Event-Triggered Multiagent System Rendezvous With Graph Maintenance in Varied Hybrid Formulations: A Comparative Study

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***Abstract*—This article explores the rendezvous problem for a multiagent system (MAS) with distance-limited, intermittent communication and sensing. Unlike previous works that provide specific event-triggered controllers, we provide a framework that characterizes a family of distributed event-triggered controllers leveraging nonsingular edge-potentials to achieve approximate rendezvous while maintaining the initial distance-limited graph. The proposed framework excludes the possibility of Zeno behavior and accommodates the development of self-triggered controllers. The combination of continuous and impulsive dynamics results in a hybrid system, where the closed-loop dynamics of the MAS is presented and analyzed using hybrid differential inclusions. In particular, three distinct hybrid system formulations are presented, where the advantages and disadvantages of each construction are discussed in terms of their solution spaces. The approximate rendezvous problem is recast into a set stabilization problem, and sufficient conditions of the rendezvous set are obtained through a Lyapunov-based analysis. Simulation results are provided to validate the development, where a specific instance of an event trigger mechanism satisfying the requirements of the proposed design achieves approximate rendezvous while preserving the edges of an initially connected communication graph.**

***Index Terms*—Consensus control, decentralized control, hybrid systems, Lyapunov methods, multi-agent systems, network systems, network theory (graphs).**

I. Introduction

NETWORK communication is often expressed in terms of distance-based constraints: two agents may communicate if they are sufficiently close. Network connectivity is typically necessary for achieving multiagent system (MAS) objectives. Therefore, the network is often assumed to remain connected so that continuous communication along edges is always possible [1], [2], [3], [4], [5], [6]. Ensuring connectivity in real time poses nontrivial challenges, even in the presence of continuous communication [7]. Centralized and distributed variants of the connectivity maintenance problem under continuous interagent communication have been extensively studied [7], [8], [9], [10], [11], [12], as well as in discrete time [13], where connectivity maintenance can be addressed inductively. Conversely, few results on distributed event-triggered control (ETC) with connectivity maintenance are available [14], [15], [16], all based on some version of consensus dynamics (i.e., $\dot{x} = -\mathbf{L}x$, where $\mathbf{L}$ is a weighted graph Laplacian). In general, there is no straightforward reduction of the event-triggered version of this problem to either the smooth or discrete-time settings because variations in the length of time intervals between communication events may be necessary. The result in [14] presents an ETC approach to approximate rendezvous for a MAS. Piecewise continuous state-dependent gains on the standard (unweighted) consensus dynamics are used to preserve the edges of a distance-based communication graph. However, the proposed event trigger requires continuous access to interagent displacements, which implies continuous demand for sensing and/or communication, where only the controller updates are intermittent. The result in [15] investigates formation control with event-triggered connectivity preservation. The objective is achieved with intermittent sensing and broadcasting. Maintenance of the initial graph is achieved using unbounded edge-tension functions, originally presented in [17], where the Laplacian is weighted by the tension functions. In practice, unbounded tension functions impose unnecessary restrictions on the space of initial configurations, due to actuation limitations. We propose an alternative treatment, inspired by Boskos and Dimarogonas [12], using new edge tension functions designed to support the connectivity maintenance argument while avoiding restrictions on the admissible set of initial configurations. Recently, Dong and Xu [16] developed an event-triggered controller and observer that achieve leader–follower consensus while maintaining the edges of an initially connected graph. Edge preservation is achieved through the use of bounded edge-potentials and intermittent communication determined by event triggers.

The results in [15] and [16] provide inroads to MAS coordination with graph maintenance under intermittent feedback.

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These results share an approach of matching specific preselected time-based triggers to the MAS dynamics as a means to guarantee the desired properties (graph-maintenance, continuous-time completeness, and stability). In contrast, we derive characterizations of trigger properties required to guarantee each of these desiderata directly from the event-triggered consensus dynamics. In particular, the triggers developed in this article need not depend on time, unlike methods, such as in [15] and [16]. This article extends the preliminary result reported in [18], which studies the event-triggered rendezvous problem while also seeking to preserve the edges of an initially connected graph for a MAS consisting of single-integrator agents. Our result differs from [14], [15], and [16] in that we characterize a family of distributed event/self-triggered controllers that

1) utilize edge-potentials of class C^1 for graph maintenance;
2) ensure completeness of maximal solutions while excluding Zeno behavior;
3) given a user-defined parameter $\nu > 0$, exponentially drive the MAS to ν-approximate rendezvous[1].

In addition to a more detailed exposition of the results of [18], this article provides an in-depth analysis of properties of solutions relating to well-posedness and robustness issues arising from different hybrid differential inclusion (HDI) models of the same ETC design. Casting the problem within the HDI framework of [19] provides a number of advantages. Using hybrid time enables a particularly concise analysis of the interactions between the clocks of the individual agents, event triggers, and dynamical properties of the overall system. Our theoretical guarantees are independent of the particular form of the trigger, allowing for a variety of trigger designs. Three hybrid structure models are explored in detail, with an emphasis on comparing their solution spaces. Specifically, this article expands on [18] by studying how different methods for enforcing jumps (in response to triggering events) affect nominal well-posedness and robustness to perturbations. A standard way of designing a nominally well-posed HDI with a robust attractor—in addition to ensuring certain regularity properties of the flow and jump maps, jointly known as the hybrid basic conditions (HBCs)—is to design the flow and jump sets as closed sets.[2] Under such a construction, a flow interval of a solution may only culminate in a jump if the flow and jump sets intersect. However, an initial condition lying in the jump set may be viable (for the given flow map), creating a potential conflict between the notion of a solution and the use of triggers as tools for stabilizing system behavior: if triggered jumps are not executed (under some admissible solution), then the stability analysis based on the provided triggers may no longer be valid. Moreover, the distributed nature of our MAS setup introduces additional complexities to this problem. For example, different agents' event mechanisms may trigger synchronously (but independently), raising questions about how to properly design the jump map to make it stable under small perturbations. Such perturbations may disrupt the synchrony of independent triggers, causing cascades of multiple triggering events to happen, possibly, in different orders and with different outcomes. One way of enforcing triggered jumps while satisfying HBC is to design the triggers to be C^1-smooth functions with nonzero gradients at points of intersection of the flow and jump sets, while ensuring that jumps occur only on the boundary of the flow set. Then, a no-flow condition is presented, ensuring that the flow at such points is not viable. This strategy is compared with two alternative designs, which, although violating HBC by requiring an open flow set, result in enforcing jumps without necessarily demanding the imposition of regularity requirements on the triggers and no-flow conditions on the flow map. Section IV studies the formal connections between solutions of the three different hybrid models, with an emphasis on local compactness properties (nominal well-posedness). This more general analysis is facilitated by [20, Prop. 2.10], which, under certain conditions that are satisfied by our design, remains applicable even when HBC is violated, and may be used to establish meaningful properties of trajectories, such as continuous-time completeness and non-Zeno behavior. Furthermore, upon establishing the ν-approximate rendezvous subset of the system state space as an attractor of the designed HDIs, a detailed comparative analysis of the robustness properties of this attractor for the considered designs is provided in Section VII. The main challenges overcome in this analysis are the noncompactness of the attractor combined with the dearth of direct robustness results for HDIs that do not satisfy HBC. The inherent translation-invariance of the problem investigated in this work was leveraged to distill natural and unrestrictive conditions on the design (specifically, the event triggers must be invariant under translations of the MAS), guaranteeing the existence of a quotient HDI with a compact attractor. From there, standard theory is deployed to deal with the HBC-compliant designs, and Krasovskii regularization is used to analyze the designs, which violate HBC. The developed methods form a rather general blueprint for adapting an HBC-noncompliant design $\mathcal{H}$ with noncompact attractor $\mathcal{A}$ for use with the standard robustness framework of [19, Ch. 6 and 7], provided sufficient symmetry is present to yield a quotient $\mathcal{H}_{\#}$ with a compact attractor $\mathcal{A}_{\#}$, and that the original attractor $\mathcal{A}$ is robustly stable (in the applicable sense) for the Krasovskii regularization $\widehat{\mathcal{H}}$.

Finally, this article expands the result of [12] in two significant ways. First, the new proposed edge-tension function removes the restriction from [12] on the length of the initial edges (which are required to be significantly shorter than the communication radius). This change clarifies the tradeoff between how close an edge is to breaking and the control effort required to preserve the edge. Second, we relax the need for continuous communication/sensing by employing an event/self-triggered control strategy. Simulation results for a specific instance of an event trigger using our theorems are presented to validate the development. In addition to a more detailed exposition of the theoretical results, this extended study provides a more complete analysis of the simulation results published earlier in [18]. The rest of this article is organized as follows. Section II introduces notation, reviews the basics of algebraic graph theory used in MASs, and recalls notions from the HDI framework necessary

[1]By ν-approximate rendezvous, we mean a state of the MAS where all agent states are within a distance of at most ν from each other.

[2]This component of HBC is necessary for enabling compactness arguments using graphical convergence.

for the development in Section IV. Section III introduces the design problem in formal terms, including the control objective, and discusses three possible HDI-based models of the desired MAS behavior. Section IV establishes formal properties of the different HDI designs proposed in Section III, with an emphasis on similarities between their respective solution spaces. The designs not satisfying HBC are shown, nevertheless, to have many of the critical properties enjoyed by the HBC-compliant design. Section IV is one of the two main contributions of this article over the preliminary conference result [18]. Section V establishes criteria for continuous-time completeness and graph maintenance under the proposed designs, while Section VI derives additional sufficient conditions for practical exponential stability of the rendezvous objective. Following these analyses is the second contribution of this article beyond the conference version. Section VII establishes conditions for robustness for the three hybrid designs. Section VIII provides examples of event- and self-triggered mechanisms satisfying the derived criteria, and Section IX presents simulation results. Finally, Section X concludes this article with an overall discussion of the results.

II. Preliminaries

A. Notation

For any sets A, B, a function f of A with values in B is denoted by $f : A \to B$, whereas $f : A \rightrightarrows B$ refers to a *set-valued* function $f : A \to 2^B$. If $A' \subseteq A$, then $f|_{A'}$ denotes the restriction of f to the domain A'. The set-complement of A is denoted by $A^{\complement}$. For $p, q, n \in \mathbb{Z}_{>0}$, the $p \times q$ zero matrix and the $p \times 1$ zero column vector are denoted by $0_{p\times q}$ and 0_p, respectively. The $p \times p$ identity matrix and the $p \times 1$ column vector of 1s are denoted by $\mathbf{I}_p$ and $\mathbb{1}_p$, respectively. The diagonal matrix with main diagonal elements $w \in \mathbb{R}^p$ is denoted by $\mathtt{diag}(w) \in \mathbb{R}^{p\times p}$. For $x \in \mathbb{R}^p$, $\|x\|$ always denotes the Euclidean norm, $\|x\| = \sqrt{x^\top x}$. The inner product between $x, y \in \mathbb{R}^p$ is denoted by $\langle x, y\rangle = x^\top y$. The infinity norm of $\mathbf{x} = (x_i)_{i\in S}$, where S is some finite indexing set and $x_i \in \mathbb{R}^p$ for each $i \in S$, is denoted by $\|\mathbf{x}\|_\infty = \max_{i\in S} \|x_i\|$. The distance of a point $x \in \mathbb{R}^p$ to a nonempty set $\mathcal{A} \subset \mathbb{R}^p$ is given by $|x|_{\mathcal{A}} \triangleq \inf\{\|x - y\| : y \in \mathcal{A}\} \in \mathbb{R}_{\geq 0}$. Let e_k denote the kth standard basis vector in $\mathbb{R}^N$. The closure of a set $C \subset \mathbb{R}^p$ is denoted by $\overline{C}$, and the tangent cone of C at a point $x \in \mathbb{R}^p$ is denoted by $\mathbf{T}_C(x)$ [19, Def. 5.12].

B. Graphs

Let $\mathcal{V}$ be a finite nonempty set of cardinality N, and let n be a fixed positive integer. The *configuration space over* $\mathcal{V}$ is defined as $\mathtt{Conf}(\mathcal{V}) \triangleq (\mathbb{R}^n)^{\mathcal{V}}$. We will refer to $\mathbf{x} \triangleq (x_p)_{p\in\mathcal{V}} \in \mathtt{Conf}(\mathcal{V})$ as a *configuration of particles in* $\mathbb{R}^n$. For $R > 0$, the *R-threshold graph* $\mathcal{G}_R(\mathbf{x})$ on a configuration $\mathbf{x}$ is the undirected graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}_R(\mathbf{x})$ defined by setting $pq \in \mathcal{E}_R(\mathbf{x})$ if and only if $\|x_p - x_q\| \leq R$, where we denote $pq \triangleq \{p, q\}$ for all $p, q \in \mathcal{V}$, $p \neq q$. Recall, for any graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a path in $\mathcal{G}$ connecting a vertex p to a vertex q is a sequence of vertices $(v_0 = p, \ldots, v_k = q)$ where $k \in \mathbb{Z}_{\geq 0}$ and $v_{s-1}v_s \in \mathcal{E}$ for all $s = 1, 2, \ldots, k$. The graph $\mathcal{G}$ is connected, if every two vertices $p, q \in \mathcal{V}$ may be connected by a path in $\mathcal{G}$. The *neighborhood* $\mathcal{N}_p$ of a vertex $p \in \mathcal{V}$ is the set of all $q \in \mathcal{V}$ with $pq \in \mathcal{E}$. The *degree* of p is $d_p \triangleq |\mathcal{N}_p|$, and $\Delta(\mathcal{G})$ denotes the maximum degree in $\mathcal{G}$. Let $\mathcal{A} \triangleq [a_{pq}] \in \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$ denote the adjacency matrix of $\mathcal{G}$, where $a_{pq} = 1$ if and only if $pq \in \mathcal{E}$ and $a_{pq} = 0$, otherwise. Within this work, no self-loops are considered. Therefore, $a_{pp} \triangleq 0$ for all $p \in \mathcal{V}$. The degree matrix Δ of $\mathcal{G}$ is the diagonal matrix whose pth diagonal entry is $\sum_{q\in\mathcal{V}} a_{pq}$. The Laplacian matrix of $\mathcal{G}$ is defined as $\mathbf{L} \triangleq \Delta - \mathcal{A} \in \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$. More generally, if $c = [c_{pq}] \in \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$ is a nonnegative symmetric matrix, the weighted Laplacian is defined as $\mathbf{L}_c \triangleq \Delta_c - \mathcal{A}_c$, where[3] $\mathcal{A}_c \triangleq \mathcal{A} \odot c$, $\Delta_c \triangleq \mathrm{diag}(\mathcal{A}_c \cdot 1_N)$, and known to be positive semidefinite. Let $\{\lambda_i(\mathbf{L}_c)\}_{i=1}^N$ denote the eigenvalues of $\mathbf{L}_c$ in a nondecreasing order, and let $\lambda_i(\mathcal{G}) \triangleq \lambda_i(\mathbf{L})$. If $\mathcal{G}$ is connected, then $\lambda_1(\mathcal{G}) = 0$ is a simple eigenvalue, and $\lambda_2(\mathcal{G})$, known as the *Fiedler value* of $\mathcal{G}$, is positive. Also, $\lambda_N(\mathbf{L}_c)$ coincides with the operator norm $\|\mathbf{L}_c\|$, since $\mathbf{L}_c$ is self-adjoint.

