\theta} - \beta\,\text{sgn}(x)$, where $\hat{\theta} : \mathbb{R}_{\geq t_0} \to \mathbb{R}^L$ denotes an estimate of the vector of unknown parameters, θ, k, $\beta \in \mathbb{R}_{>0}$ are positive constant control gains, and $\text{sgn}(\cdot)$ is the signum function. The estimate, $\hat{\theta}$, is obtained from the update law $\dot{\hat{\theta}} = (Y_{\rho(x,t)}(x))^T x$. For each $\sigma \in \mathbb{N}$, the closed-loop error system can be expressed as

$$\dot{x} = -kx + Y_\sigma(x)\tilde{\theta} + d(t) - \beta\,\text{sgn}(x) \tag{24}$$

$$\dot{\tilde{\theta}} = -(Y_\sigma(x))^T x \tag{25}$$

where $\tilde{\theta} := \theta - \hat{\theta}$ denotes the parameter estimation error. The closed-loop system in (24) and (25) is discontinuous, and hence, does not admit classical solutions. Thus, the analysis will focus on generalized solutions of (24) and (25). Since the Filippov and Krasovskii regularizations of the closed-loop system in (24) and (25) are identical, they are denoted by $\mathrm{K}[\cdot]$ and the solutions of the corresponding differential inclusions are hereafter referred to as generalized solutions.

To analyze the developed controller, consider the cLf $V : \mathbb{R}^{n+L} \to \mathbb{R}_{\geq t_0}$, defined as

$$V(z) := \frac{1}{2}x^T x + \frac{1}{2}\tilde{\theta}^T\tilde{\theta} \tag{26}$$

where $z := [x; \tilde{\theta}]$. Since the cLf is continuously differentiable, the Clarke gradient reduces to the standard gradient, i.e, $\partial V(z, t) = \{z\}$. Using the calculus of $\mathrm{K}[\cdot]$ from [43], a bound on the regularization of the system in (24) and (25) can be computed as $F_\sigma(z,t) \subseteq F'_\sigma(z,t)$, where

$$F'_\sigma(z,t) = \begin{bmatrix} \left\{-kx + Y_\sigma(x)\tilde{\theta} + d(t)\right\} - \beta \, \mathrm{K}[\mathrm{sgn}](x) \\ \left\{-Y_\sigma^T(x)\, x\right\} \end{bmatrix}.$$

Using Definition 3 and the fact that $x^T \mathrm{K}[\mathrm{sgn}](x) = \{\|x\|_1\}$, a bound on the generalized time derivative of the cLf can be computed as

$$\begin{aligned} \dot{\tilde{V}}_\sigma(z,t) &= \max_{q \in F_\sigma(z,t)} z^T q, \\ &\leq \max_{q \in F'_\sigma(z,t)} z^T q, \\ &= -k\|x\|_2^2 + x^T d(t) - \beta\|x\|_1 . \end{aligned}$$

Provided $\beta > \overline{d}$

$$\dot{\tilde{V}}_\sigma(z,t) \leq -W(z) \tag{27}$$

$\forall (z,\sigma) \in \mathbb{R}^{n+L} \times \mathbb{N}$ and for almost all $t \in \mathbb{R}_{\geq t_0}$, where $W(z) = k\|x\|_2^2$ is a positive semidefinite function. Using (26), (27), and Theorem 2, all maximal generalized solutions of the switched nonsmooth system in (24) and (25) are complete, bounded, and satisfy $\|x(t)\|_2 \to 0$ as $t \to \infty$. △

Example 4: Arbitrary switching between the systems with different parameters and disturbances can be achieved in the case where the number of subsystems is finite. For example, consider the nonlinear dynamical system

$$\dot{x} = Z_{\rho(x,t)}(x,t)\,\theta_{\rho(x,t)} + d_{\rho(x,t)}(x,t) + u \tag{28}$$

where $\rho : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathcal{N}^o$ such that $\mathcal{N}^o$ is a finite set, $Z_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^{n \times L}$, are known functions, $\theta_\sigma \in \mathbb{R}^L$ are vectors of constant unknown parameters corresponding to each $\sigma \in \mathcal{N}^o$, and $d_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n$ are unknown disturbances such that for each $\sigma \in \mathcal{N}^o$, $\|d_\sigma(x,t)\|_\infty \leq \overline{d}_\sigma, \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0}$ and some $\overline{d}_\sigma > 0$. Furthermore, for each $\sigma \in \mathbb{R}^n$, $(x,t) \mapsto d_\sigma(x,t)$ and $(x,t) \mapsto Z_\sigma(x,t)$ are continuous in x, uniformly in t and Lebesgue measurable in t, $\forall x \in \mathbb{R}^n$. Let $\theta := \left[\theta_1; \theta_2; \cdots; \theta_{|\mathcal{N}^o|}\right] \in \mathbb{R}^{L|\mathcal{N}^o|}$ and let $Y_\sigma := \mathbf{1}_\sigma \otimes Z_\sigma$, where $\mathbf{1}_\sigma \in \mathbb{R}^{1 \times L}$ is a matrix defined by

$$(\mathbf{1}_\sigma)_{1,j} = \begin{cases} 1, & j = \sigma. \\ 0, & \text{otherwise.} \end{cases}$$

