Adaptive Safety with a RISE-Based Disturbance Observer

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***Abstract*—Ensuring the safety (formally, forward invariance) of control systems using control barrier functions typically requires being overly conservative when the dynamics of the control system are uncertain. In this work, we develop a disturbance observer based on the robust integral of the sign of the error (RISE) control paradigm that exponentially identifies the unknown dynamics when certain gain conditions are satisfied. The estimate of the dynamics is used in an optimization-based control law that ensures safety while expanding the operating region of the dynamical system as the dynamic model is identified. We provide conditions for when the control law is locally Lipschitz continuous. The RISE-based disturbance observer can provide safety guarantees to a secondary model of the dynamics of unknown accuracy. A simulation example is provided to demonstrate the performance of the disturbance observer and to illustrate the benefits of combining the observer with a secondary model given by a pretrained deep neural network.**

***Index Terms*—Adaptive control, control barrier functions, estimation, nonlinear systems, uncertain systems.**

I. Introduction

Control barrier functions (CBF) have emerged as a popular tool for ensuring the safety of controlled dynamical systems, where safety is typically defined as restricting the system to operate in some subset of the state space, which is determined to be safe [1], [2]. Given that model uncertainty exists in many practical applications, ensuring safety in the presence of uncertainty is of significant interest. One approach to address the challenges introduced by uncertain dynamics is to use a worst-case upper bound of the uncertainty [3], [4]. However, such methods are conservative because they restrict the system to operate in a subset of the true safe set. Motivated to reduce such conservativeness, we use an implicit learning method [5] in this article to exponentially learn the uncertain dynamics.

Two categories that describe adaptive safety results include gradient-based methods and data-driven techniques. Gradient-based methods, refashioned from Lyapunov-based adaptive control, only apply to dynamical systems with linearly parameterized uncertainty and are difficult to generalize when the safe set is defined by multiple CBF candidates due to dependencies of the parameter update law on the CBF candidate [6], [7], [8]. Moreover, the gradient-based approach introduced in [6] uses a worst-case upper bound of the uncertainty to ensure safety, which is conservative unless coupled with a data-driven method to reduce uncertainty. Other authors have developed solely data-driven methods that reduce conservativeness by identifying the unknown dynamics [9], [10], [11], [12]. Estimators, such as integral concurrent learning [11], set-membership identification [6], and the finite-time estimator in [12], are well suited for adaptive safety because they provide a computable upper bound of the estimation error that shrinks based on the quality of data. However, these methods have only been applied to systems with linearly parameterized uncertainty. Gaussian processes (GP) and deep neural networks (DNN) have been empirically found to effectively approximate a more general class of dynamics [13]. These approaches generally do not guarantee identification of the dynamics, and therefore cannot provide theoretical guarantees of safety without additional conservativeness. For example, the authors in [14] and [15] leveraged GPs to obtain probabilistic safety guarantees, but use the standard deviation of the GP in a worst-case fashion.

Disturbance observers are a class of data-driven estimators that can identify a general class of uncertainty, including time-varying disturbances [16]. Disturbance observers provide real-time compensation for uncertainty based on state or output feedback. To merge disturbance observers and CBF methods, an effective strategy is to employ an observer providing an estimation error bound that is computable in real time. For nonlinear systems, observers capable of furnishing such a bound have been developed in special cases, such as for disturbances governed by linear dynamics in [17], for robotic manipulators with constant disturbances in [18], and for partially feedback linearizable systems in [19]. Of the aforementioned designs, only the result for linear disturbance dynamics in [17] can guarantee exponential convergence of the estimation error for a nonconstant disturbance, whereas in this article, we develop an exponential disturbance observer for more general nonlinear disturbances. One notable disadvantage of disturbance observers is that they provide an estimate of the disturbance only along the current trajectory of the control system, whereas GPs and DNNs produce state-dependent models that can be used in subsequent initializations. Such state dependent approximations motivate the use of a disturbance observer in conjunction with a learned model.

Results that combine disturbance observers with CBFs appear in [20], where Alan et al. [20] used an observer that guarantees the estimation error converges to a quantifiable envelope, although the disturbance cannot be identified exactly. The set of safety-ensuring control inputs in [20] includes a robustness parameter that introduces

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conservativeness to compensate for the residual estimation error. Performance can be improved by replacing the constant parameter with a time-varying term that represents the shrinking uncertainty envelope. Similar work is presented in [21], where an observer is used to estimate the state of an unknown dynamic model. The state estimate is restricted to a conservative set that guarantees the true state remains in the safe set. An advantage of the approach in [21] is that full state feedback is not required.

The primary contribution of this work is to develop a nonlinear disturbance observer based on the robust sign of the error (RISE) paradigm (see [22] and [23]) that exponentially identifies with a computable error envelope, a general class of unstructured uncertainty. The RISE-based observer is combined with an adaptive safety controller to yield reduced conservativeness while providing deterministic guarantees of safety throughout the identification process. The observer can also provide safety guarantees to a secondary estimator, such as a GP or DNN, where the secondary estimator can reduce the gain conditions in the observer by reducing modeling error.

The rest of this article is organized as follows. In Section II, inspired by Isaly et al. [11], we describe a framework for safe control with multiple CBFs using an upper bound of the uncertain dynamics, and describe how reducing uncertainty leads to improved performance. In Section III, we define and analyze our RISE-based observer. We show how the upper bound of the estimation error obtained from the observer analysis can be integrated as a time-varying term in an implementable optimization-based control law that guarantees safety while eventually eliminating conservativeness from the set of safety-ensuring inputs. In Section IV, we provide sufficient conditions for the optimization-based control law to be locally Lipschitz. The result we provide for local Lipschitz continuity uses weaker assumptions than results that have previously appeared in the control literature, such as [24, Th. 1], and is more general than the results for quadratic programs with limited numbers of constraints in [25]. An example in Section V demonstrates the design process of the CBF-based control law, where we implement the RISE-based observer coupled with a pretrained DNN approximation of the unknown dynamics.

Notation: For vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\|x\|$ denotes the Euclidean norm, and $(x, y) \triangleq [x^T, y^T]^T$. The shorthand $[r] \triangleq \{1, 2, \ldots, r\}$ denotes the first r positive integers. For a function $\epsilon : \mathbb{R} \to \mathbb{R}$, the notation $\epsilon(t) \searrow 0$ as $t \to \infty$ means that $\epsilon(t) \to 0$ as $t \to \infty$ and $\epsilon(t) \geq 0$ for all $t \geq 0$. Given a function $B : \mathbb{R}^n \to \mathbb{R}^r$, the components are indexed as $B(x) \triangleq (B_1(x)$ and $B_2(x), \ldots, B_r(x))$ and the inequality $B(x) \leq 0$ means that $B_i(x) \leq 0$ for all $i \in [r]$. For a set $A \subset \mathbb{R}^n$, ∂A denotes its boundary, $\overline{A}$ its closure, and $U(A)$ denotes some open neighborhood of A.