C. Hybrid Systems

An HDI $\mathcal{H}$ takes the form [19]

$$\mathcal{H} : \begin{cases} \dot{z} \in F(z), & z \in C, \text{(flow constraint)} \\ z^+ \in G(z), & z \in D, \text{(jump constraint)} \end{cases}$$

where $F : C \rightrightarrows \mathbb{R}^n$ and $C \subset \mathbb{R}^n$ are the *flow map and set*, respectively, $G : D \rightrightarrows \mathbb{R}^n$ and $D \subset \mathbb{R}^n$ are the *jump map and set*, respectively, and z^+ indicates the value of the state after a jump. Solutions of $\mathcal{H}$ evolve continuously over the flow set according to the dynamics given by the flow map, and are allowed to execute discrete jumps over the jump set, constrained to the sets specified by the jump map. Formally, a set $A \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *hybrid time domain*, if there is a nondecreasing sequence of nonnegative reals $(t_j)_{j=0}^m$, $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, $t_0 = 0$, $t_m \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, such that $A = \cup_{j=1}^m (I_j \times \{j - 1\})$, where all the I_j, $j < m$ are of the form $[t_{j-1}, t_j]$, and I_m is of the form[4] $[t_{m-1}, t_m]$ or $[t_{m-1}, t_m)$ when $m < \infty$. We refer to $(t_j)_{j=0}^m$ as the *jump sequence* of the time domain A. A *hybrid arc* ϕ is a function $\phi : \mathrm{dom}\,\phi \to \mathbb{R}^n$, where $\mathrm{dom}\,\phi \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a hybrid time domain with jump sequence $(t_j)_{j=0}^m$, and ϕ is a locally absolutely continuous function on I_j, for every j. A *solution* of $\mathcal{H}$ is a hybrid arc ϕ such that, for all $j > 0$, $\phi(t, j - 1) \in C$ and $\frac{\mathrm{d}\phi}{\mathrm{d}t}(t, j - 1) \in F(\phi(t, j - 1))$ for almost all $t \in I_j$ (the *flow condition*); and $\phi(t_{j-1}, j - 1) \in D$ and $\phi(t_{j-1}, j) \in G(\phi(t_{j-1}, j - 1))$ (the *jump condition*). A solution ϕ to $\mathcal{H}$ is called *maximal* if ϕ cannot be extended, that is: if ψ is a solution with $\mathrm{dom}\,\phi \subseteq \mathrm{dom}\,\psi$, which coincides with ϕ on $\mathrm{dom}\,\phi$, then $\psi = \phi$. A solution ϕ is called *complete* if $\mathrm{dom}\,\phi$ is unbounded.

Definition 1: A solution ϕ of $\mathcal{H}$ is said to be *t-complete* if the sequence (t_j) is unbounded.[5]

[3] $\odot$ and $\otimes$ denote the Hadamard and Kronecker matrix products, respectively.
[4] Note $t_m = \infty$ is allowed when $m < \infty$; for $m = \infty$, there is no t_m.
[5] Note that a t-complete solution is complete.

A formal means of regarding solutions ϕ of $\mathcal{H}$ as functions of t is required in what follows. Given a solution ϕ of $\mathcal{H}$, let

$$\phi^* : \bigcup_{j=1}^{m} [t_{j-1}, t_j) \to \mathbb{R}^n \tag{1}$$

be defined via

$$\phi^*(t) = \phi(t, j-1) \ \Leftrightarrow \ t \in [t_{j-1}, t_j) \,. \tag{2}$$

By construction, ϕ^* is an absolutely continuous function of t except at the jump points $\{t_j\}_{j=1}^m$. Also, $\phi^*|_{[t_{j-1},t_j)}$ extends uniquely to an absolutely continuous function on $I_j = [t_{j-1}, t_j]$, for all j. Note that ϕ^* retains the information about the initial conditions for all flow intervals of ϕ, that is: $\phi^*(t_{j-1})$ is the initial condition of $\phi|_{I_j \times \{j-1\}}$. The following definitions are presented to facilitate the discussion of the solutions spaces between the different hybrid system formulations.

Definition 2: A hybrid system $\mathcal{H}$ is said to satisfy the HBCs [19, Assumption 6.5] if the following holds:

1) the sets C and D are closed;
2) F is outer semicontinuous and locally bounded relative to C, $C \subset \operatorname{dom} F$, and $F(x)$ is convex for every $x \in C$;
3) G is outer semicontinuous and locally bounded relative to D, and $D \subset \operatorname{dom} G$.

In particular, condition 2) is satisfied when F is a single-valued continuous mapping, which is the case in this article. When a system does not satisfy HBC, it can be regularized. Recall from [19, Def. 4.13] that a hybrid arc ϕ is called a Krasovskii solution to $\mathcal{H} = (C, F, D, G)$ if ϕ is a solution to the system

$$\widehat{\mathcal{H}} : \begin{cases} \dot{z} \in \widehat{F}(z), & z \in \widehat{C} \\ z^+ \in \widehat{G}(z), & z \in \widehat{D} \end{cases} \tag{3}$$

where $\widehat{C} \triangleq \overline{C}$, $\widehat{D} \triangleq \overline{D}$, and

$$\widehat{F}(z) \triangleq \bigcap_{\delta > 0} \overline{\text{con}} F((z + \delta\mathbb{B}) \cap C) \quad \text{for all } z \in \widehat{C} \tag{4}$$

$$\widehat{G}(z) \triangleq \bigcap_{\delta > 0} \overline{G((z + \delta\mathbb{B}) \cap D)} \quad \text{for all } z \in \widehat{D}. \tag{5}$$

It is known that $\widehat{\mathcal{H}}$ satisfies HBC [19, Ex. 6.6] whenever F and G are locally bounded, which will always be the case in this article. Systems satisfying HBC are important because their solution spaces are well behaved. In this context, relevant notions are recalled in the following definitions.

Definition 3: [19, Def. 5.21] A sequence $\{\phi_i\}_{i=1}^\infty$ of hybrid arcs $\phi_i : \operatorname{dom} \phi_i \to \mathbb{R}^n$ converges graphically if the sequence of sets $\{\text{gph } \phi_i\}_{i=1}^\infty$ converges in the sense of set convergence (see [19, Def. 5.1]). The graphical limit of a graphically convergent sequence $\{\phi_i\}_{i=1}^\infty$ is the mapping $M : \mathbb{R}^2 \rightrightarrows \mathbb{R}^n$ defined by gph $M = \lim_{i\to\infty}$ gph ϕ_i.

Definition 4: A hybrid system $\mathcal{H}$ is said to be nominally well-posed [19, Def. 6.2] if the following property holds: for every graphically convergent sequence $\{\phi_i\}_{i=1}^\infty$ of solutions to $\mathcal{H}$ with $\lim_{i\to\infty} \phi_i(0,0) = \xi \in \mathbb{R}^n$:

1) if the sequence $\{\phi_i\}_{i=1}^\infty$ is locally eventually bounded, then the sequence $\{\text{length}(\phi_i)\}_{i=1}^\infty$ converges in $[0, \infty]$, and $\phi = \text{gph-lim}_{i\to\infty}\phi_i$ is a solution to $\mathcal{H}$ with $\phi(0,0) = \xi$ and $\text{length}(\phi) = \lim_{i\to\infty} \text{length}(\phi_i)$;
2) otherwise, there are $m \in (0, \infty)$, $(t_i, j_i) \in \operatorname{dom} \phi_i$, such that $\lim_{i\to\infty} \|\phi_i(t_i, j_i)\| = \infty$, and $\phi \triangleq \text{gph-lim}_{i\to\infty}\phi_i|_{t+j<m}$ is a maximal solution of $\mathcal{H}$ with $\text{length}(\phi) = m$ and $\lim_{t\to t^*} \|\phi(t, j^*)\| = \infty$, where $(t^*, j^*) = \sup(\operatorname{dom} \phi)$.

Note that, in a situation where all solutions of $\mathcal{H}$ are known to be t-complete, only the first possibility described in Definition 4 may occur.

III. Problem Formulation and Controller Design

Consider a cooperative MAS composed of $N \in \mathbb{Z}_{>0}$ agents indexed by a set $\mathcal{V}$, with states $x_p \in \mathbb{R}^n$ and $p \in \mathcal{V}$. Any two agents $p, q \in \mathcal{V}$ are capable of exchanging information with each other whenever the distance between them does not exceed $R > 0$. Then, the possible connections among the agents are encoded by the R-threshold graph $\mathcal{G}_R(\mathbf{x})$. For each $p \in \mathcal{V}$, the model of agent p is a single integrator, $\dot{x}_p = u_p$, where $u_p \in \mathbb{R}^n$ denotes a control input.

Assumption 1: The initial R-threshold graph, $\mathcal{G} \triangleq \mathcal{G}_R(\mathbf{x}(0))$, is connected, and every edge pq in $\mathcal{E} \triangleq \mathcal{E}_R(\mathbf{x}(0))$ satisfies $\|x_p - x_q\| < R$.

Assumption 2: For each $p \in \mathcal{V}$ and $q \in \mathcal{N}_p$, agent p is capable of measuring $x_p - x_q$ for all $t \geq 0$.

Definition 5: Let $\nu > 0$. The MAS achieves ν-approximate rendezvous if $\|x_p - x_q\| \leq \nu$ for all $p, q \in \mathcal{V}$.

A distributed controller is developed for driving the MAS to ν-approximate rendezvous while maintaining the initial graph structure throughout the process, in the sense that $\mathcal{E} \subseteq \mathcal{E}_R(\mathbf{x}(t))$ holds for all $t \geq 0$. The developed controller limits communications and/or sensing to edges of the graph $\mathcal{G}$ to remove the need for continual monitoring of R-neighborhoods. Instead, each agent can rely on peer-to-peer communication with a fixed set of neighbors, which was established at time $t = 0$, as long as this communication can be guaranteed. In addition, any properties of the communication graph, such as the spectrum of $\mathbf{L}_{\mathcal{G}}$, may be computed in advance at time $t = 0$.

A. Potential Functions

Inspired by the work in [12], edge-potentials are employed to preserve the edges of configurations $\mathbf{x}$, which support a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in the sense that $\mathcal{E} \subseteq \mathcal{E}_R(\mathbf{x})$. Let $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a nondecreasing continuous function with $r(0) > 0$. Furthermore, let $P : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be given by

$$P(\rho) \triangleq \int_0^\rho r(s) s \mathtt{d}s, \quad \rho \in \mathbb{R}_{\geq 0}. \tag{6}$$

The potential $V_{pq} : \mathtt{Conf}(\mathcal{V}) \to \mathbb{R}_{\geq 0}$ for each $pq \in \mathcal{E}$ is

$$V_{pq}(\mathbf{x}) \triangleq P\left(\|x_p - x_q\|\right), \ w_{pq} \triangleq r\left(\|x_p - x_q\|\right) \tag{7}$$

noting $V_{pq} = V_{qp}$, and that $w \triangleq [w_{pq}] \in \mathbb{R}_{\geq 0}^{\mathcal{V}\times\mathcal{V}}$ is a state-dependent symmetric matrix. The function r is specified as follows. First, $\varepsilon \in \mathbb{R}$ is selected so that $\widetilde{R} \triangleq R(1 - \varepsilon)$ satisfies

$$R > \widetilde{R} > \frac{2}{3}R \ \Leftrightarrow \ \frac{1}{3} > \varepsilon > 0. \tag{8}$$

Next, let $\mu, \omega > 0$, and let $r(s)$ be selected as

$$r(s) \triangleq \mu \cdot \begin{cases} 1, & s \in [0, \widetilde{R}] \\ 1 + \omega(s^2 - \widetilde{R}^2), & s \in [\widetilde{R}, R] \\ 1 + \omega(R^2 - \widetilde{R}^2), & s \geq R. \end{cases} \tag{9}$$

After some algebra, we obtain

$$\omega \geq \frac{2\,|\mathcal{E}|\,(1-\varepsilon)^2}{R^2\varepsilon^2(2-\varepsilon)^2} \implies |\mathcal{E}|\,P(\widetilde{R}) \leq P(R). \tag{10}$$

Selecting ω according to (10) is required for graph maintenance (see Th. 1). As in [12, Prop. 3.2], $\widetilde{R}$ plays the role of a safe communication distance below which the edge potential is the usual square of the distance. The goal is to prove that the initial graph will be preserved if all its edges are initially safe in this sense. However, our design differs from those considered in [12] in that r ramps up quadratically in the unsafe zone $[\widetilde{R}, R]$, whereas the latter are either held constant or grow linearly there, which results in lower bounds on the buffer: $\varepsilon \geq 1 - |\mathcal{E}|^{-1/2}$ for constant r and $\varepsilon \geq 1 - (\frac{2}{3|\mathcal{E}|-1})^{1/3}$ for linearly growing r. In contrast, our design enables the selection of a small enough ε at initialization.