The adaptive controller designed to satisfy the control objective is

$$u = -k_{\rho(x,t)} x - Y_{\rho(x,t)}(x,t)\,\hat{\theta} - \beta_{\rho(x,t)} \,\mathrm{sgn}(x)$$

where $\beta_\sigma \in \mathbb{R}_{>0}$ and $k_\sigma \in \mathbb{R}_{>0}$ are control gains corresponding to $\sigma \in \mathcal{N}^o$ and $\hat{\theta} : \mathbb{R}_{\geq t_0} \to \mathbb{R}^{L|\mathcal{N}^o|}$ is updated according to $\dot{\hat{\theta}} = (Y_{\rho(x,t)}(x,t))^T x$. A stability analysis similar to Example 3 can then be utilized to conclude the asymptotic convergence of the state x to the origin provided $\beta_\sigma > \overline{d}_\sigma, \forall \sigma \in \mathcal{N}^o$. △

X. Conclusion

Motivated by applications in switched adaptive control, the generalized LaSalle–Yoshizawa corollary in [30] is extended to switched nonsmooth systems. The extension facilitates the analysis of the asymptotic characteristics of a switched system based on the asymptotic characteristics of its subsystems where a nonstrict common Lyapunov function can be constructed for the subsystems. The application of the developed extension to a switched adaptive system is demonstrated through simple examples. Motivated by results such as [44], further research could potentially extend the developed method to utilize indefinite Lyapunov functions.

In Lemma 1, it is shown that arbitrary locally Lipschitz-continuous regular functions can be used to reduce the differential inclusion to a smaller set of admissible directions. This observation indicates that there may be a smallest set of admissible directions corresponding to each differential inclusion. Further research is needed to establish the existence of such a set and to find a representation of it that facilitates computation.

The developed method requires a strong convergence result for the subsystems, i.e., the existence of a common cLf that satisfies (17) implies that all maximal generalized solutions of the subsystems are bounded and asymptotically converge to the origin. Future research will focus on the development of results for switched nonsmooth systems where only weak convergence results (that is, only a subset of the maximal generalized solutions of the subsystems are bounded and asymptotically converge to the origin) are available for the subsystems.

Appendix

Proof of Theorem 1: Similar to the proof of [30, Corollary 1], it is established that the bound on $\dot{\tilde{V}}_F$ in (15) implies that the cLf is nonincreasing along all the maximal solutions of (13). The nonincreasing property of the cLf is used to establish boundedness of x, which is used to prove the existence and uniform continuity of complete solutions. Barbălat's lemma [17, Lemma 8.2] is then used to conclude the proof.

To show that the cLf is nonincreasing, let $x : \mathcal{I} \to \mathbb{R}^n$ be a maximal solution of (13) such that $x(t_0) \in \Omega_c := \{x \in \overline{\mathrm{B}}(0,r) | \overline{W}(x) \leq c\}$. Define $T > t_0$ be the first exit time of x from $\mathcal{D}$, i.e., $T := \min(\sup \mathcal{I}, \inf\{t \in \mathcal{I} \mid x(t) \notin \mathcal{D}\})$, where $\inf \emptyset$ is assumed to be ∞. If V is locally Lipschitz-continuous but not regular, then [45, Proposition 4] (see also, [46, Th. 2]) can be used to conclude that, for almost every $t \in [t_0, T)$, the time derivative $\dot{V}(x(t),t)$ exists, and $\exists p_0 \in \partial V(x(t),t)$ such that $\dot{V}(x(t),t) = p_0^T[\dot{x}(t); 1]$. Thus, (15) and (17) imply that $\dot{V}(x(t),t) \leq -W(x(t))$ for almost every $t \in [t_0, T)$. If V is regular, then the relaxation in Footnote 8 and [36, eq. (22)] can be used to conclude that for almost every $t \in [t_0, T)$, the time derivative $\dot{V}(x(t),t)$ exists and $\dot{V}(x(t),t) \leq -W(x(t))$. The conclusion that

$$V(x(t_0), t_0) \geq V(x(t), t) \quad \forall t \in [t_0, T) \tag{29}$$

then follows from [30, Lemma 2].

Using (29), it can be shown that (see, e.g., [17, Th. 4.8]) every solution of (13) that starts in Ω_c stays in $\overline{\mathrm{B}}(0,r)$ on every interval of its existence. Therefore, all the maximal solutions of (13) such that $x(t_0) \in \Omega_c$ are precompact [24, Definition 2.3] and $T = \sup \mathcal{I}$. In the following, the arguments similar to [47, Proposition 2] are used to show that the precompact maximal solutions are complete.

For the sake of contradiction, assume that $T < \infty$. Since F is locally bounded, uniformly in t, over Ω, and $x(t) \in \overline{\mathrm{B}}(0,r)$ on $[t_0,T)$, the map $t \mapsto F(x(t),t)$ is uniformly bounded on $[t_0,T)$. Hence, (14) implies that $\dot{x} \in \mathcal{L}_\infty([t_0,T),\mathbb{R}^n)$. Local absolute continuity of $t \mapsto x(t)$ implies that $\forall t_1,t_2 \in [t_0,T)$, $\|x(t_2)-x(t_1)\|_2 = \|\int_{t_1}^{t_2}\dot{x}(\tau)\mathrm{d}\tau\|_2$. Since $\dot{x} \in \mathcal{L}_\infty([t_0,T),\mathbb{R}^n)$, $\|\int_{t_1}^{t_2}\dot{x}(\tau)\mathrm{d}\tau\|_2 \leq \int_{t_1}^{t_2} M\mathrm{d}\tau$, for some $M>0$. Thus, $\|x(t_2)-x(t_1)\|_2 \leq M|t_2-t_1|$, and hence, $t \mapsto x(t)$ is uniformly continuous on $[t_0,T)$. Therefore, x can be extended into a continuous function $x' : [t_0,T] \to \mathbb{R}^n$. Invoking [32, p. 83, Th. 5], x' can be extended into a solution of (13) on the interval $[t_0,T')$ for some $T' > T$, which contradicts the maximality of x. Hence, $T=\infty$, i.e., all the precompact maximal solutions of (13) are complete.