II. Problem Statement

A. Preliminaries

We consider the problem of developing an estimate of the unknown dynamics for a control-affine system with input constraints

$$\dot{x} = f(x) + g(x)u + d(x), \quad u \in \Psi(x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, the functions f and g are known functions, and d represents the unknown dynamics. The function f represents a priori knowledge while the disturbance d represents modeling error. In our framework, it is acceptable to include time as a state so that d could model a time-varying disturbance. The set $\Psi(x) \triangleq \{u \in \mathbb{R}^m : \psi(x, u) \leq 0\}$ represents state-dependent constraints $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ on the control input, such as limits on its magnitude. Given a controller $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ with $\kappa(x) \in \Psi(x)$ for all $x \in \mathbb{R}^n$, we refer to the closed-loop dynamics defined by (1) and κ as $\dot{x} = f_{cl}(x) \triangleq f(x) + g(x)\kappa(x) + d(x)$. The following assumption on the dynamics will be used throughout our development.

Assumption 1: The functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^{n \times m}$, and $d : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

The following definition of a CBF is a specialization of the one in [3, Def. 2] to the dynamics considered in this article. Our framework allows the safe set to be defined by multiple functions through vector-valued CBF candidates. A function $B : \mathbb{R}^n \to \mathbb{R}^r$ is called a CBF candidate defining the safe set $\mathcal{S} \subset \mathbb{R}^n$ if $\mathcal{S} = \{x \in \mathbb{R}^n : B(x) \leq 0\}$. For each $i \in [r]$, we also define the sets $\mathcal{S}_i \triangleq \{x \in \mathbb{R}^n : B_i(x) \leq 0\}$ and $M_i \triangleq \{x \in \partial\mathcal{S} : B_i(x) = 0\}$.

Definition 1: A continuously differentiable CBF candidate $B : \mathbb{R}^n \to \mathbb{R}^r$ defining the set $\mathcal{S} \subset \mathbb{R}^n$ is a CBF for (1) and $\mathcal{S}$ on a set $\mathcal{O} \subset \mathbb{R}^n$ with respect to a function $\gamma : \mathbb{R}^n \to \mathbb{R}^r$ if 1) there exists a neighborhood of the boundary of $\mathcal{S}$ such that $U(\partial\mathcal{S}) \subset \mathcal{O}$, 2) the function γ is such that, for each $i \in [r]$, $\gamma_i(x) \geq 0$ for all $x \in U(M_i) \backslash \mathcal{S}_i$, and 3) the set

$$K_c(x) \triangleq \{u \in \Psi(x) : \Gamma(x, u) \leq -\gamma(x)\} \tag{2}$$

is nonempty for every $x \in \mathcal{O}$, where, for each $i \in [r]$, $\Gamma_i(x, u) \triangleq \nabla B_i^T(x)(f(x) + g(x)u + d(x))$.

Remark 1: For each i, the function γ_i is required to be nonnegative only on the region $U(M_i)\backslash\mathcal{S}_i$, which corresponds to a region outside the safe set nearby points where $B_i(x) = 0$. The set $\mathcal{O}$ describes the region on which the mapping K_c is nonempty, which must contain at a minimum a neighborhood of the boundary of the safe set $U(\partial\mathcal{S})$.

The set-valued mapping $K_c : \mathcal{O} \rightrightarrows \mathbb{R}^m$ represents, at each state $x \in \mathcal{O}$, a set $K_c(x) \subset \mathbb{R}^m$ of safety-ensuring control inputs. Unlike [26], safety here is characterized by forward invariance of the safe set $\mathcal{S}$. Given a controller κ, the set $\mathcal{S}$ is forward invariant for the closed-loop dynamics $\dot{x} = f_{cl}(x)$ if every maximal solution starting from $\mathcal{S}$ remains in $\mathcal{S}$ for all time. We also refer to the notion of forward preinvariance, which allows for maximal solutions that are not complete, meaning that they are defined on a bounded time domain [3, Def. 3]. The following specialization of [3, Th. 1 and 2] shows that continuous selections of K_c (i.e., continuous functions with $\kappa(x) \in K_c(x)$ for all $x \in \mathcal{O}$) render the safe set $\mathcal{S}$ forward (pre)-invariant. The proposition is a special case of the results in [3] to the case when the dynamics are not set valued and there are no state constraints.

Proposition 1: [3, Th. 1 and 2] Let Assumption 1 hold. If $B : \mathbb{R}^n \to \mathbb{R}^r$ is a CBF for (1) and $\mathcal{S}$ on $\mathcal{O}$ with respect to γ, and $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathcal{O}$ with $\kappa(x) \in K_c(x)$ for all $x \in \mathcal{O}$, then $\mathcal{S}$ is forward (pre)-invariant for the closed-loop system $\dot{x} = f_{\text{cl}}(x)$ defined by (1) and κ. Moreover, if in addition κ is continuous on $\mathcal{O} \cup \mathcal{S}$, and either 1) $\mathcal{S}$ is compact, 2) f_{cl} is bounded on $\mathcal{S}$, or **3)**f_{cl} has linear growth on $\mathcal{S}$ in the sense that there exists $c > 0$ such that, for all $x \in \mathcal{S}$, $|f_{\text{cl}}(x)| \leq c(|x| + 1)$, then $\mathcal{S}$ is forward invariant for $\dot{x} = f_{\text{cl}}(x)$.

Frequently, a selection of K_c is obtained by solving the following optimization problem at each $x \in \mathcal{O}$:

$$\kappa^*(x) \triangleq \underset{u \in \mathbb{R}^m}{\arg\min}\, Q(x, u)$$
$$\text{s.t. } \mathcal{C}(x, u) \leq 0 \tag{3}$$

where $Q : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a cost function and $\mathcal{C} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l$ defines the constraints. When K_c in (2) is nonempty on a set $\mathcal{O} \subset \mathbb{R}^n$, selecting $\mathcal{C}(x, u) \triangleq (\Gamma(x, u) + \gamma(x), \psi(x, u))$ implies that $\kappa^*(x) \in K_c(x)$ for every $x \in \mathcal{O}$. The optimization problem in (3) can be thought of as a state-feedback controller that yields a safety-ensuring control input at each state.

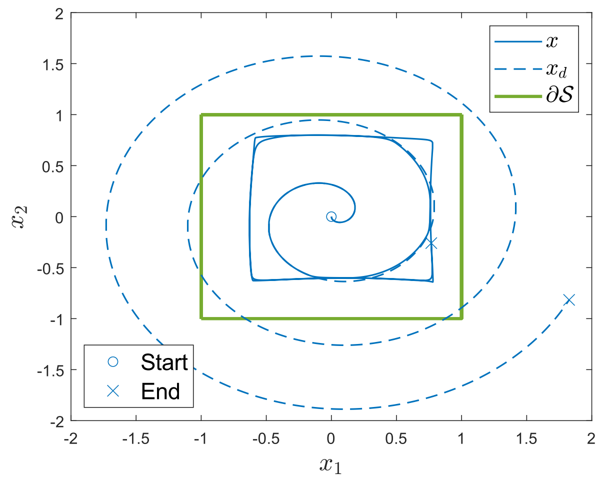

Fig. 1. Illustration of the effect of conservativeness in upper bounds of the disturbance. The trajectory x attempts to track the desired trajectory x_d, but must deviate from it to ensure safety. It is desirable for the trajectory to be allowed near the boundary of the safe set $\mathcal{S}$.