B. Hybrid Controller and Closed-Loop Dynamics

Let $\mathcal{X} \triangleq \mathtt{Conf}(\mathcal{V}) \times \mathtt{Conf}(\mathcal{V}) \times [0, T]^{\mathcal{V}}$ denote the *extended configuration space* of the MAS, where $T > 0$ is a user-defined parameter. The controller of agent $p \in \mathcal{V}$ is designed as $u_p \triangleq \eta_p$, where $\eta_p \in \mathbb{R}^n$ and $\tau_p \in [0, T]$ are auxiliary variables subject to the hybrid closed-loop dynamics $\mathcal{H}_p$ with flow map

$$\begin{bmatrix} \dot{x}_p \\ \dot{\tau}_p \\ \dot{\eta}_p \end{bmatrix} = \begin{bmatrix} \eta_p \\ 1 \\ 0_n \end{bmatrix}, \quad \mathbf{T}_p(\boldsymbol{\xi}) > 0 \text{ and } \tau_p < T \tag{11}$$

and with jump map

$$\begin{bmatrix} x_p^+ \\ \tau_p^+ \\ \eta_p^+ \end{bmatrix} = \begin{bmatrix} x_p \\ 0 \\ \sum_{q \in \mathcal{N}_p} w_{pq}(x_q - x_p) \end{bmatrix}, \quad \mathbf{T}_p(\boldsymbol{\xi}) \leq 0 \text{ or } \tau_p = T \tag{12}$$

where the following holds:

1) $\boldsymbol{\xi} \triangleq [\mathbf{x}^\top, \boldsymbol{\eta}^\top, \boldsymbol{\tau}^\top]^\top \in \mathcal{X}$, $\boldsymbol{\eta} \triangleq (\eta_p)_{p \in \mathcal{V}} \in \mathtt{Conf}(\mathcal{V})$ denotes the stacked vector of auxiliary variables η_p, and $\boldsymbol{\tau} \triangleq (\tau_p)_{p \in \mathcal{V}} \in [0, T]^{\mathcal{V}}$ is a vector of *personal clocks*, each of which resets whenever agent p triggers ($\mathbf{T}_p = 0$), or when the flow time limit is reached ($\tau_p = T$);
2) The *triggers*,[6] $\mathbf{T}_p : \mathcal{X} \to \mathbb{R}$ are continuous functions satisfying $\mathbf{T}_p(\boldsymbol{\xi}^+) > 0$ whenever $\mathbf{T}_p(\boldsymbol{\xi}) \leq 0$ or $\tau_p = T$, for every $p \in \mathcal{V}$;

and where it is understood that $x_p^+ = x_p$, $\eta_p^+ = \eta_p$, and $\tau_p^+ = \tau_p$ hold for any jump not triggered by the jump condition of agent p. It will be convenient to define vectors $\boldsymbol{\tau}_p^+ \in [0, T]^{\mathcal{V}}$ and $\boldsymbol{\eta}_p^+ \in \mathtt{Conf}(\mathcal{V})$ as

$$(\boldsymbol{\tau}_p^+)_q \triangleq \begin{cases} \tau_q, & q \neq p, \\ \tau_p^+, & q = p, \end{cases} \quad (\boldsymbol{\eta}_p^+)_q \triangleq \begin{cases} \eta_q, & q \neq p \\ \eta_p^+, & q = p \end{cases} \tag{13}$$

corresponding to the updates in the timer and actuation variables being executed only for agent p.

All functions of $\mathbf{x}$ will be regarded as functions of $\boldsymbol{\xi}$. Also, for any particular value of $\boldsymbol{\xi}$, we denote the projection of $\boldsymbol{\xi}$ to the first component of $\mathcal{X}$ by $\mathbf{x}(\boldsymbol{\xi})$ and similarly for the other components. Since the $\mathbf{T}_p$ are continuous, the sets

$$C \triangleq \bigcap_{p \in \mathcal{V}} ([\mathbf{T}_p > 0] \cap [\tau_p < T])$$

$$D \triangleq \bigcup_{p \in \mathcal{V}} ([\mathbf{T}_p \leq 0] \cup [\tau_p = T]) \tag{14}$$

the flow and jump sets of $\mathcal{H}$ are open and closed,[7] respectively, with $\overline{C} \subseteq C \cup D$. Both the flow and jump maps are single valued and continuous, and solutions to the flow equations are global and unique.

Remark 1: From an implementation perspective, as a solution evolves over time, the hybrid controller constructed in (11) and (12) is implementable as long as all the edges of the initial graph $\mathcal{G}$ remain unbroken (agent p cannot update η_p if its $\mathcal{G}$-neighbors are out of communication range). The resulting closed-loop dynamical system is a valid model of the required MAS behavior as long as $\mathcal{G}$ is maintained. An invariance result characterizing the set of initial conditions from which all trajectories maintain $\mathcal{G}$ for all time is therefore required and is provided in Theorem 1.

Using the system for agent p defined in (11) and (12), we can construct the hybrid system for the entire MAS. Let

$$\mathcal{H} : \begin{cases} (\dot{\mathbf{x}}, \dot{\boldsymbol{\eta}}, \dot{\boldsymbol{\tau}}) = (\boldsymbol{\eta}, 0_{nN}, \mathbb{1}_N), & \boldsymbol{\xi} \in C \\ (\mathbf{x}^+, \boldsymbol{\eta}^+, \boldsymbol{\tau}^+) = (\mathbf{x}, \boldsymbol{\eta}^+_{\text{sync}}, \boldsymbol{\tau}^+_{\text{sync}}), & \boldsymbol{\xi} \in D \end{cases} \tag{15}$$

where $\boldsymbol{\eta}^+_{\text{sync}} \triangleq (\eta_p^+)_{p \in \mathcal{V}}$ and $\boldsymbol{\tau}^+_{\text{sync}} \triangleq (\tau_p^+)_{p \in \mathcal{V}}$. This formulation ensures that satisfying a trigger condition results in an appropriate jump in the component of the extended state corresponding to the triggering agent. The fact that the flow and jump sets are disjoint forces solutions to jump immediately when a triggering event occurs, including multiple synchronous triggering events. This conforms to the requirements of ETC, where indeterminism regarding whether or not to respond to triggering events is not allowed. However, the flow set being open in (15) makes it impossible to apply standard results in hybrid systems theory requiring HBC, which calls for an alternative analysis, presented in Section IV. A more standard design, in the spirit of [21], is

$$\mathcal{K} : \begin{cases} (\dot{\mathbf{x}}, \dot{\boldsymbol{\eta}}, \dot{\boldsymbol{\tau}}) = (\boldsymbol{\eta}, 0_{nN}, \mathbb{1}_N), & \boldsymbol{\xi} \in \overline{C} \\ (\mathbf{x}^+, \boldsymbol{\eta}^+, \boldsymbol{\tau}^+) \in \{(\mathbf{x}, \boldsymbol{\eta}_p^+, \boldsymbol{\tau}_p^+) : p \in \mathcal{V}(\boldsymbol{\xi})\}, & \boldsymbol{\xi} \in D \end{cases} \tag{16}$$

where $\mathcal{V}(\boldsymbol{\xi})$ is the set of $p \in \mathcal{V}$ satisfying the jump condition from (12). Note that $\mathcal{H}$ and $\mathcal{K}$ share the same single-valued flow map, $F(\boldsymbol{\xi}) \triangleq (\boldsymbol{\eta}, 0_{nN}, \mathbb{1}_N)$, which is well defined and smooth throughout $\mathcal{X}$. The jump maps for $\mathcal{H}$ and $\mathcal{K}$ can be written as

$$G_1(\boldsymbol{\xi}) \triangleq (\mathbf{x}, \boldsymbol{\eta}^+_{\text{sync}}, \boldsymbol{\tau}^+_{\text{sync}}) \tag{17}$$

$$G_2(\boldsymbol{\xi}) \triangleq \{(\mathbf{x}, \boldsymbol{\eta}_p^+, \boldsymbol{\tau}_p^+) : p \in \mathcal{V}(\boldsymbol{\xi})\} \tag{18}$$

[6] An example of a distributed trigger function candidate is provided in (36).

[7] Continuous preimages of open/closed sets are open/closed, respectively.

respectively, where G_1 inherits the degree of smoothness of r (9), and G_2 is outer semicontinuous while having compact (in fact, finite) values.

Remark 2: In $\mathcal{K}$, the HBC is satisfied by construction, but at the cost of indeterminism in the jump structure. Also, for this design, it is necessary to guarantee that solutions with initial conditions in the jump set cannot flow[8]. In the presence of k distinct synchronous triggering events at hybrid time (t, j), this property will result in $k!$ distinct solutions, each of which jumps repeatedly in zero continuous time (k times) until all triggering events have been addressed at hybrid time $(t, j+k)$. Observe that the result of the last jump for any such sequence of jumps does not depend on the order in which updates are carried out by the individual agents (in discrete hybrid time), and coincides with the jump map of (15). In other words, the behaviors of solutions of the two systems as (possibly discontinuous) functions of continuous time are identical except at jump times.

An intermediate design enforcing jumps at triggering events by keeping the open flow set of $\mathcal{H}$ while using the nondeterministic jump map of $\mathcal{K}$ is

$$\mathcal{L}: \begin{cases} (\dot{\mathbf{x}}, \dot{\boldsymbol{\eta}}, \dot{\boldsymbol{\tau}}) = (\boldsymbol{\eta}, 0_{nN}, \mathbb{1}_N), & \boldsymbol{\xi} \in C \\ (\mathbf{x}^+, \boldsymbol{\eta}^+, \boldsymbol{\tau}^+) \in \{(\mathbf{x}, \boldsymbol{\eta}_p^+, \boldsymbol{\tau}_p^+) : p \in \mathcal{V}(\boldsymbol{\xi})\}, & \boldsymbol{\xi} \in D \end{cases} \tag{19}$$

for which the observations of Remark 2 hold true without additional regularity properties of the triggers. It follows by direct computation that $\mathcal{K}$ coincides with the Krasovskii regularization $\widehat{\mathcal{L}}$ of $\mathcal{L}$. This fact will be used in Section VII-B. Section IV explores the differences between $\mathcal{H}, \mathcal{K}$, and $\mathcal{L}$ in detail.

C. MAS Control Objective

A configuration $\mathbf{x}$ is a *rendezvous state*, if $x_p = x_q$ for all $p, q \in \mathcal{V}$. The set $\Delta_\mathcal{V}$ of all rendezvous states is a linear subspace of $\texttt{Conf}(\mathcal{V})$. Henceforth, let $\mathbf{x} = \tilde{\mathbf{x}} + \mathbf{x}^\perp$ denote the orthogonal decomposition of $\mathbf{x}$ with $\tilde{\mathbf{x}} \in \Delta_\mathcal{V}$ and $\mathbf{x}^\perp \in \Delta_\mathcal{V}^\perp$, and let $\Delta\mathbf{x} \triangleq (x_p - x_q)_{pq\in\mathcal{E}}$ for an arbitrarily selected orientation on the edges of $\mathcal{E}$ (to eliminate doubles)[9]. It is well known that

$$\tilde{\mathbf{x}} = \mathbb{1}_N \otimes \frac{1}{N} \sum_{p\in\mathcal{V}} x_p. \tag{20}$$

Also, as shown in Appendix A, for all $\mathbf{x}$, one has

$$\frac{1}{\sqrt{N}} \|\mathbf{x}^\perp\| \le \|\Delta\mathbf{x}\|_\infty \le \sqrt{2} \|\mathbf{x}^\perp\|. \tag{21}$$

Achieving rendezvous is equivalent to globally[10] stabilizing the set $\mathcal{R} \triangleq \{\boldsymbol{\xi} \in \mathcal{X} : \|\Delta\mathbf{x}\|_\infty = 0\}$. Moreover, ν-approximate rendezvous is achieved whenever $\|\Delta\mathbf{x}\|_\infty \le \nu$. Therefore, it suffices to show that every trajectory of $\mathcal{H}, \mathcal{K}$, and $\mathcal{L}$ satisfying Assumption 1 is eventually contained in the set

$$\mathcal{R}_\nu \triangleq \{\boldsymbol{\xi} \in \mathcal{X} : \|\Delta\mathbf{x}\|_\infty \le \nu\}. \tag{22}$$

In addition, we require maintenance of the initial communication graph. To this end, if $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a connected graph, for any $\rho > 0$, let

$$\mathcal{C}_\rho(\mathcal{G}) \triangleq \{\boldsymbol{\xi} \in \mathcal{X} : \mathcal{E} \subseteq \mathcal{E}_\rho(\mathbf{x})\} \tag{23}$$

and note that Assumption 1 means $\boldsymbol{\xi}(0,0) \in \text{int}(\mathcal{C}_R(\mathcal{G}))$. Given an initial configuration $\mathbf{x}(0)$ satisfying Assumption 1, the initial condition $\boldsymbol{\xi}(0,0) = \phi(0,0)$ is set to satisfy

$$\phi(0,0) = (\mathbf{x}(0), (\mathbf{L}_w \otimes I_n)\mathbf{x}(0), 0_N) \tag{24}$$

and a controller is provided—that is, the values of ε, μ, and ω are determined—to guarantee $\boldsymbol{\xi}(t,j) \in \mathcal{C}_R(\mathcal{G})$ for all $(t,j) \in \text{dom}\,\phi$, for the corresponding maximal solution ϕ of $\mathcal{H}$.