The continuity of $x \mapsto W(x)$ and compactness of $\overline{\mathrm{B}}(0,r)$ imply that $x \mapsto W(x)$ is uniformly continuous on $\overline{\mathrm{B}}(0,r)$. Since $t \mapsto x(t)$ is uniformly continuous on $\mathbb{R}_{\geq t_0}$, $t \mapsto W(x(t))$ is uniformly continuous on $\mathbb{R}_{\geq t_0}$. Furthermore, $t \mapsto \int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau$ is monotonically increasing and from (17) and the fact that V is positive definite

$$\int_{t_0}^{t} W(x(\tau))\,\mathrm{d}\tau \leq V(x(t_0),t_0) - V(x(t),t) \leq V(x(t_0),t_0).$$

Hence, $\lim_{t\to\infty}\int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau$ exists and is finite. By Barbălat's Lemma [17, Lemma 8.2], $\lim_{t\to\infty} W(x(t)) = 0$. ■

Proof of Proposition 1 for Filippov Regularization: Fix $(x,t) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0}$, select $\delta^* > 0$ such that $|\rho(\mathrm{B}(x,\delta^*),t)| < \infty$, and let $\mathcal{N} := \rho(\mathrm{B}(x,\delta^*),t)$. Similar to the proof for the Krasovskii regularization, the proof proceeds in three steps. First, it is observed that

$$\begin{aligned}&\bigcap_{\delta>0}\bigcap_{\mu(N)=0}\overline{\mathrm{co}}\{f_{\rho(y,t)}(y,t) \mid y \in \mathrm{B}(x,\delta)\setminus N\}\\ &\subseteq \bigcap_{\delta>0}\bigcap_{\mu(N)=0} A_N^\delta(x,t)\end{aligned} \tag{30}$$

where $A_N^\delta := \overline{\mathrm{co}}\bigcup_{\sigma\in\mathcal{N}}\{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\setminus N\}$. Second, it is established that

$$\bigcap_{\delta>0}\bigcap_{\mu(N)=0} A_N^\delta(x,t) \subseteq \bigcap_{\delta>0}\bigcap_{\mu(N)=0} B_N^\delta(x,t) \tag{31}$$

where $B_N^\delta(x,t) := \mathrm{co}\bigcup_{\sigma\in\mathcal{N}} B_{N\delta\sigma}(x,t)$ and $B_{N\delta\sigma}(x,t) := \overline{\mathrm{co}}\{f_\sigma(y,t) \mid y \in \mathrm{B}(x,\delta)\setminus N\}$. Finally, it is shown that $\forall x \in \mathbb{R}^n$ and almost all $t \in \mathbb{R}_{\geq t_0}$

$$\bigcap_{\delta>0}\bigcap_{\mu(N)=0} B_N^\delta(x,t) \subseteq \mathrm{co}\bigcup_{\sigma\in\mathcal{N}}\bigcap_{\delta>0}\bigcap_{\mu(N)=0} B_{N\delta\sigma}(x,t). \tag{32}$$

The conclusion of the proposition then follows. Apart from the technical detail required to handle the exclusion of measure-zero sets in the Filippov inclusion, the methods utilized to prove (31) and (32) are similar to those used in the proof for the Krasovskii inclusions. Thus, in the following, only the techniques used to handle the exclusion of measure-zero sets are illustrated.

The containment in (30) is self-evident. To prove (31), define $\overline{\mathscr{N}}(\delta) := \{N \subset \mathrm{B}(x,\delta) \mid \mu(N)=0\}$, and let $N^*(\delta) \subset 2^{\mathrm{B}(x,\delta)}$ be a collection of sets of zero measure such that $\sup\{\|\theta\| \mid \theta \in A_N^\delta\} < \infty$, $\forall N \in N^*(\delta)$. Since the functions $f_\sigma(x,t)$ are locally essentially bounded, uniformly in t and σ, the collection $N^*(\delta)$ is nontrivial. Fix $N \in N^*(\delta)$ and $z \in A_N^\delta$. Using the arguments similar to Part 1 of the proof it can be shown that the point z is a convex combination of points from $B_{N\delta\sigma_j}(x,t)$. That is, $z \in \mathrm{co}\,B_N^\delta(x,t)$, and hence

$$\bigcap_{N\in N^*(\delta)} A_N^\delta(x,t) \subseteq \bigcap_{N\in N^*(\delta)} B_N^\delta(x,t). \tag{33}$$

To establish (31) the intersection in (33) needs to include all of $\overline{\mathscr{N}}(\delta)$, not just the subset $N^*(\delta)$. Since $N^*(\delta) \subseteq \overline{\mathscr{N}}(\delta)$, the inclusion $\bigcap_{N\in N^*(\delta)} A_N^\delta(x,t) \subseteq \bigcap_{N\in\overline{\mathscr{N}}(\delta)} A_N^\delta(x,t)$ follows. Let $M \in \overline{\mathscr{N}}(\delta)$. There exist $N^1 \in \overline{\mathscr{N}}(\delta)\setminus N^*(\delta)$ and $N^0 \in N^*(\delta)$ such that $M = N^1 \cup N^0$. Since $N^0 \subseteq M$, $A_M^\delta(x,t) \subseteq A_{N^0}^\delta(x,t)$. Therefore, $\bigcap_{N\in\overline{\mathscr{N}}(\delta)} A_N^\delta(x,t) \subseteq \bigcap_{N\in N^*(\delta)} A_N^\delta(x,t)$, which implies $\bigcap_{N\in N^*(\delta)} A_N^\delta(x,t) = \bigcap_{N\in\overline{\mathscr{N}}(\delta)} A_N^\delta(x,t)$. A similar reasoning for $B_N^\delta(x,t)$ yields $\bigcap_{N\in N^*(\delta)} B_N^\delta(x,t) = \bigcap_{N\in\overline{\mathscr{N}}(\delta)} B_N^\delta(x,t), \forall\delta \in (0,\delta^*]$, which proves (31).