B. Robust Safety With Improved Performance

The function Γ defining K_c in (2) cannot be directly computed because it depends on the disturbance d, so it is not possible to implement the controller κ^* in (3). Suppose, for each $i \in [r]$, there exists an upper bound $\chi_i : \mathbb{R}^n \to \mathbb{R}$ of the unknown term $\nabla B_i^T(x)d(x)$. One can then implement an optimization-based control law with feasible set defined by the mapping

$$\bar{K}_c(x) \triangleq \left\{u \in \Psi(x) : \bar{\Gamma}(x, u) \leq -\gamma(x)\right\} \tag{4}$$

where $\bar{\Gamma}_i(x, u) \triangleq \nabla B_i^T(x)(f(x) + g(x)u) + \chi_i(x)$ for each $i \in [r]$. It holds that $\bar{\Gamma}(x, u) \geq \Gamma(x, u)$ for any $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$, and hence, $\bar{K}_c(x) \triangleq \{u \in \Psi(x) : \bar{\Gamma}(x, u) \leq -\gamma(x)\}$ is a subset of $K_c(x)$. Thus, any continuous controller $\kappa(x) \in \bar{K}_c(x)$ is also a selection of K_c, and therefore using Proposition 1 such controllers render the safe set $\mathcal{S}$ forward preinvariant. This approach requires $\bar{K}_c(x)$ to be nonempty for all $x \in \mathcal{O}$. As an example, an upper bound of the disturbance $\bar{d} \in \mathbb{R}_{>0}$ with $\|d(x)\| \leq \bar{d}$ for all $x \in \mathbb{R}^n$ could be used to define $\chi_i(x) \triangleq \bar{d}\|\nabla B_i(x)\|$.

When the upper bound χ is conservative, the mismatch between $\bar{\Gamma}$ and Γ is restrictive because trajectories of the closed-loop dynamics under a selection of $\bar{K}_c$ are able to explore less of the safe set than if inputs could be selected from K_c. To see how conservativeness degrades performance, note that if $\bar{\Gamma}_i(x, u) \leq 0$, as is required near the boundary of the safe set by Definition 1, then $\Gamma_i(x, u) \leq -\epsilon_i(x)$, where $\epsilon_i(x) \triangleq \chi_i(x) - \nabla B_i^T(x)d(x) \geq 0$ denotes the mismatch between $\bar{\Gamma}_i$ and Γ_i. When Γ_i is strictly negative near the boundary of the safe set, the control law will force trajectories into the interior of the safe set, potentially creating an asymptotically stable subset of the safe set. Fig. 1 illustrates a trajectory that is overly restricted due to a conservative upper bound. To remedy the mismatch between $\bar{\Gamma}$ and Γ, adaptive upper bounds that solve the following problem are desirable.

Problem 1: For each $i \in [r]$, find an upper bound $\chi_i : \mathbb{R}^n \to \mathbb{R}$ of the unknown disturbance term such that along any trajectory $t \mapsto x(t)$ of the closed-loop system, $\chi_i(x(t)) - \nabla B_i^T(x(t))d(x(t)) \searrow 0$ as $t \to \infty$.

Remark 2: Problem 1 has been solved in certain special cases. For example, Isaly et al. [11] assumed a finite-excitation condition and linearly parameterized uncertainty $d(x) = Y(x)\theta$, with $\theta \in \mathbb{R}^p$ denoting a vector of unknown constant model parameters. The approach uses integral concurrent learning to produce an exponentially convergent estimate $\hat{\theta} : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ of the unknown parameters and an exponentially decaying upper bound $\tilde{\theta}_{\text{UB}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ of the estimation error with $\tilde{\theta}_{\text{UB}}(t) \geq \|\tilde{\theta}(t)\|$ for all time and $\tilde{\theta}_{\text{UB}}(t) \to 0$ as $t \to \infty$, where $\tilde{\theta} = \theta - \hat{\theta}$. Problem 1 is solved by defining the term $\chi_i(x, t) \triangleq \nabla B_i^T(x)Y(x)\hat{\theta}(t) + \|\nabla B_i^T(x)Y(x)\|\tilde{\theta}_{\text{UB}}(t)$.

III. Estimation of Disturbance

A. RISE-Based Disturbance Observer

Motivated by Problem 1 and results such as [27] and [28], the following RISE-based disturbance observer is developed:

$$\dot{\hat{x}} = f(x) + g(x)u + \hat{d} + \alpha\tilde{x} \tag{5}$$

$$\dot{\hat{d}} = k_d\left(\dot{\tilde{x}} + \alpha\tilde{x}\right) + \tilde{x} + \beta \operatorname{dir}\left(\tilde{x}\right) \tag{6}$$

where $\tilde{x} \triangleq x - \hat{x}$; $\alpha, \beta,$ and $k_d > 0$ are control gains; and $\operatorname{dir}(\tilde{x}) \triangleq \tilde{x}/\|\tilde{x}\|$ if $\tilde{x} \neq 0$ and $\operatorname{dir}(\tilde{x}) \triangleq 0$ otherwise. We assume that state feedback is possible, namely, $\tilde{x}$ is measurable, and it is therefore always possible to set $\hat{x}(0) = x(0)$ leading to $\tilde{x}(0) = 0$. We also define $\tilde{d} \triangleq d - \hat{d}$. Since the derivative of $\tilde{x}$ is unavailable in many applications, an implementable form of the disturbance estimate is obtained by integrating (6) and using the facts that $\int_0^t \dot{\tilde{x}}(\tau)\, d\tau = \tilde{x}(t) - \tilde{x}(0)$ and $\tilde{x}(0) = 0$ to obtain $\hat{d}(t) = \hat{d}(0) + k_d\tilde{x}(t) + \int_0^t [(k_d\alpha + 1)\tilde{x}(\tau) + \beta \operatorname{dir}(\tilde{x}(\tau))]\, d\tau$.

Remark 3: The RISE paradigm typically uses the component-wise sign of the error $\tilde{x}$ in place of the term $\operatorname{dir}(\tilde{x})$ [22, Eq. (6)]. The term $\operatorname{dir}(\tilde{x})$ indicates the direction of the error vector $\tilde{x}$ and is advantageous because it is discontinuous only when $\tilde{x} = 0$, whereas the component-wise sign is discontinuous on each coordinate axis, i.e., when $\tilde{x}_i = 0$ for some $i \in [n]$.

To facilitate the subsequent development, we impose the following assumption on the total derivative of the disturbance term. Assumptions on the first derivative are common in the disturbance observer literature [16], [20], while it is less common to impose boundedness of the second derivative. The assumption that the derivatives exist will require some additional regularity of the dynamics beyond the continuity of Assumption 1.