Remark 3: The inequalities in (21) offer an alternative interpretation of ν-approximate rendezvous in terms of the Euclidean distance to the rendezvous set $\mathcal{R}$ in $\mathcal{X}$. Consider the following sets, defined for $\theta > 0$

$$\mathcal{R}'_\theta \triangleq \{\boldsymbol{\xi} \in \mathcal{X} : |\boldsymbol{\xi}|_\mathcal{R} \le \theta\}. \tag{25}$$

Since $|\boldsymbol{\xi}|_\mathcal{R} = \|\mathbf{x}^\perp\|$, for any $\theta, \nu > 0$, one has

$$\mathcal{R}'_{\nu/\sqrt{2}} \subseteq \mathcal{R}_\nu, \quad \mathcal{R}_{\theta/\sqrt{N}} \subseteq \mathcal{R}'_\theta. \tag{26}$$

Thus, sets of the form $\mathcal{R}_\nu$ and $\mathcal{R}'_\theta$ may be used interchangeably as targets for stabilization. Neither $\mathcal{R}_\nu$ nor $\mathcal{R}'_\theta$ are compact, yet both have compact quotients under translations of the MAS and projection to the MAS state component.

IV. Comparison of $\mathcal{H}, \mathcal{K}$, and $\mathcal{L}$

The models $\mathcal{H}$, $\mathcal{L}$, and $\mathcal{K}$ are different formalizations of the same desired behavior. This section studies the relative strengths and weaknesses these models exhibit and establishes basic properties of the solutions spaces. Much of the analysis is not specific to the current application and may be applied more broadly to MASs with individual agent dynamics given by hybrid differential equations whose flow maps induce globally defined C^1 flows and where the agent positions evolve continuously over time (e.g., see the proof of Lemma 1). Arguments invoking additional properties specific to our treatment of the rendezvous problem (as opposed to those applicable in more generality) will be highlighted. Let the projection to the $\mathbf{x}$ component be denoted by

$$\pi : \mathcal{X} \to \texttt{Conf}(\mathcal{V}), \ \pi(\mathbf{x}, \boldsymbol{\eta}, \boldsymbol{\tau}) \triangleq \mathbf{x}. \tag{27}$$

An important property of solutions of $\mathcal{H}$, $\mathcal{K}$, and $\mathcal{L}$ is that the $\mathbf{x}$ component of a solution ϕ does not jump, namely: $\pi(\phi(t_{j+1}, j)) = \pi(\phi(t_{j+1}, j+1))$ for all $j \ge 0$, where $(t_j)_{j=1}^\infty$ is the sequence of jump times in $\text{dom}\,\phi$. As a result, the expression $\mathbf{x}(\phi(t,j))$ is a well-defined continuous[11] function of t. Moreover, it is crucial to observe that the flow equation $\dot{\boldsymbol{\xi}} = F(\boldsymbol{\xi})$ has global solutions for any initial condition, which facilitates the following result.

Lemma 1: Every initial condition $\phi(0,0) \in \overline{C}$ determines one and only one maximal solution ϕ of the hybrid system $\mathcal{H}$ given in (15). Moreover, every maximal solution of $\mathcal{H}$ is either t-complete or Zeno.

Proof: By [19, Prop. 2.10] and the preceding observation, maximal solutions of $\mathcal{H}$ are complete. The condition on the triggers gives $G(D) \subset D^\complement$, implying that no solution of $\mathcal{H}$ has a pair of consecutive jumps. ■

[8]The required guarantee clearly necessitates sufficient regularity of the event triggers, which we explore in Lemma 2.

[9]It is well known that weighted Laplacian operators are independent of the choice of orientation on the edges of the graph [9, Sec. 2.3.3]

[10]Over the space of configurations with connected communication graphs.

[11]In fact, even piecewise smooth.

A similar result for $\mathcal{K}$, also relating its solutions to those of $\mathcal{H}$, may be derived from the fact that $\mathcal{K}$ satisfies HBC, provided the following no-flow condition is satisfied:

$$(\dagger) \quad F(\boldsymbol{\xi}) \notin \mathbf{T}_C(\boldsymbol{\xi}) \text{ for all } \boldsymbol{\xi} \in \partial C.$$

Remark 4: Condition (†) is always satisfied when $\tau_p(\boldsymbol{\xi}) = T$. Therefore, from a design perspective, condition (†) may be enforced by selecting the triggers to satisfy the following criteria. First, for each $p \in \mathcal{V}$, $\mathbf{T}_p$ is a C^1-smooth function with regular value 0. Second, for every $\boldsymbol{\xi} \in \partial C$, there is $p \in \mathcal{V}$ such that $\langle F(\boldsymbol{\xi}), \nabla \mathbf{T}_p(\boldsymbol{\xi})\rangle < 0$.

Lemma 2: Every initial condition $\phi(0,0) \in \overline{C}$ extends to a complete solution ϕ of the hybrid system $\mathcal{K}$ given in (16). Moreover, every maximal solution of $\mathcal{K}$ is either t-complete or Zeno. In addition, if (†) is satisfied, then the $\mathbf{x}$ component of any such solution, regarded as a function of t, coincides with that of the unique solution for $\mathcal{H}$ with the same initial condition.

Proof: Since $\mathcal{K}$ satisfies HBC and $C \setminus D$ is open, the result of [19, Prop. 6.10] applies. Note that $G(D) \subseteq C \cup D$ and that no flow solutions escape in finite time. Therefore, every $\phi(0,0) \in C \cup D$ extends to a complete solution of $\mathcal{K}$. To show that every maximal solution is either t-complete or Zeno, it suffices to verify that $\mathcal{K}$ has no complete discrete solutions. Were ϕ a complete discrete solution, at least one agent p would have to experience infinitely many jumps. At most one of these jumps is due to the flow timer for agent p running out. Consider two of the remaining jumps, say, at times $(0,j)$ and $(0,k)$, $j < k$, with $\tau_p = 0$ for both. However, in the absence of flow intervals, one must have $\mathbf{T}_p(\phi(0,k)) = \mathbf{T}_p(\phi(0,j+1)) > 0$, in contradiction of the jump condition for agent p at time $(0,k)$.

Now, assuming (†) and applying [22, Prop. 4.2.2] with $P(\boldsymbol{\xi}) = \overline{C}$ for every $\boldsymbol{\xi}$ implies that no initial condition in ∂C extends to a nontrivial solution of $\dot{\boldsymbol{\xi}} = F(\boldsymbol{\xi})$ that is contained in $\overline{C}$. Therefore, any nontrivial solution with an initial condition $\phi(0,0) \in \partial C$ must jump. Since $\mathcal{K}$ is an autonomous system, this reasoning extends to initial conditions at arbitrary times $(t,j) \in \operatorname{dom}\phi$. Suppose that $\phi(t,j) = (\mathbf{x}, \boldsymbol{\eta}, \boldsymbol{\tau}) \in D$, and let $k(t,j) \triangleq |\mathcal{V}(\phi(t,j)| \neq 0$. Note that $1 \le k \le N$. Since (t,j) is a jump point, $(t,j+1) \in \operatorname{dom}\phi$, and there is a $q \in \mathcal{V}(\phi(t,j)$ such that $\phi(t,j+1) = (\mathbf{x}, \boldsymbol{\eta}_q^+, \boldsymbol{\tau}_q^+)$. By Property 2 of triggers, $k(t,j+1) = k(t,j) - 1$. By induction, $k(t, j + k(t,j)) = 0$, and therefore, $\phi(t, j + k(t,j)) \in C$. In other words, once ϕ arrives at a jump point $\phi(t,j)$, it experiences $1 \le k(t,j) \le N$ jumps before returning into the interior C of the flow set $\overline{C}$ of $\mathcal{K}$. Moreover, observe that

$$\phi(t, j + k(t,j)) = (\mathbf{x}, \boldsymbol{\eta}^+_{\text{sync}}, \boldsymbol{\tau}^+_{\text{sync}}). \tag{28}$$

Since the flow dynamics of $\mathcal{H}$ and $\mathcal{K}$ coincide, we conclude that solutions of $\mathcal{H}$ and $\mathcal{K}$ with the same initial conditions in $\overline{C}$ coincide, as functions of t, except, possibly at jump points. More formally, for all $\boldsymbol{\xi}_0 \in \overline{C}$, if ϕ and ψ are solutions of $\mathcal{K}$ and $\mathcal{H}$, respectively, satisfying $\phi(0,0) = \psi(0,0) = \boldsymbol{\xi}_0$, the maps ϕ^* and ψ^* coincide. The proof proceeds by induction on the jump index, j. If $\boldsymbol{\xi}_0 \in C$, then both ϕ and ψ can only flow from $\boldsymbol{\xi}_0$ under the dynamics $\dot{\boldsymbol{\xi}} = F(\boldsymbol{\xi})$. Therefore, $\phi|_{[0,t_1]\times\{0\}} = \psi|_{[0,t_1]\times\{0\}}$ and hence $\phi^*|_{[0,t_1)} = \psi^*|_{[0,t_1)}$. Applying the observation in (28) yields $\phi^*|_{[0,t_1]} = \psi^*|_{[0,t_1]} \in C$, and now the same argument may be repeated for the next interval. If $\boldsymbol{\xi}_0 \in \partial C$, then (28) is applied directly to obtain $\phi^*(0) = \psi^*(0)$ as the (common) value of the two solutions after the initial jumps. ■

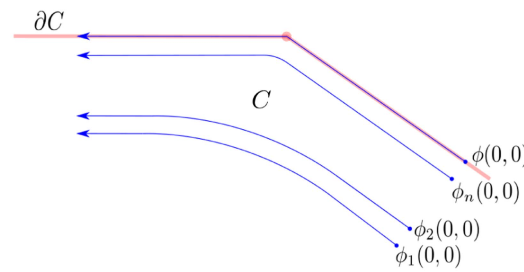

Fig. 1. $\mathcal{L}$ may not be nominally well-posed. In this illustration, a putative sequence ϕ_n of jump-free solutions of $\mathcal{K}$ contained in C converges to a solution ϕ of $\mathcal{K}$ contained in ∂C. By Lemma 3 and the fact that the ϕ_n do not reach the jump set D, the ϕ_n are also solutions of $\mathcal{L}$. However, their limit ϕ is not.

Solutions of the hybrid system $\mathcal{L}$ enjoy a middle-ground status between those of $\mathcal{H}$ and those of $\mathcal{K}$.

Lemma 3: Every initial condition $\phi(0,0) \in \overline{C}$ extends to a maximal solution ϕ of the hybrid system $\mathcal{L}$ given in (19), which is also a maximal solution of $\mathcal{K}$. In particular, every maximal solution of $\mathcal{L}$ is either t-complete or Zeno.

Proof: By [19, Prop. 2.10] and the preceding observation, maximal solutions of $\mathcal{L}$ are complete. The jump map G_2 does not provide $G_2(D) \subset D^{\complement}$ (compare with the proof of Lemma 1); however, the fact that the flow and jump sets are disjoint implies a solution ϕ of $\mathcal{L}$ with $\phi(t,j) \in D$ must satisfy $(t, j+1) \in \operatorname{dom}\phi$. In other words, upon arrival in the jump set, any solution of $\mathcal{L}$ must execute a jump. Therefore, we may apply the argument in the second part of the proof of Lemma 2 to conclude that no solution ϕ of $\mathcal{L}$ executes more than N consecutive jumps, proving that ϕ must be either t-complete or Zeno. Finally, every solution of $\mathcal{L}$ is a solution of $\mathcal{K}$, although, possibly not the other way around unless the no-flow condition of Lemma 2 is satisfied. ■

The preceding analysis has implications for the solutions of all three systems.

Corollary 1: Suppose that (†) is satisfied. Then, the solution spaces of $\mathcal{L}$ and $\mathcal{K}$ coincide. In particular, $\mathcal{L}$ is nominally well-posed.

Proof: The analysis in Lemma 3 implies that a solution of $\mathcal{K}$ is a solution of $\mathcal{L}$ if it executes a jump whenever it arrives at a jump state. The latter is guaranteed by (†). Nominal well-posedness of an HDI is a property of the solution space of the HDI. Therefore, $\mathcal{L}$ is nominally well-posed because $\mathcal{K}$ is nominally well-posed. ■

Delicate questions remain regarding designs in which the triggers $\mathbf{T}_p$ do not satisfy the no-flow condition (†). In these cases, the systems $\mathcal{L}$ and $\mathcal{H}$ have the clear advantage of enforcing the ETC paradigm despite nominally violating HBC (the flow set C is not closed), while $\mathcal{K}$ allows agents to ignore events to maintain HBC (a solution of $\mathcal{K}$ may visit a point of D without executing a jump at that point). In such a situation, $\mathcal{L}$ may not be nominally well-posed (see Fig. 1). Nevertheless, the connection between the designs allows for some positive results.

Corollary 2: Bounded solutions of $\mathcal{H}$, $\mathcal{K}$, and $\mathcal{L}$ are non-Zeno.