As an intermediate step toward proving (32), the containment

$$\bigcap_{\mu(N)=0} B_N^\delta(x,t) \subseteq \mathrm{co}\bigcup_{\sigma\in\mathcal{N}}\bigcap_{\mu(N)=0} B_{N\delta\sigma}(x,t) \qquad \forall\delta>0 \tag{34}$$

is established in the following. Let $z \in \bigcap_{\mu(N)=0} B_N^\delta(x,t)$. The objective now is to show that

$$z \in \mathrm{co}\left(\bigcap_{\mu(N)=0} B_{N\delta 1}(x,t) \cup \bigcap_{\mu(N)=0} B_{N\delta 2}(x,t) \cup \cdots\right).$$

Since the functions $(x,t) \mapsto f_\sigma(x,t)$ are Lebesgue measurable, the functions $x \mapsto f_\sigma(x,t)$ are Lebesgue measurable $\forall(\sigma,t) \in \mathcal{N}\times\mathbb{R}_{\geq t_0}$. Using [42, Lemma 1], it can be concluded that $\forall(x,t,\delta,\sigma) \in \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \times \mathbb{R}_{>0} \times \mathcal{N}$, there exists a measure-zero set N_σ such that, $\bigcap_{\mu(N)=0} B_{N\delta\sigma}(x,t) = B_{N_\sigma\delta\sigma}(x,t)$. Define $N^* := \bigcup_{\sigma\in\mathcal{N}} N_\sigma$. Since N^* is a countable union of measure-zero sets, $\mu(N^*) = 0$. The fact that $z \in \bigcap_{\mu(N)=0} B_N^\delta(x,t)$ implies that $z \in B_{N^*}^\delta(x,t)$ and hence, by the Carathéodory's Theorem [38, p. 103], there exist $\{z_1,\ldots,z_m\}$ such that each $z_j \in B_{N^*\delta\sigma_j}(x,t)$ for some $\sigma_j \in \mathcal{N}$, and the positive real numbers $\{a_1,\ldots,a_m\}$ with $\sum_{j=1}^m a_j = 1$, such that $z = \sum_{j=1}^m a_j z_j$. By definition of N^*, $N_\sigma \subseteq N^*$, $\forall\sigma \in \mathcal{N}$. As a result, $B_{N^*\delta\sigma}(x,t) \subseteq B_{N_\sigma\delta\sigma}(x,t)$, $\forall\sigma\in\mathcal{N}$, and hence, $B_{N^*\delta\sigma}(x,t) \subseteq \bigcap_{\mu(N)=0} B_{N\delta\sigma}(x,t)$, $\forall\sigma\in\mathcal{N}$. Hence, for each $j \in \{1,\ldots,m\}$, $z_j \in \bigcap_{\mu(N)=0} B_{N\delta\sigma_j}(x,t)$ for some $\sigma_j \in \mathcal{N}$, which implies (34). Using a nesting argument similar to the proof for Krasovskii inclusions, the containment in (32) follows $\forall(x,t) \in \mathbb{R}_n \times \mathbb{R}_{\geq t_0}$. ■

Proof of Lemma 1: The proof closely follows [37, Lemma 1]. Let $x : \mathcal{I} \to \mathbb{R}^n$ be a solution of (13) such that $x(t_0) \in \mathcal{D}$. Consider the set of times $\mathcal{T} \subseteq [t_0,T)$ where $\dot{x}(t)$ is defined, $\dot{x}(t) \in F(x(t),t)$, and $\dot{V}_i(x(t),t)$ is defined $\forall i \geq 0$. Since x is a solution of (13) and the functions V_i

are locally Lipschitz-continuous, $\mu([t_0,T)\setminus\mathcal{T})=0$, where μ denotes the Lebesgue measure on $\mathbb{R}$. The idea is to show that $\dot{x}(t)\in\tilde{F}(x(t),t)$, not just $F(x(t),t)$. Indeed since V_i is locally Lipschitz-continuous, for $t\in\mathcal{T}$ its time derivative can be expressed as

$$\dot{V}_i\left(x\left(t\right),t\right)=\lim_{h\to 0}\frac{\left(V_i\left(x\left(t\right)+h\dot{x}\left(t\right),t+h\right)-V_i\left(x\left(t\right),t\right)\right)}{h}.$$

Since each V_i is regular, for $i\geq 1$, $\dot{V}_i(x(t),t)=V_{i+}'([x(t);t],[\dot{x}(t);1])=V_i^o([x(t);t],[\dot{x}(t);1])=\max(p^T[\dot{x}(t);1],p\in\partial V_i(x(t),t))$, and $\dot{V}_i(x(t),t)=V_{i-}'([x(t);t],[\dot{x}(t);1])=V_i^o([x(t);t],[\dot{x}(t);1])=\min(p^T[\dot{x}(t);1],p\in\partial V_i(x(t),t))$, where V_+' and V_-' denote the right and left directional derivatives and V^o denotes the Clarke-generalized derivative [35, p. 39]. Hence, $p^T[\dot{x}(t);1]=\dot{V}_i(x(t),t)$, $\forall p\in\partial V_i(x(t),t)$, which implies $\dot{x}(t)\in G_i(x(t),t)$ for each i. Therefore, $\dot{x}(t)\in\tilde{F}(x(t),t)$. Hence, (22), along with the fact that $\dot{V}(x(t),t)=p^T[\dot{x}(t);1]$, $\forall p\in\partial V(x(t),t)$, implies that $\forall t\in\mathcal{T}$, $\dot{V}(x(t),t)\leq -W(x(t))$. Since $\mu([t_0,T)\setminus\mathcal{T})=0$, $\dot{V}(x(t),t)\leq -W(x(t))$ for almost all $t\in[t_0,T)$. ■

In the following, three technical Lemmas are stated to facilitate the proof of Corollary 1.