Assumption 2: Given a set $\mathcal{R} \subset \mathbb{R}^n$ and a controller $\kappa : \mathbb{R}^n \to \mathbb{R}^m$, assume there exist constants c_1 and $c_2 > 0$ such that any solution $x : \operatorname{dom} x \to \mathbb{R}^n$, where $\operatorname{dom} x \subset \mathbb{R}_{\geq 0}$, to the closed-loop system $\dot{x} = f_{cl}(x)$ defined by (1) and κ with $x(0) \in \mathcal{R}$ satisfies $\|\dot{d}(x(t))\| \leq c_1$ for all $t \in \operatorname{dom} x$ and $\|\ddot{d}(x(t))\| \leq c_2$ for almost all $t \in \operatorname{dom} x$, where we have assumed that $\dot{d}$ exists everywhere and $\ddot{d}$ exists almost everywhere.

We next justify Assumption 2 by providing a sufficient condition under common assumptions relevant to the safety application. To aid in the upcoming proofs, for a given controller $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ recall that $\dot{d}(x) \triangleq J_d(x)f_{\text{cl}}(x)$, where $J_d : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ denotes the Jacobian of d.

Lemma 1: Consider the functions f, g, and d in (1) and a controller $\kappa : \mathbb{R}^n \to \mathbb{R}^m$. Let $\mathcal{R} \subset \mathbb{R}^n$ be forward preinvariant for $\dot{x} = f_{\text{cl}}(x)$ and suppose that f, g, d, and κ are Lipschitz and bounded on $\mathcal{R}$, and $J_d : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ exists and is Lipschitz (in the sense of the induced matrix norm) on $\mathcal{R}$. Then, Assumption 2 is satisfied for $\mathcal{R}$ and κ.

Proof: Since d is Lipschitz and J_d exists on $\mathcal{R}$, J_d is bounded on $\mathcal{R}$. If $x : \operatorname{dom} x \to \mathbb{R}^n$ is a solution starting from $\mathcal{R}$, it remains in $\mathcal{R}$ for all $t \in \operatorname{dom} x$ via forward preinvariance, so that $t \mapsto \dot{x}(t)$ is bounded on $\operatorname{dom} x$ via boundedness of the dynamics on $\mathcal{R}$. Thus, since $\dot{d}(x(t)) = J_d(x(t))\dot{x}(t)$, there exists a constant $c_1 > 0$ such that $\|\dot{d}(x(t))\| \leq c_1$ for all $t \in \operatorname{dom} x$.

The function $x \mapsto \dot{d}(x)$, being an additive/multiplicative combination of Lipschitz and bounded functions, is Lipschitz on $\mathcal{R}$. Under the assumptions of the lemma, any solution $t \mapsto x(t)$ to $\dot{x} = f_{cl}(x)$ starting from $\mathcal{R}$ is continuously differentiable with bounded derivative [29, Sec. 3.1], and therefore Lipschitz. Thus, the function $t \mapsto \dot{d}(x(t))$ is Lipschitz on $\mathrm{dom}\, x$, which implies that there exists $c_2 > 0$ such that $\|\ddot{d}(x(t))\| \leq c_2$ for almost all $t \in \mathrm{dom}\, x$. ∎

Remark 4: The assumptions of Lemma 1 will also guarantee uniqueness of solutions to the closed-loop dynamics [29, Th. 3.2]. In the case that $\mathcal{R}$ is compact, the assumptions of Lemma 1 reduce to assuming that f, g, and κ are locally Lipschitz and d is continuously differentiable with locally Lipschitz Jacobian.

Next, we provide conditions under which the disturbance observer in (5) and (6) exponentially identifies the unknown disturbance in (1) with a quantifiable rate of decay. Our result applies even when the system in (1) does not have complete or bounded solutions. However, the resulting observer in (5) and (6) may not be bounded. In practice, developing a bounded observer requires ensuring that the closed-loop system has bounded solutions, which can be accomplished via, e.g., designing a control law that ensures forward invariance of a compact safe set.

Theorem 1: Let Assumption 1 hold and let $\mathcal{R} \subset \mathbb{R}^n$ and $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ be such that Assumption 2 holds with bounds c_1 and c_2. If the control gains α, β, and k_d in (5) and (6) satisfy

$$\beta > c_1 + \frac{c_2}{\max\{1, \alpha - k_d\}} \tag{7}$$

then for each solution to the closed-loop system defined by (1), (5), (6), and κ with $x(0) \in \mathcal{R}$ and $\hat{x}(0) = x(0)$, the dynamic estimator is exponentially convergent such that $\|\tilde{d}(t)\| \leq \|\tilde{d}(0)\| e^{-\lambda t}$ and $\|\tilde{x}(t)\| \leq \|\tilde{d}(0)\| e^{-\lambda t}$ for all $t \in \mathrm{dom}\, x \subset \mathbb{R}_{\geq 0}$, where

$$\lambda = \min\{\alpha - 1, k_d\}/2. \tag{8}$$

Proof: Using the transformations $\tilde{d} \triangleq d(x) - \hat{d}$ and $\tilde{x} \triangleq x - \hat{x}$, every solution to the closed-loop system corresponds to a solution to the following transformed system:

$$\dot{\tilde{x}} = \tilde{d} - \alpha \tilde{x} \tag{9}$$

$$\dot{\tilde{d}} \in \dot{d}(x) - k_d \tilde{d} - \tilde{x} - \beta \mathrm{DIR}\,(\tilde{x}) \tag{10}$$

where (10) is derived by noticing that (9) implies $\dot{\hat{d}} = k_d \tilde{d} + \tilde{x} + \beta \mathrm{dir}(\tilde{x})$. We have taken the Filippov regularization of the original dynamics to ensure that the analysis is robust to small noise, which results in the set-valued function $\mathrm{DIR}(\tilde{x}) \triangleq \tilde{x}/\|\tilde{x}\|$ if $\tilde{x} \neq 0$ and $\mathrm{DIR}(\tilde{x}) = \mathbb{B}$ if $\tilde{x} = 0$, with $\mathbb{B} \subset \mathbb{R}^n$ being the closed unit ball. The error system in (9) and (10) has the same form as the one in [22, Eqs. (3) and (7)] with $N_B(x) \triangleq \dot{d}(x)$.

The analysis proceeds similarly to [22], while we provide a summary here due to minor differences. We first design an auxiliary term $P \in \mathbb{R}$ with the following set-valued dynamics:

$$\dot{P} \in -\lambda_P P + \tilde{d}^T \left(\beta \mathrm{DIR}\,(\tilde{x}) - \dot{d}(x)\right) \tag{11}$$

with $\lambda_P > 0$. Defining a composite state $\zeta \triangleq (x, \tilde{x}, \tilde{d}, P) \in \mathbb{R}^{3n+1}$, we analyze a differential inclusion with dynamics defined by (1), (9), (10), and (11), respectively. Define the function $V : \mathbb{R}^{3n+1} \to \mathbb{R}$ as $V(\zeta) \triangleq \frac{1}{2}\tilde{x}^T\tilde{x} + \frac{1}{2}\tilde{d}^T\tilde{d} + P$. Consider the set $D \triangleq \{\zeta \in \mathbb{R}^{3n+1} : \tilde{x} = 0, \tilde{d} \neq 0\}$. For $\zeta \in \mathbb{R}^{3n+1} \backslash D$, we have

$$\dot{V}\,(\zeta) = -\alpha\|\tilde{x}\|^2 - k_d\|\tilde{d}\|^2 - \lambda_P P \leq -\lambda_V V\,(\zeta)$$

with $\lambda_V \triangleq \min\{\alpha, k_d, \lambda_P\}$. Let $\zeta : \mathrm{dom}\,\zeta \to \mathbb{R}^{3n+1}$ be a solution to the differential inclusion defined above with $x(0) \in \mathcal{R}$. Due to the dynamics defining $\tilde{x}$, it can be shown that the set of time instants $\{t \in \mathrm{dom}\,\zeta : \zeta(t) \in D\}$ has Lebesgue measure zero (cf. [22, Lemma 1]). It follows that $\dot{V}(\zeta(t)) \leq -\lambda_V V(\zeta(t))$ for almost all $t \in \mathrm{dom}\,\zeta$. Thus

$$V\,(\zeta(t)) \leq V\,(\zeta(0))\, e^{-\lambda_V t} \quad \forall t \in \mathrm{dom}\,\zeta. \tag{12}$$