Proof: All bounded solutions of $\mathcal{K}$ are non-Zeno since $\mathcal{K}$ is nominally well-posed: by [23, Th. 1], no bounded solution of $\mathcal{K}$

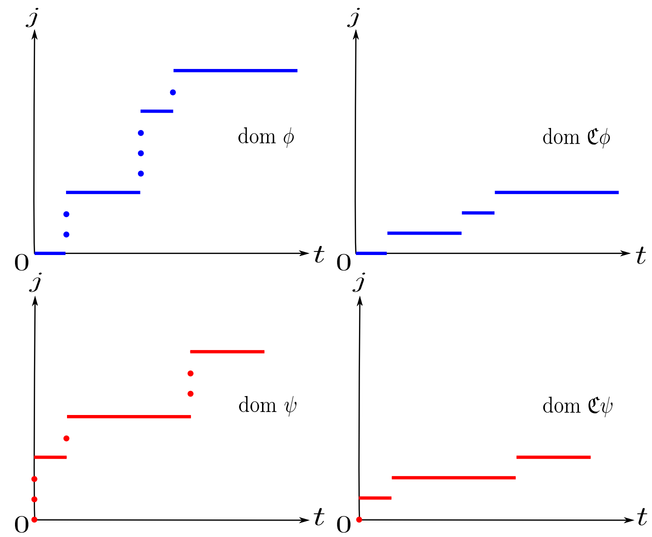

Fig. 2. Comparing hybrid time domains before (dom ϕ) and after (dom $\mathfrak{C}\phi$) crushing (Definition 6) in two cases of interest: ϕ starts with a flow segment, and ψ begins with a jump.

could have vanishing time between jumps, because the proof of Lemma 2 shows $\mathcal{K}$ does not have complete discrete solutions. Any bounded solution of $\mathcal{L}$ is a bounded solution of $\mathcal{K}$, and is, therefore, non-Zeno. Finally, by the proof of Lemma 2, any bounded solution of $\mathcal{H}$ has the same jump times as a bounded solution of $\mathcal{K}$, omitting the repetitions corresponding to multiple jumps being executed at the same continuous time. ■

The question remains whether or not anything can be said about unbounded solutions: since $\overline{C}$ is not compact, its forward invariance under either of the hybrid dynamics does not imply that solutions must be bounded. Addressing this issue is outside the scope of the discussion in this section, which focuses on comparing the formal properties of the three designs $\mathcal{H}$, $\mathcal{K}$, and $\mathcal{L}$.

To discuss the differences between $\mathcal{H}$ and $\mathcal{L}$ in more detail, we introduce the following notion.

Definition 6: Let $\mathcal{X}^\dagger$ denote the space of hybrid arcs in $\mathcal{X}$. The *crushing map* $\mathfrak{C} : \mathcal{X}^\dagger \to \mathcal{X}^\dagger$ is defined as follows. Given $\phi \in \mathcal{X}^\dagger$, let $(t_j)_{j=0}^m$ denote its sequence of jump times. A map $c_\phi : \mathbb{Z} \cap [1, m) \to \mathbb{N}$ is defined inductively by setting $c_\phi(1) \triangleq 1$, and, for all $j \in \mathbb{Z} \cap [1, m)$

$$c_\phi(j+1) \triangleq \begin{cases} c_\phi(j)+1, & \text{if } t_{j-1} < t_j \text{ or } j = 1 \\ c_\phi(j), & \text{otherwise.} \end{cases} \tag{29}$$

Then, the hybrid time domain dom $\mathfrak{C}\phi$ is obtained from dom ϕ by eliminating repetitions in the sequence of jump times, see Fig. 2:

$$(t, k-1) \in \operatorname{dom} \mathfrak{C}\phi \;\Leftrightarrow\; \exists_{j \ge 1} \begin{cases} (t, j-1) \in \operatorname{dom} \phi \\ k = c_\phi(j) \end{cases} \tag{30}$$

and $\mathfrak{C}\phi$ is defined on dom $\mathfrak{C}\phi$ to ensure $(\mathfrak{C}\phi)^* = \phi^*$, namely

$$(\mathfrak{C}\phi)(t, k-1) \triangleq \phi(t, j(k)-1) \tag{31}$$

for all $t \in [t_{j(k)-1}, t_{j(k)}]$, where $j(k) = \max c_\phi^{-1}(k)$.

The considerations in the proofs of Lemmas 2 and 3 motivate this definition, and are summarized in the following result, whose main role in this article is to facilitate the robustness analysis in Section VII.

Corollary 3: Let ϕ be a maximal solution of $\mathcal{H}$, and ψ be a maximal solution of $\mathcal{L}$ with $\psi(0,0) = \phi(0,0) \in \overline{C}$. Then, $\phi = \mathfrak{C}\psi$. Similarly, if (†) is satisfied and ψ is, instead, a maximal solution of $\mathcal{K}$, then also $\phi = \mathfrak{C}\psi$.

Remark 5: The map $\mathfrak{C} : \mathcal{X}^\dagger \to \mathcal{X}^\dagger$ is not continuous. In other words, it does not, in general, preserve graphical convergence. For example, consider a sequence ϕ_n of hybrid arcs with dom ϕ_n being the union of the sets $[0, 1] \times \{0\}$, $[1, 1 + \frac{1}{n}] \times \{1\}$, and $[1 + \frac{1}{n}, \infty) \times \{2\}$. If $\phi_n \to \phi$, then dom ϕ must be the union of $[0, 1] \times \{0\}$, $\{1\} \times \{1\}$, and $[1, \infty) \times \{2\}$. In particular, $\mathfrak{C}\phi \neq \phi$ despite the fact that $\mathfrak{C}\phi_n = \phi_n$ for all n. This example illustrates the barrier to $\mathcal{H}$ being nominally well-posed, even in the presence of (†), when $\mathcal{L}$ is nominally well-posed: the limit of a graphically convergent sequence of $\mathcal{H}$-solutions is necessarily an $\mathcal{L}$-solution, but may still need to be crushed if one were to obtain an $\mathcal{H}$-solution. At the same time, the observation that solutions of $\mathcal{H}$, $\mathcal{L}$, and $\mathcal{K}$ with the same initial condition differ only in their values at isolated jumps suggests the possibility that a weaker topology (than the topology of graphical convergence) on the space of hybrid arcs—or possibly a quotient thereof discounting information about isolated jumps—may be more suitable for discussing well-posedness and robustness properties of HDIs. Whether or not such a toolkit exists remains to be seen.

The fact that $\mathcal{H}$ does not satisfy HBC additionally motivates the study of its Krasovskii regularization $\widehat{\mathcal{H}}$, which, in turn, can be compared to $\mathcal{K}$, leading to the following results.

Lemma 4: For every $\boldsymbol{\xi} \in D$, let $J(\boldsymbol{\xi})$ denote the set of all nonempty subsets $P \subseteq \mathcal{V}(\boldsymbol{\xi})$. Then, the Krasovskii regularization $\widehat{\mathcal{H}}$ of $\mathcal{H}$ is given by

$$\widehat{\mathcal{H}} : \begin{cases} (\dot{\mathbf{x}}, \dot{\boldsymbol{\eta}}, \dot{\boldsymbol{\tau}}) = (\boldsymbol{\eta}, 0_{nN}, \mathbb{1}_N), & \boldsymbol{\xi} \in \overline{C} \\ (\mathbf{x}^+, \boldsymbol{\eta}^+, \boldsymbol{\tau}^+) \in \widehat{G}_1(\mathbf{x}, \boldsymbol{\eta}, \boldsymbol{\tau}), & \boldsymbol{\xi} \in D \end{cases} \tag{32}$$

and $\widehat{G}_1(\boldsymbol{\xi}) \subseteq \{(\mathbf{x}, \boldsymbol{\eta}_P^+, \boldsymbol{\tau}_P^+)\}_{P \in J(\boldsymbol{\xi})}$, where $\boldsymbol{\tau}_P^+ \in [0, T]^{\mathcal{V}}$ and $\boldsymbol{\eta}_P^+ \in \mathtt{Conf}(\mathcal{V})$ are defined as

$$(\boldsymbol{\tau}_P^+)_q \triangleq \begin{cases} \tau_q, & q \notin P, \\ \tau_p^+, & q \in P, \end{cases} \quad (\boldsymbol{\eta}_P^+)_q \triangleq \begin{cases} \eta_q, & q \notin P \\ \eta_p^+, & q \in P. \end{cases} \tag{33}$$

Note that the flow set C of $\mathcal{H}$ is replaced with its closure $\overline{C}$, the flow map F and the jump set $D = \overline{D}$ remain unchanged, but the jump map is altered and becomes nondeterministic. Note also that $J(\boldsymbol{\xi}) \neq \varnothing$ if and only if $\boldsymbol{\xi} \in D$.

Proof of Lemma 4: For any $\boldsymbol{\xi}$, let $\mathcal{V}(\boldsymbol{\xi})$ denote the set of all $p \in \mathcal{V}$ such that $\mathbf{T}_p(\boldsymbol{\xi}) \le 0$ or $\tau_p(\boldsymbol{\xi}) \ge T$. In particular, the elements of $J(\boldsymbol{\xi})$ are the nonempty subsets of $\mathcal{V}(\boldsymbol{\xi})$, and $\boldsymbol{\xi} \in D$ if and only if $\mathcal{V}(\boldsymbol{\xi}) \neq \varnothing$. Recall from (17) that the jump map G_1 of $\mathcal{H}$ is single valued with $\mathbf{x}^+ = \mathbf{x}$ at all jump points $\boldsymbol{\xi} = (\mathbf{x}, \boldsymbol{\eta}, \boldsymbol{\tau})$. Therefore, the regularized map $\widehat{G}_1$ will satisfy the same property. It remains to compute the $\boldsymbol{\eta}$ and $\boldsymbol{\tau}$ components of $\widehat{G}_1(\boldsymbol{\xi})$. By (5), a point $\boldsymbol{\xi}' = (\mathbf{x}, \boldsymbol{\eta}', \boldsymbol{\tau}')$ lies in $\widehat{G}_1(\boldsymbol{\xi})$ if and only if it is the limit of a convergent sequence of points of the form $G_1(\boldsymbol{\xi}^k) = (\mathbf{x}, \bar{\boldsymbol{\eta}}^k, \bar{\boldsymbol{\tau}}^k)$, where $\boldsymbol{\xi}^k = (\mathbf{x}, \boldsymbol{\eta}^k, \boldsymbol{\tau}^k)$ satisfies $\boldsymbol{\xi}^k \in D$ for all $k \in \mathbb{Z}_{\ge 0}$ and $\boldsymbol{\xi}^k \to \boldsymbol{\xi}$ as $k \to \infty$. Given such a sequence, let $P^k \triangleq \mathcal{V}(\boldsymbol{\xi}^k)$ for $k \in \mathbb{Z}_{\ge 0}$ and let $p \in \mathcal{V}$. If $p \in P^k$ for infinitely many values of k, denote these values by $(k_s)_{s=1}^\infty$, in an increasing order. Recall that $\bar{\boldsymbol{\eta}}_p^{k_s} = g_p(\boldsymbol{\xi}^{k_s})$, where g_p is the continuous function given by the expression for η_p^+ in (12). Since $\boldsymbol{\xi}^{k_s} \to \boldsymbol{\xi}$ and $\bar{\boldsymbol{\eta}}^{k_s} \to \boldsymbol{\eta}'$ as $s \to \infty$, this implies that $\boldsymbol{\eta}_p' = g_p(\boldsymbol{\xi})$. Similarly, since $\bar{\boldsymbol{\tau}}^{k_s} = 0$ for all s, one has $\boldsymbol{\tau}_p' = 0$. Otherwise, if there exists $K \in \mathbb{Z}_{\geq 0}$ such that $p \notin P^k$ for all $k \geq K$, then $\bar{\boldsymbol{\eta}}_p^k = \boldsymbol{\eta}_p^k$ and $\bar{\boldsymbol{\tau}}_p^k = \boldsymbol{\tau}_p^k$ for all such k, implying that $\boldsymbol{\eta}_p' = \boldsymbol{\eta}_p$ and $\boldsymbol{\tau}_p' = \boldsymbol{\tau}_p$. It remains to verify that the set of $p \in \mathcal{V}$ satisfying $p \in P^k$ for infinitely many values of k is contained in $\mathcal{V}(\boldsymbol{\xi})$. Using the previous notation, if $p \in P^{k_s}$, then the points $\boldsymbol{\xi}^{k_s}$ satisfy the closed condition $\mathbf{T}_p(\boldsymbol{\xi}^{k_s}) \leq 0$ or $\tau_p(\boldsymbol{\xi}^{k_s}) \geq T$, for all $s \in \mathbb{Z}_{\geq 0}$. Therefore, so does their limit, $\boldsymbol{\xi}$, resulting in $p \in \mathcal{V}(\xi)$, as desired. ∎

Remark 6: It follows from the computation in the above proof that $\widehat{G}_1(\boldsymbol{\xi}) \supseteq G_2(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in D$ as long as the triggers satisfy the following weak general position requirement for all $\boldsymbol{\xi} \in D$: for every agent $p \in \mathcal{V}(\boldsymbol{\xi})$, there is a sequence $\boldsymbol{\xi}^k \to \boldsymbol{\xi}$ such that $\mathcal{V}(\boldsymbol{\xi}^k) = \{p\}$. In particular, under this very mild condition on the triggers, any solution of $\mathcal{K}$ becomes a Krasovskii solution of $\mathcal{H}$, and this connection does not require any transversality conditions, such as (†).

Most importantly, Lemma 4 makes it possible to apply the arguments in the proof of Lemma 2 to the system $\widehat{\mathcal{H}}$ instead of the system $\mathcal{K}$, resulting in the following corollary.