Lemma 2: If $\{F_\sigma:\mathbb{R}^n\times\mathbb{R}_{\geq t_0}\rightrightarrows\mathbb{R}^n\mid\sigma\in\mathbb{N}\}$ is a collection of locally bounded, continuous, compact-valued, and convex-valued maps, then the set-valued map $F:=(x,t)\mapsto\overline{\text{co}}\bigcup_{\sigma\in\mathbb{N}}F_\sigma(x,t)$ is continuous.

Proof: Let $H:\mathbb{R}^n\times\mathbb{R}_{\geq t_0}\rightrightarrows\mathbb{R}^n$ be defined as $H(x,t)=\text{co}(F_1(x,t)\cup F_2(x,t))$. If $N\subseteq\mathbb{R}^n$ is an open set containing $H(x,t)$, then $\exists\epsilon>0$ such that $H(x,t)+\text{B}((x,t),\epsilon)\subset N$. Since F_1 and F_2 are upper semicontinuous (USC), there exist open sets $M_1,M_2\subseteq\mathbb{R}^n\times\mathbb{R}_{\geq t_0}$ such that $(x,t)\in M_1\cap M_2$, $F_1(M_1)\subset H(x,t)+\text{B}((x,t),\epsilon)$, and $F_2(M_2)\subset H(x,t)+\text{B}((x,t),\epsilon)$. Therefore, $F_1(x,t)\cup F_2(x,t)\subset H(x,t)+\text{B}((x,t),\epsilon)$. Since $H(x,t)+\text{B}((x,t),\epsilon)$ is convex, $\text{co}(F_1(x,t)\cup F_2(x,t))\subset H(x,t)+\text{B}((x,t),\epsilon)$. Thus, H is USC.

It is easy to see that $(x,t)\mapsto F_1(x,t)\cup F_2(x,t)$ is lower semicontinuous (LSC). Using [41, Th. 5.9 (c)], H is also LSC. Inductively, the map $(x,t)\mapsto\text{co}\cup_{k=1}^K F_k(x,t)$ is continuous $\forall K<\infty$. Thus, the collection $\{F_k\}_{k\in\mathbb{N}}$ defined as $F_k(x,t)=\text{co}\cup_{\sigma=1}^k F_\sigma(x,t)$ is a collection of nondecreasing continuous set-valued maps. By [41, Exercise 4.3], the sequence $\{F_k\}_{k\in\mathbb{N}}$ converges pointwise to the map $(x,t)\mapsto\overline{\cup_{k\in\mathbb{N}}F_k(x,t)}$. Since the sets $\{F_k\}$ are nested, $\overline{\cup_{k\in\mathbb{N}}F_k(x,t)}=\overline{\text{co}\cup_{\sigma\in\mathbb{N}}F_\sigma(x,t)}$. Hence, by [41, Th. 5.48 (a)], the map $(x,t)\mapsto\overline{\text{co}}\cup_{\sigma\in\mathbb{N}}F_\sigma(x,t)$, is continuous.[12] ■

Lemma 3: Let $g:\mathbb{R}^n\to\mathbb{R}$ be continuous and let $F:\mathbb{R}^n\times\mathbb{R}_{\geq t_0}\rightrightarrows\mathbb{R}^n$ be a locally bounded, continuous, and compact-valued map. If $\phi:=(x,t)\mapsto\max_{q\in F(x,t)}g(q)$, then ϕ is continuous at (x,t), $\forall(x,t)\in\mathbb{R}^n\times\mathbb{R}_{\geq t_0}$.

Proof: If not, then $\exists\epsilon>0$ such that $\forall\delta>0$, $\exists(y,\tau)\in\text{B}((x,t),\delta)$ such that $|\phi(y,\tau)-\phi(x,t)|\geq\epsilon$. If $\phi(y,\tau)-\phi(x,t)\geq\epsilon$ then

$$\operatorname*{arg\,max}_{q\in F(y,\tau)\cup F(x,t)}g(q)\subseteq F(y,\tau)\setminus F(x,t).$$

If $\phi(x,t)-\phi(y,\tau)\geq\epsilon$, then $\arg\max_{q\in F(y,\tau)\cup F(x,t)}g(q)\subseteq F(x,t)\setminus F(y,\tau)$. That is, $\arg\max_{q\in F(y,\tau)\cup F(x,t)}g(q)\subseteq F(x,t)\triangle F(y,\tau)$. Let $\beta>0$. If $\{(y_k,\tau_k)\}_{k\in\mathbb{N}}\subset\overline{\text{B}}((x,t),\beta)$ is a sequence converging to (x,t) such that $|\phi(y_k,\tau_k)-\phi(x,t)|\geq\epsilon$, then, $\forall k\in\mathbb{N}$, $\max_{q\in F(y_k,\tau_k)\cup F(x,t)}g(q)=\max_{q\in F(x,t)\triangle F(y_k,t_k)}g(q)$. Since g and F are continuous and F is locally bounded, the sequence $\{\max_{q\in F(y_k,\tau_k)\cup F(x,t)}g(q)\}_{k\in\mathbb{N}}$ is a bounded sequence. On the other hand, since F is continuous, the sequence $\{F(x,t)\triangle F(y_k,\tau_k)\}_{k\in\mathbb{N}}$ converges to the null set, and hence, the sequence $\{\max_{q\in F(y_k,\tau_k)\cup F(x,t)}g(q)\}_{k\in\mathbb{N}}$ converges to $-\infty$, which is a contradiction. ■