Next, we show that the initial condition for the state P can be selected so that $P(t) \geq 0$ for all $t \in \mathrm{dom}\,\zeta$. Given a solution ζ with $x(0) \in \mathcal{R}$, consider the function $P(t) = (P(0) - \beta\|\tilde{x}(0)\| + \tilde{x}^T(0)\dot{d}(x(0)))e^{-\lambda_P t} + \beta\|\tilde{x}(t)\| - \tilde{x}^T(t)\dot{d}(x(t)) + e^{-\lambda_P t} * [(\alpha - \lambda_P)(\beta\|\tilde{x}(t)\| - \tilde{x}^T(t)\dot{d}(x(t))) - \tilde{x}^T(t)\ddot{d}(x(t))]$, with $p(t) * q(t) \triangleq \int_0^t p(t-\tau)q(\tau)\,d\tau$ denoting the convolution operation. A similar analysis to [22, Lemma 3] shows that P is the unique solution to (11) corresponding to a given trajectory ζ. We select $P(0) = \beta\|\tilde{x}(0)\| - \tilde{x}(0)^T\dot{d}(x(0))$, although any larger value of $P(0)$ is also sufficient. Using the bounds in Assumption 2, we find that $P(t) \geq (\beta - c_1)\|\tilde{x}(t)\| + e^{-\lambda_P t} * [(\alpha - \lambda_P)(\beta\|\tilde{x}(t)\| - c_1\|\tilde{x}(t)\|) - c_2\|\tilde{x}(t)\|]$. Thus, $P(t)$ is nonnegative for all $t \in \mathrm{dom}\,\zeta$ if $\alpha > \lambda_P$ and $\beta > c_1 + \frac{c_2}{\alpha - \lambda_P}$. To simplify parameter selection, we choose $\lambda_P = \min\{\alpha - 1, k_d\}$, which leads to (7). Since $P(t) \geq 0$ for all $t \in \mathrm{dom}\,\zeta$ and $V(\zeta(0)) = \frac{1}{2}\|\tilde{x}(0)\|^2 + \frac{1}{2}\|\tilde{d}(0)\|^2 + P(0)$, it follows from (12) that:

$$\|\tilde{d}(t)\| \leq \sqrt{\|\tilde{x}(0)\|^2 + \|\tilde{d}(0)\|^2 + 2P(0)}\; e^{-\lambda_V t/2} \tag{13}$$

for all $t \in \mathrm{dom}\,\zeta$, showing that the disturbance estimation error is exponentially convergent. The same upper bound can be shown to hold for $\|\tilde{x}(t)\|$. The right-hand side of (13) becomes $\|\tilde{d}(0)\|e^{-\lambda t}$ when $\tilde{x}(0) = 0$ and $\lambda \triangleq \lambda_V/2$, leading to (8). In addition, the composite solution $t \mapsto \zeta(t)$ has the same domain of definition as the component $t \mapsto x(t)$ (i.e., $\mathrm{dom}\,\zeta = \mathrm{dom}\,x$) because the components $(\tilde{x}, \tilde{d}, P)$ cannot exhibit finite-time escape due to (12). ∎

Remark 5: The Lyapunov analysis of Theorem 1 suggests that faster convergence of the disturbance observer can be obtained by increasing both of the gains k_d and α [see (8)]. It is also observed in (7) that the lower bound on β imposed by (7) is smaller when α is significantly larger than k_d. In the continuous-time setting, the RISE gain β can be selected arbitrarily large to satisfy (7). However, large values of α, β, or k_d can lead to performance degradation in discrete-time implementations of the algorithm.

Remark 6: The gain condition in (7) requires the gain β to be large enough relative to the rates of change of the disturbance. The constants c_1 and c_2 defined in Assumption 2 can be approximated using some a priori knowledge of the dynamics, such as conservative upper bounds based on physics or estimates from identification experiments. It should also be noted that the performance of the estimator can be verified online using the measurable quantity $\tilde{x}$, which converges according to the bound $\|\tilde{x}(t)\| \leq \|\tilde{d}(0)\|e^{-\lambda t}$ provided β is sufficiently large.

B. Implementation in Control Law

Theorem 1 applies even when the disturbance is unbounded (although its rate of change should be bounded according to Assumption 2). However, the bound $\|\tilde{d}(t)\| \leq \|\tilde{d}(0)\|e^{-\lambda t}$ cannot be computed without information about the initial estimation error $\tilde{d}(0)$. Bounds on $\|\tilde{d}(0)\|$ are typically obtained under the assumption that the disturbance is bounded on the safe set.

Assumption 3: Given the safe set $\mathcal{S} \subset \mathbb{R}^n$, there exists $\bar{d} \in \mathbb{R}_{\geq 0}$ such that $\|d(x)\| \leq \bar{d}$ for all $x \in \mathcal{S}$.

The disturbance observer is injected into the original dynamic model in (1) as follows. Define the vector $z \triangleq (x, \hat{x}, \hat{d}, t) \in \mathbb{R}^{3n} \times \mathbb{R}_{\geq 0} \triangleq \mathcal{Z}$. Recalling Section II-B, we define the term $\bar{\Gamma}_i(z, u) \triangleq \nabla B_i^T(x)(f(x) + g(x)u) + \chi_i(z)$ where $\chi_i(z) \triangleq \min\{\bar{d}\|\nabla B_i(x)\|, \nabla B_i^T(x)\hat{d} + \tilde{d}_{\mathrm{UB}}(t)\|\nabla B_i(x)\|\}$ for each $i \in [r]$, and $\tilde{d}_{\mathrm{UB}}(t) \triangleq 2\bar{d}\, e^{-\lambda t}$, with λ defined according to (8). Under the assumptions of Theorem 1, the term χ_i solves Problem 1, so that the set $\bar{K}_c(z) \triangleq \{u \in \Psi(x) : \bar{\Gamma}(z, u) \leq -\gamma(x)\}$ converges to the true set of safety-ensuring control inputs as $t \to \infty$. We next show that safety is ensured using an optimization-based control law where the unknown function Γ is replaced with $\bar{\Gamma}$.