Corollary 4: Every initial condition $\phi(0,0) \in \overline{C}$ extends to a complete solution ϕ of the hybrid system $\widehat{\mathcal{H}}$ given in (32). Moreover, every maximal solution of $\widehat{\mathcal{H}}$ is either t-complete or Zeno. In addition, if (†) is satisfied, then the $\mathbf{x}$ component of any such solution, regarded as a function of t, coincides with that of the unique solutions for $\mathcal{H}$ and $\mathcal{K}$ with the same initial condition.

V. Graph Maintenance and t-Completeness

The edge weights w_{pq} in (7) give rise to the weighted graph Laplacian matrix of the MAS, $\mathbf{L}_w$, defined in Section II-B, using the weights w_{pq} introduced in (7). Set

$$\boldsymbol{\zeta} \triangleq -(\mathbf{L}_w \otimes \mathbf{I}_n)\,\mathbf{x} - \boldsymbol{\eta}. \tag{34}$$

Writing $\boldsymbol{\zeta} = (\zeta_p)_{p\in\mathcal{V}} \in \mathtt{Conf}(\mathcal{V})$, observe that

$$\zeta_p = \sum_{q\in\mathcal{N}_p} w_{pq}\,(x_q - x_p) - \eta_p \tag{35}$$

is the error between the instantaneous consensus term over $\mathcal{N}_p$ and the sampled consensus term for agent p.

Definition 7: Let $\mathbf{T}_p : \mathcal{X} \to \mathbb{R}$, $p \in \mathcal{V}$ be a continuously differentiable function, let $\sigma \in (0,1]$, and let

$$f_p(\boldsymbol{\xi}) \triangleq \|\eta_p\|^2 - \|\zeta_p\|^2 + \sigma K \widetilde{R}^2\,,\;\; K \triangleq \frac{\lambda_2^2(\mathcal{G})\mu^2}{2N}. \tag{36}$$

$\mathbf{T}_p$ is an *admissible trigger*, if $\mathbf{T}_p \leq f_p$ throughout $\mathcal{X}$ and there exist $\mathfrak{h}, \mathfrak{m} > 0$ such that (a) $\mathbf{T}_p(\boldsymbol{\xi}^+) \geq \mathfrak{h}$ at each jump of $\mathcal{H}$, and (b) $\langle \nabla \mathbf{T}_p(\boldsymbol{\xi}), F(\boldsymbol{\xi})\rangle \geq -\mathfrak{m}$ holds for all $\boldsymbol{\xi} \in \mathcal{X}$[12].

Note that admissibility implies that agent p must update η_p and τ_p at some time earlier than dictated by the condition $f_p \leq 0$, which is only state dependent (and not time dependent). Also, the parameter σ, to be determined later, allows the user to control the degree of rendezvous approximation (see Theorem 2).

Theorem 1: Given $\widetilde{R} = R(1-\varepsilon)$ satisfying (8) and a connected graph $\mathcal{G}$, if $\{\mathbf{T}_p\}_{p\in\mathcal{V}}$ is a collection of admissible triggers, then every solution of $\mathcal{H}, \mathcal{K}, \mathcal{L}$, or $\widehat{\mathcal{H}}$ satisfying (24) and initiating from $\mathcal{C}_{\widetilde{R}}(\mathcal{G})$ remains in $\mathcal{C}_R(\mathcal{G})$ and is t-complete. Moreover, the controllers u_p are bounded, with $\|u_p\| \leq \Delta(\mathcal{G})Rr(R)$ for all time.

Proof: Using (7), define the *total potential* $V_{\mathcal{G}} : \mathcal{X} \to \mathbb{R}_{\geq 0}$ as

$$V_{\mathcal{G}}(\boldsymbol{\xi}) \triangleq \sum_{p\in\mathcal{V}}\sum_{q\in\mathcal{N}_p} V_{pq}(\mathbf{x}) = 2\sum_{pq\in\mathcal{E}} V_{pq}(\mathbf{x}). \tag{37}$$

Note that $V_{\mathcal{G}}(\boldsymbol{\xi}) = 0$ for $\boldsymbol{\xi} \in \mathcal{R}$ and $V_{\mathcal{G}}(\boldsymbol{\xi}) > 0$ otherwise. Let $\phi : \operatorname{dom}\phi \to \mathcal{X}$ be a solution of $\mathcal{H}, \mathcal{K}, \mathcal{L}$, or $\widehat{\mathcal{H}}$ with initial condition $\phi(0,0) \in \mathcal{C}_{\widetilde{R}}(\mathcal{G})$. We will abuse notation by writing $\boldsymbol{\xi} = \phi(t,j)$ for $(t,j) \in \operatorname{dom}\phi$.

First, we show that $\dot{V}_{\mathcal{G}}(\boldsymbol{\xi}) \leq 0$ whenever $\|x_p - x_q\| \geq \widetilde{R}$ for some $pq \in \mathcal{E}$. During flows, the change in $V_{\mathcal{G}}$ is given by $\dot{V}_{\mathcal{G}}(\boldsymbol{\xi}) = \langle \nabla V_{\mathcal{G}}(\boldsymbol{\xi}), F(\boldsymbol{\xi})\rangle$, where F is the flow map of $\mathcal{H}$. It is known [9, Sec. 7.2] that $\nabla V_{\mathcal{G}}(\boldsymbol{\xi}) = 2(\mathbf{L}_w \otimes I_n)\mathbf{x}$ follows from (7). Therefore, upon substituting (34) we obtain

$$\begin{aligned}\dot{V}_{\mathcal{G}}(\boldsymbol{\xi}) &= 2\mathbf{x}^\top(\mathbf{L}_w \otimes I_n)^\top \boldsymbol{\eta}\\ &= -2\mathbf{x}^\top\left(\mathbf{L}_w^2 \otimes I_n\right)\mathbf{x} - 2\mathbf{x}^\top\left(\mathbf{L}_w \otimes I_n\right)\boldsymbol{\zeta}.\end{aligned}$$

Splitting the leading term and applying (34) twice yields

$$\begin{aligned}\dot{V}_{\mathcal{G}}(\boldsymbol{\xi}) =&\; -\mathbf{x}^\top\left(\mathbf{L}_w^2 \otimes I_n\right)\mathbf{x} - \|\boldsymbol{\zeta}\|^2 - \|\boldsymbol{\eta}\|^2\\ &-2\boldsymbol{\eta}^\top\boldsymbol{\zeta} - 2\mathbf{x}^\top\left(\mathbf{L}_w \otimes I_n\right)\boldsymbol{\zeta}\\ =&\; -\mathbf{x}^\top\left(\mathbf{L}_w^2 \otimes I_n\right)\mathbf{x} - \|\boldsymbol{\zeta}\|^2 - \|\boldsymbol{\eta}\|^2\\ &-2(\boldsymbol{\eta} + (\mathbf{L}_w \otimes I_n)\,\mathbf{x})^\top\boldsymbol{\zeta}\\ =&\; -\mathbf{x}^\top\left(\mathbf{L}_w^2 \otimes I_n\right)\mathbf{x} - \|\boldsymbol{\eta}\|^2 + \|\boldsymbol{\zeta}\|^2.\end{aligned}$$

Given (7) and (9), $w_{pq} \geq \mu$ for all $pq \in \mathcal{E}$, which implies that $\lambda_2(\mathbf{L}_w) \geq \mu\lambda_2(\mathcal{G})$. Together with (21), this yields

$$\begin{aligned}\dot{V}_{\mathcal{G}}(\boldsymbol{\xi}) &\leq -\lambda_2^2(\mathcal{G})\,\mu^2\|\mathbf{x}^\perp\|^2 - \|\boldsymbol{\eta}\|^2 + \|\boldsymbol{\zeta}\|^2\\ &\leq -\left(NK\|\Delta\mathbf{x}\|_\infty^2 + \|\boldsymbol{\eta}\|^2 - \|\boldsymbol{\zeta}\|^2\right).\end{aligned} \tag{38}$$

Using $\|\Delta\mathbf{x}\|_\infty \geq \widetilde{R}$, $\sigma K\widetilde{R}^2 \leq K\widetilde{R}^2$, and (36), we obtain

$$\begin{aligned}\dot{V}_{\mathcal{G}}(\boldsymbol{\xi}) &\leq -\sum_{p\in\mathcal{V}}(\sigma K\widetilde{R}^2 + \|\eta_p\|^2 - \|\zeta_p\|^2)\\ &\leq -\sum_{p\in\mathcal{V}} f_p(\boldsymbol{\xi}) \leq -\sum_{p\in\mathcal{V}} \mathbf{T}_p(\boldsymbol{\xi}) \leq 0\end{aligned} \tag{39}$$

by the admissibility of the $\mathbf{T}_p$ and since all the $\mathbf{T}_p$ are nonnegative during flows.

We now show that the edges of $\mathcal{G}$ are maintained. Proceeding by contradiction, let $(s_1, j_1) \in \operatorname{dom}\phi$ be a point with $\phi(s_1,j_1) \notin \mathcal{C}_R(\mathcal{G})$, and let

$$(s_0, j_0) \triangleq \sup\left\{(t,j) \in \operatorname{dom}\phi \;\middle|\; \begin{array}{c}(t,j) \leq (s_1,j_1)\\ \phi(t,j) \in \mathcal{C}_{\widetilde{R}}(\mathcal{G})\end{array}\right\}$$

using the order on $\mathbb{R}^2$ given by $\mathbf{a} \leq \mathbf{b} \Leftrightarrow \mathbf{b} - \mathbf{a} \in \mathbb{R}^2_{\geq 0}$. Since $\operatorname{dom}\phi$ is closed in $\mathbb{R}^2$ under this order, $(s_0,j_0) \in \operatorname{dom}\phi$. Let $V_1 \triangleq V_{\mathcal{G}}(\phi(s_1,j_1))$ and $V_0 \triangleq V_{\mathcal{G}}(\phi(s_0,j_0))$. Also, note $j_0 \leq j_1$ and $s_0 \in [t_{j_0}, t_{j_0+1})$. Since $\mathbf{x}$ evolves continuously over time,

[12]Condition (b) is equivalent to $\dot{\mathbf{T}}_p \geq -\mathfrak{m}$ along solutions of $\mathcal{H}, \mathcal{K}, \mathcal{L}$, and $\widehat{\mathcal{H}}$ during flows.

the expression $\mathbf{z}(t) \triangleq \mathbf{x}(\phi(t,j))$ is a well-defined function of $t \in J \triangleq [s_0, s_1]$. Moreover, it is C^1-smooth at every $t \in J$ except for the points $t = t_j, j_0 \le j \le j_1$. Since $V_\mathcal{G}$ is a C^1-function of $\mathbf{x}$, the function $v(t) \triangleq V_\mathcal{G}(\mathbf{z}(t))$ is C^1-smooth at every $t \in J$ except for the points $t = t_j$ and $j_0 \le j \le j_1$. Therefore, $V_1 - V_0 = v(s_1) - v(s_0) = \int_{s_0}^{s_1} \dot{v}(t)\mathrm{d}t$. Since $\phi(t,j) \notin \mathcal{C}_{\widetilde{R}}(\mathcal{G})$ for $t \in (s_0, s_1]$, the integrand is nonpositive by (39), which results in $V_1 \le V_0$. Since $\mathbf{x}$ evolves continuously with t, and $\mathcal{C}_{\widetilde{R}}(\mathcal{G})$ is closed, we have $\phi(s_0, j_0) \in \mathcal{C}_{\widetilde{R}}(\mathcal{G})$. Therefore, by (10) and (37), $V_0 = 2\sum_{pq\in\mathcal{E}} V_{pq}(\phi(s_0, j_0)) \le 2|\mathcal{E}|P(\widetilde{R}) \le 2P(R)$. Since $\mathbf{x}(s_1)$ has at least one edge of length greater than R, $V_0 \le 2P(R) < V_1$—contradiction.

To prove the second assertion of the theorem, we suppose that the solution ϕ is maximal and verify that ϕ is t-complete. By Lemmas 1–3 and Corollary 4, a maximal solution $\phi : \mathrm{dom}\,\phi \to \mathcal{X}$ of $\mathcal{H}, \mathcal{K}, \mathcal{L}$, and $\widehat{\mathcal{H}}$, respectively, is either t-complete (and we are done)—or it has an infinite sequence of jump times t_j, $j \in \mathbb{Z}_{\ge 0}$. For each $p \in \mathcal{V}$, let J_p denote the set of indices $j > 0$ such that agent p experiences a jump at time t_j along ϕ, with the addition of $j = 0$. In particular, $\mathbf{T}_p(\phi(t_j, j-1)) = 0$ or $\tau_p(\phi(t_j, j-1)) = T$ (but note that this is not a sufficient condition for a jump when ϕ is a solution of $\mathcal{K}$). Since $\mathcal{V}$ is finite, we may select $p \in \mathcal{V}$ such that J_p is infinite, and denote the *hybrid jump times* $(t_j, j-1)$, $j \in J_p$ as $((t_i^p, j_i^p - 1))_{i=1}^\infty$, in an increasing order. Also, let $t_0^p \triangleq 0$. It suffices to find $\delta > 0$ such that $t_{i+1}^p - t_i^p \ge \delta$ for all $i \ge 0$. We claim that $\delta = \min\{\mathfrak{h}/\mathfrak{m}, T\}$ satisfies our needs. For each $i \ge 1$, $\mathbf{T}_p(\phi(t_i^p, j_i^p - 1)) = 0$ or $\tau_p(\phi(t_i^p, j_i^p - 1)) = T$. Also, $\mathbf{T}_p(\phi(t_i^p, j_i^p)) \ge \mathfrak{h}$ by Definition 7(a), and $\tau_p(\phi(t_i^p, j_i^p)) = 0$ by (12). Moreover, the continuity of $\mathbf{T}_p(\phi(t,j))$ during flows implies that $t_{i+1}^p - t_i^p > 0$ (in other words, the time interval between t_i^p and t_{i+1}^p is nondegenerate). Proceeding by contradiction, assume that $t_{i+1}^p - t_i^p < \delta$ and denote $\boldsymbol{\xi} = \phi(t_{i+1}^p, j_i^p)$ for the rest of the argument. Then, it must be the case that $\mathbf{T}_p(\boldsymbol{\xi}) = 0$, because $t_{i+1}^p - t_i^p < T$. Substituting $\boldsymbol{\xi}$ into $\mathbf{T}_p$, together with Definition 7(b) and the continuity of $\dot{\mathbf{T}}_p(\phi(s, j_i^p))$ over all but finitely many $s \in [t_i^p, t_{i+1}^p]$, produces

$$\mathbf{T}_p(\boldsymbol{\xi}) = \mathbf{T}_p(\phi(t_i^p, j_i^p)) + \int_{t_i^p}^{t_{i+1}^p} \dot{\mathbf{T}}_p(\phi(s, j_i^p))\mathrm{d}s$$
$$\ge \mathfrak{h} - \mathfrak{m}\left(t_{i+1}^p - t_i^p\right) > 0$$

contradicting the fact that $\mathbf{T}_p(\boldsymbol{\xi}) = 0$.