Lemma 4: Let $g:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ be a continuous function and let $F:\mathbb{R}^n\times\mathbb{R}_{\geq t_0}\rightrightarrows\mathbb{R}^n$ be a locally bounded, USC, and compact-valued map. Let $h:=(p,x,t)\mapsto\max_{q\in F(x,t)}g(p,q)$. If $C_x\subset\mathbb{R}^n\times\mathbb{R}_{\geq t_0}$ and $C_p\subset\mathbb{R}^n$ are compact, then h is continuous in p, uniformly in (x,t) over $C_p\times C_x$.

Proof: Since g is continuous, and $F(C_x)$ and C_p are compact,[13] it is uniformly continuous on $C_p\times F(C_x)$. Thus, given $\epsilon>0$, $\exists\delta>0$, independent of (p,x,t), such that $\forall p,p_0\in C_p$ and $\forall q,q_0\in F(C_x)$, $\|p-p_0\|<\delta\wedge\|q-q_0\|<\delta\Rightarrow g(p_0,p_0)<g(p,q)+\epsilon$. In particular, $\|p-p_0\|<\delta\Rightarrow g(p_0,p_0)<g(p,q_0)+\epsilon$. For any fixed $p_0\in C_p$ and $(x,t)\in C_x$, $\exists q_0\in F(x,t)$ such that $h(p_0,x,t)=g(p_0,p_0)$, and hence, $h(p_0,x,t)<g(p,q_0)+\epsilon$. Since $g(p,q_0)\leq h(p,x,t)$ by definition, $h(p_0,x,t)<h(p,x,t)+\epsilon$. That is, $\forall p,p_0\in C_p$ and $\forall(x,t)\in C_x$, $\|p-p_0\|<\delta\Rightarrow h(p_0,x,t)<h(p,x,t)+\epsilon$. By symmetry, $|h(p_0,x,t)-h(p,x,t)|<\epsilon$. ■

Proof of Corollary 1: The Rademacher's theorem [48, Th. 3.2] and [35, Proposition 2.3.6 (d)] imply that ∂V is single-valued for almost all $(x,t)\in\mathbb{R}^n\times\mathbb{R}_{\geq t_0}$. As a result, for almost all $(x,t)\in\mathbb{R}^n\times\mathbb{R}_{\geq t_0}$, $\dot{\overline{V}}_F(x,t)=\underline{\dot{V}}_F(x,t)$. By Proposition 2, for any $(x,t)\in\mathbb{R}^n\times\mathbb{R}_{\geq t_0}$ and $\beta>0$, there exists a sequence $\{(y_k,\tau_k)\}_{k\in\mathbb{N}}\subset\overline{\text{B}}((x,t),\beta)$, converging to (x,t) such that $\partial V(y_k,\tau_k)=\{\nabla V(y_k,\tau_k)\}=:\{p_k\}$ and $\max_{q\in F(y_k,\tau_k)}p_k^T[q;1]\leq -W(y_k)$.

Let $q_k\in\arg\max_{q\in F(y_k,\tau_k)}p_k^T[q;1]$. Since the set-valued map F is locally bounded and USC, the sequence $\{q_k\}_{k\in\mathbb{N}}$ is bounded, and hence, admits a convergent subsequence $\{q_{k_l}\}_{l\in\mathbb{N}}$ converging to some $q^*\in\mathbb{R}^n\times\mathbb{R}_{\geq t_0}$. Since ∂V is locally bounded and USC (cf.[49, p. 4]), the sequence $\{p_{k_l}\}_{l\in\mathbb{N}}$ is bounded. Hence, there exists a subsequence $\{p_{k_{l_m}}\}_{m\in\mathbb{N}}$ converging to some $p^*\in\mathbb{R}^n$. Hence,

$$\left(p^*\right)^T\left[q^*;1\right]\leq\lim_{m\to\infty}-W\left(y_{k_{l_m}}\right)=-W\left(x\right). \qquad (35)$$

Using the characterization of the generalized gradient from [35, p. 11, eq. (4)], $p^*\in\partial V(x,t)$. From Lemma 2, F is continuous, and hence, $q^*\in F(x,t)$.

Let $h:=(p,x,t)\mapsto\max_{q\in F(x,t)}p^T[q;1]$. To prove the corollary, it needs to be established that $h(p^*,x,t)=$

[12]By [41, Th. 5.7 (c)], the notion of LSC in this paper is equivalent to the notion of the inner semicontinuity in [41]. Since all the maps under consideration are locally bounded and compact valued, by [41, Th. 5.19], the notion of USC in this paper is equivalent to the notion of the outer semicontinuity in [41].

[13]$F(C_x)$ is bounded by [32, Lemma 15, p. 66], and since F is USC and C_x is compact, $F(C_x)$ is also closed by [41, Th. 5.25 (a)].