Theorem 2: Consider the dynamical system in (1) for which Assumptions 1 and 3 hold. Let $B : \mathbb{R}^n \to \mathbb{R}^r$ be a continuously differentiable CBF candidate defining $\mathcal{S} \subset \mathbb{R}^n$ and let the set $\mathcal{O} \subset \mathbb{R}^n$ and function $\gamma : \mathbb{R}^n \to \mathbb{R}^r$ respectively satisfy conditions 1) and 2) of Definition 1. Let $(\hat{x}, \hat{d})$ be updated according to the estimator in (5) and (6) with $\hat{x}(0) = x(0)$, $\|\tilde{d}(0)\| \leq \bar{d}$, and where the observer gains satisfy (7). Define the controller $\kappa^* : \mathcal{Z} \to \mathbb{R}^m$ as

$$\begin{aligned} \kappa^*(z) \triangleq\ & \underset{u \in \mathbb{R}^m}{\arg\min}\ Q(x, u) \\ & \text{s.t. } \bar{\Gamma}(z, u) \leq -\gamma(x) \\ & \psi(x, u) \leq 0 \end{aligned} \tag{14}$$

for the cost function $Q : \mathcal{O} \times \mathbb{R}^m \to \mathbb{R}$. Assume that $\bar{K}_c(z) = \{u \in \Psi(x) : \bar{\Gamma}(z, u) \leq -\gamma(x)\}$ is nonempty for all $z \in \mathcal{O} \times \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}$, and κ^* is single-valued and continuous[1] on $\mathcal{O} \times \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}$. If Assumption 2 holds with $\mathcal{R} = \mathcal{S}$ and $\kappa = \kappa^*$ and the RISE gain β satisfies (7), then $\mathcal{S}$ is forward preinvariant for the closed-loop dynamics $\dot{x} = f_{\mathrm{cl}}(z) \triangleq f(x) + g(x)\kappa(z) + d(x)$. If additionally κ^* is continuous on $(\mathcal{O} \cup \mathcal{S}) \times \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}$ and either $\mathcal{S}$ is compact or $z \mapsto f_{\mathrm{cl}}(z)$ satisfies Condition 2) or 3) of Proposition 1 on $\mathcal{S} \times \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}$, then $\mathcal{S}$ is forward invariant for $\dot{x} = f_{\mathrm{cl}}(z)$.

Proof: Let $t \mapsto z(t) = (x(t), \hat{x}(t), \hat{d}(t))$ be a solution to the closed-loop dynamics in (1), (5), and (6) with $x(0) \in \mathcal{S}$. Assumption 2 and Theorem 1 show that $\|\tilde{d}(t)\| \leq \|\tilde{d}(0)\| e^{-\lambda t}$ for all $t \in \mathrm{dom}\, z$. Since $\|\tilde{d}(0)\| \leq 2\bar{d}$, we have $\chi_i(z(t)) \geq \nabla B_i^T(x(t))\hat{d} + \tilde{d}_{UB}(t)\|\nabla B_i(x(t))\| \geq \nabla B_i^T(x(t))d(x(t))$ for every $i \in [r]$ and $t \in \mathrm{dom}\, z$. It follows that for every $u \in \Psi(x(t))$, $\bar{\Gamma}_i(z(t), u) \geq \Gamma_i(x(t), u)$. Thus, $\kappa^*(z(t)) \in \bar{K}_c(z(t)) \subset K_c(x(t))$ for all $t \in \mathrm{dom}\, z$. Since $\bar{K}_c(z)$ is nonempty on $\mathcal{O} \times \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}$, K_c is nonempty on $\mathcal{O}$ so that B is a CBF. Since κ^* is continuous, Proposition 1 shows that $\mathcal{S}$ is forward preinvariant for the closed-loop dynamics. The claim of forward invariance is a direct consequence of Proposition 1. ■

Remark 7: Before deploying a controller, it is important to verify that the set $\bar{K}_c(z) = \{u \in \Psi(x) : \bar{\Gamma}(z, u) \leq -\gamma(x)\}$ is nonempty on the operating region of the dynamical system. [3, Sec. V] develops sum of squares optimization tools that can be used to verify feasibility. This task is generally not possible without knowing a bound on the unknown disturbance d. Supposing that Assumption 3 holds, the feasibility verification should be performed using the set $K_{\mathrm{UB}}(x) \triangleq \{u \in \Psi(x) : \Gamma_{\mathrm{UB}}(x, u) \leq -\gamma(x)\}$ with $\Gamma_{\mathrm{UB}}(x, u) \triangleq \nabla B_i^T(x)(f(x) + g(x)u) + \bar{d}\|\nabla B_i(x)\|$. We have $K_{\mathrm{UB}}(x) \subset \bar{K}_c(z)$ for every $z \in \mathcal{Z}$. Thus, verifying the nonemptiness of K_{UB} also certifies that $\bar{K}_c$ is nonempty.

[1] The controller can be certified as single-valued and continuous using the result in [3, Th. 3].

C. Interconnection With Secondary Estimator

An important setting in adaptive safety is one where a model of a totally unknown set of dynamics has already been developed using a secondary estimator. Such models can be developed using system identification experiments. The RISE-based observer in Section III can provide safety guarantees to the secondary model without the need to precisely characterize the modeling error. While the RISE observer can serve as a stand alone estimator for unknown dynamics, employing a secondary estimator can reduce the gain condition in (7), enhancing performance with smaller gains.

Suppose the dynamics originally are $\dot{x} = f_k(x) + g(x)u$ for an unknown function f_k that is approximated using a model $\hat{f}_k : \mathbb{R}^n \to \mathbb{R}^n$. For such models, knowledge of the estimation error is typically unavailable. Defining the estimation error $\tilde{f}_k \triangleq f_k - \hat{f}_k$, the dynamics are rewritten as

$$\dot{x} = \hat{f}_k(x) + g(x)u + \tilde{f}_k(x)$$

which matches the model in (1) with the known function $f \triangleq \hat{f}_k$ and the unknown function $d \triangleq \tilde{f}_k$. The RISE observer can then be implemented as described in Section III to identify the residual estimation error $\tilde{f}_k$. In this case, the controller in (14) remains as defined with $\hat{f}_k$ taking the place of f and $\tilde{f}_k$ taking the place of d.

Based on Assumption 2 and the gain condition (7), the interconnected secondary estimator will offer an improvement over the stand alone RISE observer when the rates of change of the estimation error are smaller than those of the unknown dynamics. Whether this holds true is dependent on the application and the quality of the estimated model. Numerical analysis of these rates of change is provided in the example of Section V.

IV. Local Lipschitz Continuity of κ^*

In this section, we study the continuity of the optimization-based controller κ^* in (14). We refer to the generic definition in (3) for generality. In order to meet Assumption 2, it will often be desirable for κ^* to be locally Lipschitz, which aids in applying Lemma 1. The following result from [30] provides conditions for when the function κ^* is locally Lipschitz under the linear independence constraint qualification (LICQ). A comparable result is available in [31, Th. 5.1]. Shapiro [32] provided a result for a weaker notion of pointwise Lipschitz continuity using the weaker Mangasarian–Fromovitz constraint qualification. The Mangasarian–Fromovitz qualification is typically easier to verify than the LICQ.