Finally, since $\|\Delta\mathbf{x}\|_\infty \le R$ for all time, we see $w_{pq} \le r(R)$ by (7), and $\|u_p\| = \|\eta_p\| \le \Delta(\mathcal{G})Rr(R)$ by (9), for each $pq \in \mathcal{E}$ and $p \in \mathcal{V}$—hence bounded. ■

Remark 7: The result of Theorem 1 for $\mathcal{L}$, as well as for $\mathcal{K}$ in the presence of (†), directly follows from its validity for $\mathcal{H}$, using Corollary 3. To see this, observe that if ϕ is an $\mathcal{H}$-solution satisfying the conclusions of the theorem, then any hybrid arc ψ with $\phi = \mathfrak{C}\psi$ also satisfies the same conclusion. Rather than follow this argument in the proof of Theorem 1, a unified argument was provided that applies also to the $\mathcal{K}$-solutions, which do not become $\mathcal{H}$-solutions (via the crushing map $\mathfrak{C}$) in the absence of (†).

VI. Stability Analysis

With sufficient conditions on the triggers $\{\mathbf{T}_p\}_{p\in\mathcal{V}}$ guaranteeing graph maintenance now established, additional conditions guaranteeing the practical stability of the rendezvous set are derived below, using the total potential $V_\mathcal{G}$ as a Lyapunov function candidate (in the broader HDI sense [19, Sec. 3.2]). Of note is the fact that $V_\mathcal{G}$ is a continuous function of $\mathbf{x}$ alone, and therefore evolves continuously as a function of t along hybrid trajectories, no matter which design—$\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$, or $\widehat{\mathcal{H}}$—is considered. This fact enables a stability analysis that is agnostic to the differences between these designs, focusing on estimating the rate of change in $V_\mathcal{G}$ along flow intervals.

Theorem 2: Let $\nu > 0$, $\widetilde{R} = R(1-\varepsilon)$ satisfy (8), and $\mathcal{G}$ be a connected graph. Let

$$\mathcal{A}_\nu \triangleq \mathcal{R}_\nu \cap \{\boldsymbol{\xi} \in \mathcal{X} : \forall_{p\in\mathcal{V}} \|\eta_p\| \le \Delta(\mathcal{G})Rr(R)\}. \tag{40}$$

Suppose that $0 < \beta \le \sigma K \widetilde{R}^2$, and $\{\mathbf{T}_p\}_{p\in\mathcal{V}}$ is a collection of admissible triggers such that, over solutions of $\mathcal{H}, \mathcal{K}, \mathcal{L}$, or $\widehat{\mathcal{H}}$ satisfying (24) and initiating from $\mathcal{C}_{\widetilde{R}}(\mathcal{G})$,

$$\mathbf{T}_p + \sigma K \widetilde{R}^2 \le f_p + \beta \tag{41}$$

holds for all $p \in \mathcal{V}$. Then, any such solution satisfies

$$\|\Delta\mathbf{x}(t)\|_\infty^2 \le \frac{r(R)\,|\mathcal{E}|}{r(0)}\left(\|\Delta\mathbf{x}(0)\|_\infty^2\, \mathrm{e}^{-\frac{NKt}{r(R)|\mathcal{E}|}} + \sigma\widetilde{R}^2\right). \tag{42}$$

In particular, the set $\mathcal{A}_0$ is practically exponentially stable for systems $\mathcal{H}, \mathcal{K}, \mathcal{L}$, and $\widehat{\mathcal{H}}$.

Before tackling the proof, we consider a few observations. Recalling (21), the expressions $\|\Delta\mathbf{x}(t)\|$ and $\|\Delta\mathbf{x}(0)\|$ may be replaced with the corresponding Euclidean distances $|\boldsymbol{\xi}(t)|_{\mathcal{A}_0}$ and $|\boldsymbol{\xi}(0)|_{\mathcal{A}_0}$, as in Remark 3, yielding a stability bound in more standard form. However, the definition of ν-approximate rendezvous makes (42) a more natural statement in the context of the current problem. Due to the continuous evolution of $\mathbf{x}$ under the dynamics, and noting the fact that the length of a sequence of consecutive 0-time jumps is bounded above by N, (42) may be regarded as an exponential stability result for $\mathcal{A}_\nu$ for arbitrarily small ν—equivalently, practical exponential stability for $\mathcal{A}_0$.

Corollary 5 (Approximate rendezvous is achieved): Under the assumptions of Theorem 2, ν'-approximate rendezvous is achieved in finite time for every maximal solution, for any $\nu' > \nu$, upon selecting $\sigma \triangleq \frac{\nu^2}{|\mathcal{E}|\widetilde{R}^2} \cdot \frac{r(0)}{r(R)}$. ■

Proof of Theorem 2: Let $\phi : \mathrm{dom}\,\phi \to \mathcal{X}$ be a maximal solution of $\mathcal{H}, \mathcal{K}, \mathcal{L}$, or $\widehat{\mathcal{H}}$ with initial condition $\phi(0,0) \in \mathcal{C}_{\widetilde{R}}(\mathcal{G})$. By Theorem 1, ϕ is t-complete. Consider the same total potential $V_\mathcal{G}(\boldsymbol{\xi})$ defined in (37). Using (6), (7), (9), and (37), we can bound $V_\mathcal{G}$ as

$$r(0)\|\Delta\mathbf{x}\|^2 \le V_\mathcal{G}(\boldsymbol{\xi}) \le r(R)\|\Delta\mathbf{x}\|^2. \tag{43}$$

By (41), $\|\eta_p\|^2 - \|\zeta_p\|^2 \ge -\beta + \mathbf{T}_p$. Then, (38) yields

$$\begin{aligned}\dot{V}_\mathcal{G}(\boldsymbol{\xi}) &\le -NK\|\Delta\mathbf{x}\|_\infty^2 - \sum_{p\in\mathcal{V}}\left(\|\eta_p\|^2 - \|\zeta_p\|^2\right)\\ &\le -NK\|\Delta\mathbf{x}\|_\infty^2 - \sum_{p\in\mathcal{V}}\left(-\beta + \mathbf{T}_p\right)\\ &\le -NK\|\Delta\mathbf{x}\|_\infty^2 + N\beta.\end{aligned} \tag{44}$$

Combining (43) with $\|\Delta\mathbf{x}\| \leq \sqrt{|\mathcal{E}|}\|\Delta\mathbf{x}\|_\infty$ implies $V_\mathcal{G}(\boldsymbol{\xi}) \leq r(R)|\mathcal{E}|\|\Delta\mathbf{x}\|_\infty^2$. Then, (44) can be upper bounded as

$$\dot{V}_\mathcal{G}(\boldsymbol{\xi}) \leq -\tfrac{NK}{r(R)|\mathcal{E}|}V_\mathcal{G}(\boldsymbol{\xi}) + N\beta. \tag{45}$$

Since $V_\mathcal{G}$ only depends on $\mathbf{x}$, and $\mathbf{x}$ evolves continuously under $\mathcal{H}, \mathcal{K}, \mathcal{L}$, and $\widehat{\mathcal{H}}$, $V_\mathcal{G}$ evolves continuously as well. By Theorem 1, $\dot{V}_\mathcal{G}(\boldsymbol{\xi}), \boldsymbol{\xi} = \phi(t,j)$ has only finitely many discontinuities in any bounded subinterval of dom ϕ. Therefore, integrating (45) and recalling that $\beta \leq \sigma K \widetilde{R}^2$ yields

$$\begin{aligned} V_\mathcal{G}(\phi(t,j)) &\leq \exp\left(-\frac{NKt}{r(R)\,|\mathcal{E}|}\right)V_\mathcal{G}(\phi(0,0)) \\ &\quad + \frac{r(R)\,|\mathcal{E}|\,\beta}{K}\left(1-\exp\left(-\frac{NKt}{r(R)\,|\mathcal{E}|}\right)\right) \\ &\leq V_\mathcal{G}(\phi(0,0))\mathrm{e}^{-\frac{NKt}{r(R)|\mathcal{E}|}} + \sigma r(R)\,|\mathcal{E}|\,\widetilde{R}^2, \end{aligned}$$

where using the bounds of $V_\mathcal{G}$ in (43) and $\|\Delta\mathbf{x}\|_\infty \leq \|\Delta\mathbf{x}\| \leq \sqrt{|\mathcal{E}|}\|\Delta\mathbf{x}\|_\infty$ produces (42) for any $(t,j) \in$ dom ϕ. ■

VII. Analysis of Robustness Properties

The book [19] develops the theory of compact attractors for HDIs satisfying HBC. Focusing on $\mathcal{K}$ and $\widehat{\mathcal{H}}$, note that Theorem 2 presents two examples of HDIs satisfying HBC, however, with a noncompact attractor $\mathcal{A}_\nu$. To see this, observe that in addition to graph maintenance, Theorem 1 shows the set $\mathcal{X}_\# \subset \mathcal{X}$ defined as

$$\mathcal{X}_\# \triangleq \{\boldsymbol{\xi} \in \mathcal{X} : \forall_{p\in\mathcal{V}}\ \|\eta_p\| \leq \Delta(\mathcal{G})Rr(R)\} \tag{46}$$

is positively invariant.[13] Further, by Theorem 2, the set $\mathcal{A}_\nu \subset \mathcal{X}_\#$ is closed and uniformly attractive from the set of initial conditions in int$(\mathcal{C}_R(\mathcal{G})) \cap \mathcal{X}_\#$ satisfying (24) and Corollary 5. Comparing with [19, Def. 6.24], only the necessary conditions in the context of a compact attractor are missing from among the conclusions of Theorem 2.

Had $\mathcal{A}_\nu$ been compact, results from [19, Ch. 7] would imply its robustness under continuous perturbations of $\mathcal{K}$ and $\widehat{\mathcal{H}}$. In the noncompact case, analogous results are expected for perturbations that are uniformly continuous on the attractor. The specific cases of $\mathcal{K}$ and $\widehat{\mathcal{H}}$ allow an argument (provided in this section) not requiring the development of such general results. For perturbations invariant under translations of the MAS configuration, the robustness of the attractor to such perturbations (of $\mathcal{K}$ or $\widehat{\mathcal{H}}$) may be derived from the robustness of corresponding quotient HDIs, $\mathcal{K}_\#$ and $\widehat{\mathcal{H}}_\#$, each possessing a compact attractor as detailed below.

Briefly, we recall notions from [19, Ch. 6] required for the analyses presented in this section.

Definition 8: [19, Def. 6.27] Given a hybrid system $\mathcal{H}$ and function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, the ρ-perturbation of $\mathcal{H}$, denoted by $\mathcal{H}_\rho$, is the hybrid system

$$\mathcal{H}_\rho : \begin{cases} \dot{z} \in F_\rho(z), & z \in C_\rho \\ z^+ \in G_\rho(z), & z \in D_\rho, \end{cases} \tag{47}$$

where the flow and jump sets are

$$C_\rho \triangleq \{z \in \mathbb{R}^n : (z + \rho(z)\mathbb{B}) \cap C \neq \varnothing\},$$

$$D_\rho \triangleq \{z \in \mathbb{R}^n : (z + \rho(z)\mathbb{B}) \cap D \neq \varnothing\} \tag{48}$$

and, for $z \in C_\rho$, the flow map is defined as

$$F_\rho(z) \triangleq \rho(z)\mathbb{B} + \overline{\mathrm{con}}F((z+\rho(z)\mathbb{B}) \cap C) \tag{49}$$

whereas, for $z \in D_\rho$, the jump map is defined as

$$G_\rho(z) \triangleq \bigcup_{g\in G(D\cap(z+\rho(z)\mathbb{B}))} (g + \rho(g)\mathbb{B})\,. \tag{50}$$

For greater specificity, we adapt [19, Def. 7.18(a)] to the present setting: since all the HDIs being discussed are t-complete and the attractors $\mathcal{A}$ under consideration are noncompact continuous-time semi-globally[14] practically asymptotically stable (ctSGPAS) sets (see [24, Def. 2]), preattractivity may be replaced with attractivity, and the compactness of the attractor may be replaced by closedness.