$(p^*)^T[q^*;1]$. The inequality $h(p^*,x,t) \geq (p^*)^T[q^*;1]$ is immediate from the definitions. Also,

$$\begin{aligned} &h\left(p^*,x,t\right) \\ &\quad-\left(p^*\right)^T\left[q^*;1\right] = h\left(p^*,x,t\right) - h\left(p^*,y_{k_{l_m}},\tau_{k_{l_m}}\right) \\ &\quad+ h\left(p^*,y_{k_{l_m}},\tau_{k_{l_m}}\right) - h\left(p_{k_{l_m}},y_{k_{l_m}},\tau_{k_{l_m}}\right) \\ &\quad+ h\left(p_{k_{l_m}},y_{k_{l_m}},\tau_{k_{l_m}}\right) - \left(p^*\right)^T\left[q^*;1\right]. \end{aligned} \tag{36}$$

Let $\varepsilon > 0$. By definition of p^* and q^*, $\exists M_1 \in \mathbb{N}$ such that $\forall m \geq M_1, |h(p_{k_{l_m}},y_{k_{l_m}},\tau_{k_{l_m}}) - (p^*)^T[q^*;1]| < \frac{\varepsilon}{3}$. Since ∂V and F are USC, $\partial V(\overline{\mathrm{B}}((x,t),\beta))$ and $F(\overline{\mathrm{B}}((x,t),\beta))$ are closed by [41, Th. 5.25], and hence, compact. Since $(p,q) \mapsto p^T[q;1]$ is continuous, Lemma 4 implies that the function h is continuous in p, uniformly in (x,t), over $\partial V(\overline{\mathrm{B}}((x,t),\beta)) \times \overline{\mathrm{B}}((x,t),\beta)$. Hence, $\exists M_2 \in \mathbb{N}$ such that $\forall m \geq M_2$, $|h(p^*,y_{k_{l_m}},\tau_{k_{l_m}}) - h(p_{k_{l_m}},y_{k_{l_m}},\tau_{k_{l_m}})| < \frac{\varepsilon}{3}$. Lemma 3 implies that the function $(x,t) \mapsto h(p^*,x,t)$ is continuous. Hence, $\exists M_3 > 0$ such that $\forall m \geq M_3, |h(p^*,x,t) - h(p^*,y_{k_{l_m}},\tau_{k_{l_m}})| \leq \frac{\varepsilon}{3}$.

Thus, for $m \geq \max\{M_1,M_2,M_3\}$, $h(p^*,x,t) \leq (p^*)^T[q^*;1] + \varepsilon$. Since ε was arbitrary, $h(p^*,x,t) = (p^*)^T[q^*;1]$. Hence, from (35) and the definition of h, $\exists p^* \in \partial V(x,t)$ such that $\max_{q\in F(x,t)}(p^*)^T[q;1] \leq W(x)$, and hence, $\min_{p\in\partial V(x,t)}\max_{q\in F(x,t)} p^T[q;1] \leq -W(x)$. ■

In the following, [42, Lemma 1] is generalized to set-valued maps using the generalized Lucin's Theorem [41, Th. 14.10]. To that end, the notion of approximate continuity and its relation to Lebesgue measurability are generalized to set-valued maps.

Definition 6: A Lebesgue measurable set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called approximately continuous at $x \in \mathbb{R}^n$ if there exists a measurable set $G \subseteq \mathbb{R}^n$ such that x is a point of density 1 for G [48, Definition 1.25] and the map $F_{|G}$ is continuous at x.

Lemma 5: A Lebesgue measurable closed-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is approximately continuous at x for almost all $x \in \mathbb{R}^n$.

Proof: Let $\epsilon > 0$. The generalized Lusin's Theorem [41, Th. 14.10] implies that there exists a set E with $\mu(E^c) < \epsilon$ such that $F_{|E}$ is continuous. By the Lebesgue density theorem [48, Th. 1.35], almost every point of E is a point of density 1 for E. As a result, F is approximately continuous at almost every point of E. Since ϵ was arbitrary, F is approximately continuous almost everywhere. ■

Using Lemma 5, the results of [42, Lemma 1] can be generalized to closed-valued maps as follows.

Theorem 4: If $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is Lebesgue measurable and closed-valued, and if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable, then there exists $N_0 \subset \mathbb{R}^n$ such that $\mu(N_0) = 0$ and

$$\bigcap_{\mu(N)=0} \overline{\mathrm{co}}F\left(E \backslash N\right) = \overline{\mathrm{co}}F\left(E \setminus N_0\right).$$

Proof: If $\mu(E) = 0$ then the conclusion of the theorem trivially follows with $N_0 := E$. In the case where $\mu(E) > 0$, let $N_1 \subset \mathbb{R}^n$ denotes the set of points where F is not approximately continuous and let N_2 denotes the set of points in E that are not points of density 1 for E. By Lemma 5, $\mu(N_1) = 0$ and by the Lebesgue Density Theorem [48, Th. 1.35], $\mu(N_2) = 0$. Let $N_0 := N_1 \cup N_2$.

For all $N \subset \mathbb{R}^n$ with $\mu(N) = 0$, $\overline{\mathrm{co}}F(E \setminus (N \cup N_0)) \subseteq \overline{\mathrm{co}}F(E \setminus N_0)$. As a result, $\bigcap_{\mu(N)=0} \overline{\mathrm{co}}F(E \setminus (N \cup N_0)) \subseteq \overline{\mathrm{co}}F(E \setminus N_0)$, and since $\mu(N_0) = 0$,

$$\bigcap_{\mu(N)=0} \overline{\mathrm{co}}F\left(E \setminus N\right) \subseteq \overline{\mathrm{co}}F\left(E \setminus N_0\right).$$

To prove the reverse inclusion, let $z \in \overline{\mathrm{co}}F(E \setminus N_0)$. Then, by the Carathéodory's Theorem [38, p. 103], there exist points $\{z_1^j,\ldots,z_m^j\}$ such that $z_i^j \in F(E \setminus N_0)$, $\forall i = 1,\ldots,m$, and positive real numbers $\{a_1,\ldots,a_m\}$ with $\sum_{j=1}^m a_j = 1$, such that $\lim_{j\to 0}\sum_{i=1}^m a_i^j z_i^j = z$. Fix $N \subset \mathbb{R}^n$ such that $\mu(N) = 0$.