Definition 2: Let $\mathcal{I}(x, u) \triangleq \{i \in [l] : \mathcal{C}_i(x, u) = 0\}$ denote the active constraints of problem (3) at a given point. The LICQ is said to hold at $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ if the vectors $\{\nabla_u \mathcal{C}_i(x, u)\}_{i \in \mathcal{I}(x,u)}$ are linearly independent.

Theorem 3 (Local Lipschitz continuity of $\boldsymbol{\kappa}^*$*):* Consider the controller $\kappa^* : \mathbb{R}^n \to \mathbb{R}^m$ in (3). Given $\bar{x} \in \mathbb{R}^n$, assume there exists a neighborhood $U(\bar{x})$ of $\bar{x}$ such that κ^* is single-valued and continuous on $U(\bar{x})$ and, for each $i \in [l]$

A) The functions $(x, u) \mapsto Q(x, u)$ and $(x, u) \mapsto \mathcal{C}_i(x, u)$ are twice continuously differentiable on $U(\bar{x}) \times \mathbb{R}^m$.

B) The LICQ holds at $(\bar{x}, \kappa^*(\bar{x}))$
and for every $x \in U(\bar{x})$.

C) The function $u \mapsto \mathcal{C}_i(x, u)$ is convex on $K(x) \triangleq \{u \in \mathbb{R}^m : \mathcal{C}(x, u) \leq 0\}$ and the function $u \mapsto Q(x, u)$ is strongly convex on $K(x)$.

Then, κ^* in (3) is locally Lipschitz at $\bar{x}$.

Proof: We apply [30, Th. D.1]. [30, Conditions D.2 and D.3] hold by assumption. Since κ^* is continuous on $U(\bar{x})$, it is also bounded and (D.4) in [30] holds. Condition (D.5) holds

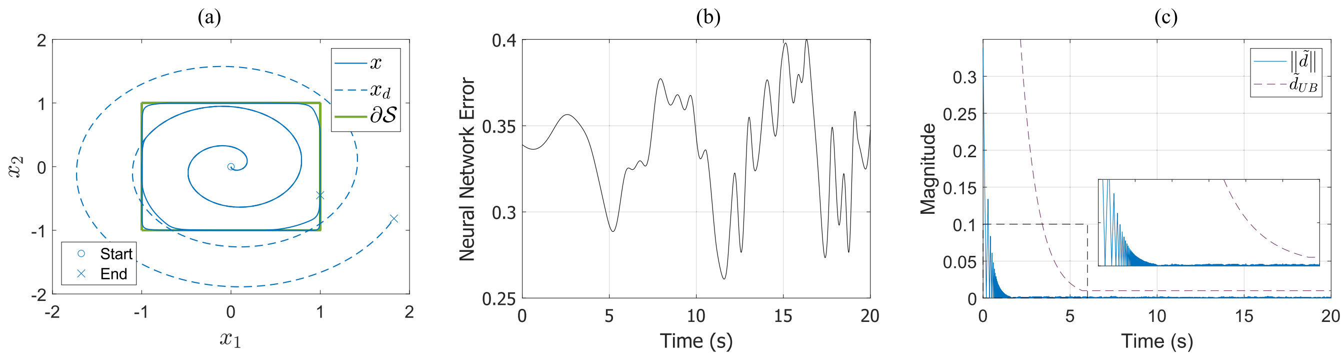

Fig. 2. (a) Trajectory of the closed-loop system in the example of Section V using a pretrained DNN coupled with the RISE-based disturbance observer. (b) Magnitude of the estimation error along the closed-loop trajectory for the pretrained DNN defined as $\|f_k - \hat{f}_k\|$. (c) Disturbance estimation error $\tilde{d}$ for the RISE observer. The safety-ensuring control law maintains the state inside the safe set although the desired trajectory leaves the set. The trajectory is allowed to closely approach the boundary of the safe set because the RISE observer accurately identifies the DNN estimation error.

when $u \mapsto Q(x,u)$ is strongly convex and the matrix $M(x) \triangleq [\nabla_u \mathcal{C}_i(x,\kappa^*(x))^T]_{i\in\mathcal{I}(x,\kappa^*(x))}$ has full rank for all x in a neighborhood of $\bar{x}$. The LICQ ensures that $M(\bar{x})$ has full rank, and continuity of the constraints and optimal value κ^* can be used to find that it remains full rank in a neighborhood of $\bar{x}$. Thus, Theorem D.1 shows that κ^* is locally Lipschitz. ∎

Remark 8: According to the conditions in [30], Assumption A of Theorem 3 can be relaxed when the problem defining κ^* is a quadratic program. Assume that $Q(x,u) = u^T H(x)u + h(x)^T u$ and $\mathcal{C}(x,u) = A(x)u + b(x)$. Then, via [30, Th. 3.1], Assumption A can be replaced with the assumption that H, h, A, and b are locally Lipschitz.

V. Simulation

A. Dynamics

The following example demonstrates the effectiveness of the RISE-based disturbance observer. The nonlinear system $\dot{x} = f_k(x) + u$ is considered where $x, u \in \mathbb{R}^2$, $\Psi(x) = \mathbb{R}^2$ for all $x \in \mathbb{R}^2$, and $f_k(x) = (\cos(x_1)\sin(x_2)\tanh(x_2) + \operatorname{sech}^2(x_1), \operatorname{sech}^2(x_1))$. To constrain the state to a safe set, a CBF candidate is defined as $B(x) \triangleq (-x_1 - c, x_1 - c, -x_2 - c, x_2 - c)$ where $c = 1$, which defines a square safe set $\mathcal{S} = \{x \in \mathbb{R}^2 : B(x) \leq 0\}$. A DNN function approximator is used to estimate the unknown term f_k in the dynamics. The DNN was pretrained using noisy data (Gaussian noise with nonzero mean) from a simulated trajectory of the system, thereby leading to imperfect function approximation. The DNN has tanh activation functions with three hidden layers and ten neurons in each layer, for a total of 272 individual weights. The weights are initially randomized for pretraining. Denoting the DNN approximation of the dynamics as $\hat{f}_k : \mathbb{R}^2 \to \mathbb{R}^2$, the dynamics are written as $\dot{x} = \hat{f}_k(x) + u + d(x)$ with $d(x)$ representing the residual estimation error $d(x) = f_k(x) - \hat{f}_k(x)$.

The RISE-based observer was used to identify d with $\alpha = 8$, $k_d = 2$, and $\beta = 1.5$. Numerical derivatives of the training dataset were used to approximate the constants c_1 and c_2 and then β was selected according to (7). The conservative upper bound of the disturbance was set to $\bar{d} = 3$ and the performance function is designed as $\gamma(x) = k_b B(x)$ for $k_b = 10$. In practice, some amount of estimation error is expected in the disturbance observer due to discretization error. To account for small errors, the upper bound of $\tilde{d}$ is defined as $\bar{d}_{\mathrm{UB}}(t) \triangleq \max(\bar{d}\, e^{-\lambda t}, 0.01)$. The control law κ^* is defined as in (14) with the cost function $Q(z,u) = \|u - \kappa_{\mathrm{nom}}(z)\|^2$, where the expanded state is $z = (x, \hat{x}, \hat{d}, t) \in \mathcal{Z} \triangleq \mathbb{R}^6 \times \mathbb{R}_{\geq 0}$ and $\kappa_{\mathrm{nom}}(z) = \dot{x}_d(t) - f_k(x) - K(x - x_d(t))$ is a nominal controller designed to track a spiral trajectory $x_d(t) = \min\{0.1t, 5\} \cdot (\sin(t), -\cos(t))$. The nominal controller is given access to the actual dynamics f_k to improve tracking performance and better highlight when the safety constraints cause κ^* to deviate from κ_{nom}. The controller κ^* is a quadratic program with constraints of the form $Au \leq b(z)$, where $A = J_B$ is the constant Jacobian of the CBF candidate B.

B. Results

The system was simulated using Euler integration with a time step of $dt = 1\text{E-}4\ s$. An example trajectory is shown in Fig. 2(a). Although the desired trajectory exits the safe set, the controller κ^* deviates from the nominal controller to keep the trajectory in the safe set. As can be seen in Fig. 2(a), the trajectory x never exits the safe set. Fig. 2(b) shows the neural network function approximation error and Fig. 2(c) shows the estimation error for the RISE observer. The RISE observer was able to identify the large approximation error in the DNN caused by noisy training data. After the disturbance was identified, the trajectory closely approached the boundary of the safe set; the maximum value for the CBFs was -0.001.

To show how the developed controller would behave without either the DNN approximation of f_k or the use of the disturbance observer, two additional simulations were performed. To remove the DNN from the simulation, the approximation of the dynamics was set to $\hat{f}_k = 0$. In this case, the trajectory remained in the safe set, but the RISE gain β had to be increased from $\beta = 1.5$ to $\beta = 6$ to compensate for the larger uncertainty. Alternatively, if the RISE terms are removed so that $\bar{\Gamma}_i(x,u) \triangleq \nabla B_i^T(x)(\hat{f}_k(x) + u)$, the trajectory escapes the safe set because the imperfect approximation of the dynamics by the DNN is not able to ensure safety, as shown in Fig. 3. The maximum value of the four CBFs with the DNN alone was 0.26. The trajectory in Fig. 1 was also generated from the dynamics in this example using a robust control approach where $\bar{\Gamma}_i(x,u) \triangleq \nabla B_i^T(x)u + \bar{d}\|\nabla B_i(x)\|$, in which case the closed-loop trajectory was not allowed to approach the boundary of the safe set as closely as adaptive approach; the maximum value of the CBFs was -0.2.

C. Discussion

The results in Sections III and IV can be applied to the example problem in this section. The feasibility and continuity of the

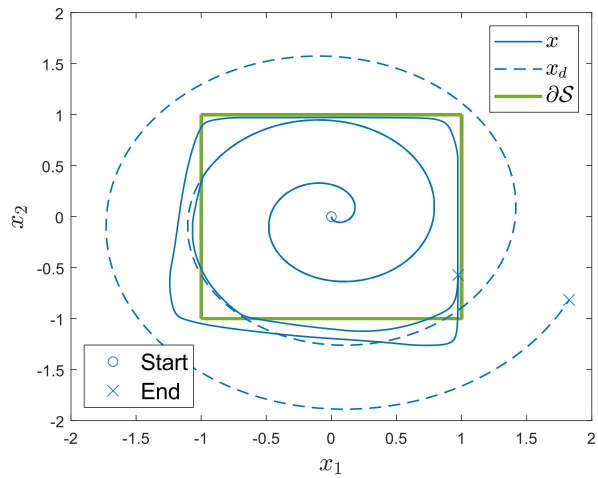

Fig. 3. Trajectory of the closed-loop system in the example of Section V using a pretrained DNN but without the RISE-based disturbance observer. Since the DNN estimate is imperfect, the trajectory is allowed to leave the safe set.

optimization-based control law can be verified using Theorem 3. Recalling Remark 7, manual computation reveals that the set $K^{\circ}_{\text{UB}}(x) \triangleq \{u \in \Psi(x) : \Gamma_{\text{UB}}(x,u) < -\gamma(x)\}$ is nonempty for all $x \in \mathbb{R}^2$ if $k_b c > \bar{d}$. Thus, $\bar{K}^{\circ}_c(z) = \{u \in \Psi(x) : \bar{\Gamma}(z,u) < -\gamma(x)\}$ is also nonempty on $\mathcal{Z}$ and [3, Th. 3] shows that κ^* is continuous. Moreover, whenever $\bar{K}^{\circ}_c(z)$ is nonempty, the LICQ must hold because the feasible set is a rectangle and only two constraints can be active at a given point. Via Theorem 3 and Remark 8, it follows that κ^* is locally Lipschitz on $\mathcal{Z}$. Since the desired trajectory is bounded and $\chi_i(z) \leq \bar{d}\|\nabla B_i(x)\|$ for every $i \in [r]$, it can be found that κ^* is Lipschitz and bounded on $\mathcal{S} \times \mathbb{R}^4 \times \mathbb{R}_{\geq 0}$. The dynamics $\hat{f}_k$, g, and $\tilde{f}_k$ are locally Lipschitz and $\mathcal{S} \subset \mathbb{R}^2$ is compact, so Lemma 1 shows that Assumption 2 holds with $\mathcal{R} = \mathcal{S}$. The next paragraph indicates that the observer gains satisfy (7), so that Theorem 1 verifies the exponential convergence of the disturbance estimation error. Since $\mathcal{S}$ is compact, Theorem 2 shows that $\mathcal{S}$ is forward invariant for the closed-loop dynamics.

The disturbance bounds in Assumption 2 were reduced significantly by the DNN. When the DNN was active, the disturbance was equal to the estimation error $d(x) = \tilde{f}(x)$. For the sake of analysis, exact knowledge of the dynamics was used to determine accurate values of the disturbance rate of change as $c_1 = 0.54$ and $c_2 = 4.02$, so that the choice of $\beta = 1.5$ determined without model knowledge was sufficient since (7) requires that $\beta > 1.19$. When the DNN was inactive, the entire dynamics were treated as a disturbance $d(x) = f(x)$, in which case $c_1 = 2.2$ and $c_2 = 15.0$, so that the gain needed to be increased significantly to $\beta = 6$ to meet the specification of $\beta > 4.7$.

The gain condition in (7) provides a sufficient condition for convergence of the estimator. It was observed in the simulation that convergence is possible for smaller values of the gain β. When the DNN was active, it was found that the estimator converged within a threshold of $\|\tilde{d}(t)\| \leq 0.01$ with β as low as 0.35. With the DNN inactive, the estimator converged within the threshold when β was as low as 2.5.

VI. Conclusion

This article developed a RISE-based observer that exponentially identifies unknown dynamics. The RISE-based disturbance estimate is integrated into an optimization-based controller, which enforces input constraints defined by a CBF. Sufficient conditions for the control law to be locally Lipschitz are presented. A simulation demonstrates the reduced conservativeness and safety guarantees offered by the observer.

Disclaimer: Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the sponsoring agency.

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