Claim For each $j \in \mathbb{N}$, we can select $\{\overline{z}_i^j\}_{i=1}^m \subset F(E \setminus (N_0 \cup N))$ such that $\|\sum_{i=1}^m a_i^j z_i^j - \sum_{i=1}^m a_i^j \overline{z}_i^j\| \leq \frac{1}{j}$.

Proof of Claim: Fix $j \in \mathbb{N}$. If $\exists i \in \{1,\ldots,m\}$ such that $z_i^j \in F(E \setminus N_0) \setminus F(E \setminus (N_0 \cup N))$, then $\exists x_i^j \in (N \setminus N_0) \cap (E \setminus N_0)$ such that $z_i^j \in F(x_i^j)$. By the definition of N_0, x_i^j is a point of density 1 for E. As a result, $\forall \epsilon > 0, \exists K \in \mathbb{N}$ such that $\forall k \geq K$

$$1 - \frac{\mu\left(\mathrm{B}\left(x_i^j,\frac{1}{k}\right) \cap E\right)}{\mu\left(\mathrm{B}\left(x_i^j,\frac{1}{k}\right)\right)} < \epsilon.$$

Particularly, $\forall k \geq K$, $\mu(\mathrm{B}(x_i^j,\frac{1}{k}) \cap E) > (1-\epsilon)\mu(\mathrm{B}(x_i^j,\frac{1}{k})) > 0$, which implies that $\forall k \in \mathbb{N}$, $\mu((\mathrm{B}(x_i^j,\frac{1}{k}) \cap E) \setminus (N_1 \cup N_2 \cup N)) > 0$. For each $k \in \mathbb{N}$, if ${}_kx_i^j$ is selected such that ${}_kx_i^j \in (\mathrm{B}(x_i^j,\frac{1}{k}) \cap E) \setminus (N_0 \cup N)$, then $\lim_{k\to\infty} {}_kx_i^j = x_i^j$.

Since F is approximately continuous on $E \setminus N_0$, $\lim_{k\to\infty} F({}_kx_i^j) = F(x_i^j)$, in the sense of Painlevé–Kuratovski convergence [39, Definition 1.1.1]. Since $F(x_i^j)$ is the set of limits of sequences $\{{}_kz_i^j\}$ such that ${}_kz_i^j \in F({}_kx_i^j)$, $\forall k \in \mathbb{N}$ [39, Proposition 1.1.2], there exists a sequence $\{{}_kz_i^j\}$ such that $\lim_{k\to\infty} {}_kz_i^j = z_i^j$ and ${}_kz_i^j \in F({}_kx_i^j)$, $\forall k \in \mathbb{N}$. Hence, $\forall \gamma > 0$, $\exists K \in \mathbb{N}$ such that $\overline{z}_i^j := {}_Kz_i^j$ satisfies $\|z_i^j - \overline{z}_i^j\| < \gamma$. Since the collection $\{z_i^j\}_{i=1}^m$ is finite, the claim is established. ■

By the triangle inequality

$$\begin{aligned} \left\|\sum_{i=1}^m a_i^j \overline{z}_i^j - z\right\| &= \left\|\sum_{i=1}^m a_i^j \overline{z}_i^j - \sum_{i=1}^m a_i^j z_i^j + \sum_{i=1}^m a_i^j z_i^j - z\right\| \\ &\leq \left\|\sum_{i=1}^m a_i^j \overline{z}_i^j - \sum_{i=1}^m a_i^j z_i^j\right\| + \left\|\sum_{i=1}^m a_i^j z_i^j - z\right\|. \end{aligned}$$

Given $\epsilon > 0$, if $J \in \mathbb{N}$ is selected large enough such that $\frac{1}{j} < \frac{\epsilon}{2}$ and $\forall j > J$, $\|\sum_{i=1}^m a_i^j z_i^j - z\| < \frac{\epsilon}{2}$, then $\|\sum_{i=1}^m a_i^j \overline{z}_i^j - z\| < \epsilon$, $\forall j > J$. That is, $\lim_{j\to 0}\sum_{i=1}^m a_i^j \overline{z}_i^j = z$. Therefore, z is the limit of a sequence comprised of elements that are convex combinations of points from $F(E \setminus (N_0 \cup N))$. That is, $z \in \overline{\mathrm{co}}F(E \setminus (N_0 \cup N))$. Since N was an arbitrary set of Lebesgue measure zero, $\overline{\mathrm{co}}F(E \setminus N_0) \subseteq \overline{\mathrm{co}}F(E \setminus (N_0 \cup N))$, $\forall N$ such that $\mu(N) = 0$. Hence, $\overline{\mathrm{co}}F(E \setminus N_0) \subseteq \bigcap_{\mu(N)=0} \overline{\mathrm{co}}F(E \setminus (N_0 \cup N))$, and since $\mu(N_0) = 0$, $\overline{\mathrm{co}}F(E \setminus N_0) \subseteq \bigcap_{\mu(N)=0} \overline{\mathrm{co}}F(E \setminus N)$. ■

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