Definition 9: Let $\mathcal{A} \subset \mathbb{R}^n$ be a closed set and $\mathcal{U} \subset \mathbb{R}^n$ be an open set containing $\mathcal{A}$. Then, $\mathcal{A}$ is robustly ctSGPAS on $\mathcal{U}$ for $\mathcal{H}$, if there exists a continuous function $\rho : \mathcal{U} \to \mathbb{R}_{\geq 0}$ that is positive on $\mathcal{U} \setminus \mathcal{A}$ such that $\mathcal{A}$ is ctSGPAS on $\mathcal{U}$ for $\mathcal{H}_\rho$.

A. Robustness of $\mathcal{K}$ and $\widehat{\mathcal{H}}$

Consider the linear projection operator $\mathbf{S}$ of $\mathtt{Conf}(\mathcal{V})$ defined by $\mathbf{S}\mathbf{x} \triangleq \mathbf{x}^\perp$. $\mathbf{S}$ maps any configuration $\mathbf{x}$ into $\Delta_\mathcal{V}^\perp$, the orthogonal complement of the agreement subspace. Consider the evolution of $\mathbf{x}^\perp$ over time. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}^\perp) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{S}\mathbf{x}) = \dot{\mathbf{S}}\mathbf{x} + \mathbf{S}\dot{\mathbf{x}} = \mathbf{S}\dot{\mathbf{x}} = \mathbf{S}\boldsymbol{\eta} \in \Delta_\mathcal{V}^\perp,$$

the flow dynamics of $\mathbf{x}^\perp$ evolves parallel to $\Delta_\mathcal{V}^\perp$. Let $C_\#$ and $D_\#$ be obtained from C and D, respectively, by intersection with the set $\Delta_\mathcal{V}^\perp \times \mathtt{Conf}(\mathcal{V}) \times [0,T]^\mathcal{V}$. Then, the variable $\boldsymbol{\xi}_\# \triangleq [\mathbf{x}^\top, \boldsymbol{\eta}^\top, \boldsymbol{\tau}^\top]^\top \in \Delta_\mathcal{V}^\perp \times \mathtt{Conf}(\mathcal{V}) \times [0,T]^\mathcal{V}$ evolves according to the flow dynamics

$$(\dot{\mathbf{x}}, \dot{\boldsymbol{\eta}}, \dot{\boldsymbol{\tau}}) = (\mathbf{S}\boldsymbol{\eta}, 0_{nN}, \mathbb{1}_N) \text{ for } \boldsymbol{\xi}_\# \in \overline{C}_\#. \tag{51}$$

The discrete dynamics of $\boldsymbol{\xi}_\#$ under the systems $\mathcal{K}$ and $\widehat{\mathcal{H}}$ evolve subject to

$$(\mathbf{x}^+, \boldsymbol{\eta}^+, \boldsymbol{\tau}^+) \in G_2(\mathbf{x}, \boldsymbol{\eta}, \boldsymbol{\tau}), \tag{52}$$

$$(\mathbf{x}^+, \boldsymbol{\eta}^+, \boldsymbol{\tau}^+) \in \widehat{G}_1(\mathbf{x}, \boldsymbol{\eta}, \boldsymbol{\tau}), \tag{53}$$

respectively, for $\boldsymbol{\xi}_\# \in D_\#$. Noting $\eta_p^+ = -(\mathsf{e}_p\mathbf{L}_w \otimes I_n)\mathbf{x}^\perp$ and that w_{pq} is determined by $\|x_p - x_q\| = \|\mathbf{x}_p^\perp - \mathbf{x}_q^\perp\|$, the jump equations for η_p are well defined in terms of the variables composing $\boldsymbol{\xi}_\#$.

To ensure that the jump conditions also only depend on $\boldsymbol{\xi}_\#$, one must impose the additional requirement that the triggers $\{\mathbf{T}_p\}_{p\in\mathcal{V}}$ are invariant under translations of the MAS configuration. In other words, the condition

$$(\ddagger)\ \mathbf{T}_p(\mathbf{x}, \boldsymbol{\eta}, \boldsymbol{\tau}) = \mathbf{T}_p(\mathbf{x} + \boldsymbol{\varrho}, \boldsymbol{\eta}, \boldsymbol{\tau}) \text{ for all } \boldsymbol{\varrho} \in \Delta_\mathcal{V}$$

must be satisfied over $\mathcal{X}$. The condition ($\ddagger$) is not overly restrictive since the triggers need only depend on interagent measurements, and not on the MAS position in the workspace. Note that the triggers introduced in Section VIII do satisfy ($\ddagger$).

[13] In fact, positively invariant under $\mathcal{H}, \mathcal{K}, \mathcal{L}$, and $\widehat{\mathcal{H}}$.

[14] The result is semi-global because the initial configuration of the MAS dictates the size of $\widetilde{R}$, and the user-defined performance parameter ν determines other controller constants.

To summarize, (51) and (52) give rise to a quotient $\mathcal{K}_\#$ of $\mathcal{K}$, whereas (51) and (53) give rise to a quotient $\widehat{\mathcal{H}}_\#$ of $\widehat{\mathcal{H}}$. It then follows from Theorem 2 that the compact set:

$$\mathcal{A}_\# \triangleq \{0_{nN}\} \times \{\boldsymbol{\eta} : \|\boldsymbol{\eta}\|_\infty \leq \Delta(\mathcal{G})Rr(R)\} \times [0,T]^{\mathcal{V}}$$

is ctSGPAS for $\mathcal{K}_\#$ and $\widehat{\mathcal{H}}_\#$ from the set of initial conditions in $\text{int}(\mathcal{C}_R(\mathcal{G})) \cap \mathcal{X}_\#$ satisfying (24). The result [19, Th. 7.21] guarantees the robustness of $\mathcal{A}_\#$ under continuous perturbations of $\mathcal{K}_\#$ as well as $\widehat{\mathcal{H}}_\#$, which, in turn, guarantees the robustness of $\mathcal{A}$ under continuous perturbations of $\mathcal{K}$ and $\widehat{\mathcal{H}}$ that are invariant under translations of the MAS[15].

B. Robustness of $\mathcal{L}$ and $\mathcal{H}$

Recalling that $\mathcal{K} = \widehat{\mathcal{L}}$, and that the robustness—according to Definition 9—of the attractor $\mathcal{A}$ to translation-invariant perturbations for $\mathcal{K}$ and $\widehat{\mathcal{H}}$ is already established, the robustness of $\mathcal{A}$ to the same perturbations for $\mathcal{L}$ and $\mathcal{H}$, respectively, follows as a consequence of the next result.

Lemma 5: Let $\mathcal{H}$ be any HDI on $\mathbb{R}^n$, and let ρ be a continuous perturbation. If $\mathcal{A} \subset \mathbb{R}^n$ is a ctSGPAS attractor for $\widehat{\mathcal{H}}_\rho$, then $\mathcal{A}$ is robustly ctSGPAS for $\mathcal{H}$.

Proof: Any solution ϕ of $\mathcal{H}_\rho$ is a solution of $\widehat{\mathcal{H}}_\rho$. Therefore, any upper bound on $|\phi(t,j)|_\mathcal{A}$ that is valid for solutions of $\widehat{\mathcal{H}}_\rho$ remains valid for solutions of $\mathcal{H}_\rho$. ∎

Applying Lemma 5 to the attractor $\mathcal{A}_0$ shows that $\mathcal{A}_0$ is robustly ctSGPAS for the HDIs $\mathcal{L}$ and $\mathcal{H}$, because it is robustly ctSGPAS for their respective Krasovskii regularizations, $\mathcal{K}$ and $\widehat{\mathcal{H}}$. In summary, the fact that the quotient system $\widehat{\mathcal{H}}_\#$ has the compact attractor $\mathcal{A}_\#$ and satisfies HBC produces, through the established theory, a translation-invariant perturbation ρ, demonstrating the robustness of $\mathcal{A}_0$ for $\widehat{\mathcal{H}}$, while Lemma 5 shows the same perturbation certifies the robustness of $\mathcal{A}_0$ for $\mathcal{H}$.

VIII. Trigger Design

It remains to produce examples of admissible triggers satisfying the condition (41) of Theorem 2. Recalling that the purpose of introducing triggers was to limit the deviations $\{\|\zeta_p\|\}_{p\in\mathcal{V}}$ of the sampled local consensus errors from the actual local consensus errors, the following result establishes such a bound for admissible triggers. This enables the design of the desired examples provided in (56) and (57).

Lemma 6: Let $\widetilde{R}$, r, and ω satisfy the requirements in Section III-A, and let $\mathcal{G}$ be a connected graph. If $\{\mathbf{T}_p\}_{p\in\mathcal{V}}$ is a collection of admissible triggers and ϕ is a solution of $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$, or $\widehat{\mathcal{H}}$ with $\phi(0,0) \in \mathcal{C}_{\widetilde{R}}(\mathcal{G})$ satisfying (24), then the following holds over every flow interval of agent p:

$$\|\zeta_p(\phi(t,j))\| \leq Z\tau_p(\phi(t,j)),$$

where $Z \triangleq 2\Delta(\mathcal{G})^2 Rr(R)(2\mu\omega R^2 + r(R)) > 0$.

Proof: Using the notation from the proof of Theorem 1, consider any $t \in [t_i^p, t_{i+1}^p]$. By (11), η_p is constant over the interval $[t_i^p, t_{i+1}^p] \times \{j_i^p\} \subset \text{dom}\,\phi$. Furthermore, Theorem 1 and (7)–(9) imply that $\|x_p - x_q\| \leq R$ and $|w_{pq}| \leq r(R)$ for all $q \in \mathcal{N}_p$, respectively. It then follows that[16]

$$\|\eta_p\| \leq d_p Rr(R) \leq \Delta(\mathcal{G})Rr(R). \tag{54}$$

Taking the time derivative of ζ_p over $[t_i^p, t_{i+1}^p] \times \{j_i^p\}$ yields

$$\dot{\zeta}_p = \sum_{q\in\mathcal{N}_p} \dot{w}_{pq}(x_q - x_p) + \sum_{q\in\mathcal{N}_p} w_{pq}(\eta_q - \eta_p). \tag{55}$$

From (9), it follows that $\dot{w}_{pq} = 0$ when $\|x_q - x_p\| \in [0, \widetilde{R}]$, and $\dot{w}_{pq} = 2\mu\omega(x_q - x_p)^\top(\eta_q - \eta_p)$ for $\|x_q - x_p\| \in [\widetilde{R}, R]$. Therefore, $|\dot{w}_{pq}| \leq 4\mu\omega\Delta(\mathcal{G})R^2 r(R)$ from (54). Combining these bounds with (55) implies

$$\begin{aligned}\|\dot{\zeta}_p\| &\leq d_p R \cdot 4\mu\omega\Delta(\mathcal{G})R^2 r(R) \\ &\quad + d_p r(R) \cdot 2\Delta(\mathcal{G})Rr(R) \leq Z,\end{aligned}$$

where we recall that d_p is the degree of agent p in $\mathcal{G}$, and $\Delta(\mathcal{G})$ is the maximal degree in $\mathcal{G}$. Over the interval $[t_i^p, t_{i+1}^p] \times \{j_i^p\}$, ζ_p is continuous as a function of t, and differentiable everywhere except $t = t_j$, $j_i^p \leq j \leq j_{i+1}^p$. Therefore, using $\|\zeta_p(\phi(t_i^p, j_i^p))\| = 0$ we may write

$$\|\zeta_p(t)\| \leq \int_{t_i^p}^t \|\dot{\zeta}_p(s)\|\mathrm{d}s \leq Z\left(t - t_i^p\right).$$

Since $\tau_p = t - t_i^p$ over $[t_i^p, t_{i+1}^p] \times \{j_i^p\}$, $\|\zeta_p(t, j_i^p)\| \leq Z\tau_p$. ∎

An immediate consequence of Lemma 6 is the following.

Corollary 6: The trigger $\mathbf{T}_p = f_p$ is admissible.

Proof: Clearly, $f_p(\boldsymbol{\xi}^+) \geq \sigma K\widetilde{R}^2$. Also, by the Cauchy–Schwarz inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t} f_p = -2\zeta_p^\top \dot{\zeta}_p \geq -4\Delta(\mathcal{G})Rr(R)Z,$$

where $\|\zeta_p\| \leq 2\Delta(\mathcal{G})Rr(R)$ and $\|\dot{\zeta}_p\| \leq Z$. ∎

Alternative trigger structures are made possible by the task-informed approach enabled by the combination of Theorems 1 and 2 and Lemma 6. While a self-trigger based on the rate bound of the ζ_p is naturally conservative, it is but only one way to satisfy the sufficient conditions derived herein. To see this, let $0 < \beta \leq \sigma K\widetilde{R}^2$ and $\gamma > 0$. Consider the triggers

$$\begin{aligned}\mathbf{T}_{p,1}(\boldsymbol{\xi}) &= \|\eta_p\|^2 - \|\zeta_p\|^2 + \alpha(\|\eta_p\|), \\ \mathbf{T}_{p,2}(\boldsymbol{\xi}) &= \|\eta_p\|^2 - Z^2\tau_p^2 + \alpha(\|\eta_p\|),\end{aligned} \tag{56}$$

where

$$\alpha(s) \triangleq \begin{cases} \beta - \frac{\beta}{\gamma}s, & s \in [0,\gamma] \\ 0, & s \geq \gamma. \end{cases} \tag{57}$$

Remark 8: $\mathbf{T}_{p,1}$ is state based, whereas $\mathbf{T}_{p,2}$ is a self-trigger.

Corollary 7: $\mathbf{T}_{p,1}$ and $\mathbf{T}_{p,2}$ are admissible and satisfy (41).

Proof: $\mathbf{T}_{p,1}$ and $\mathbf{T}_{p,2}$ share the same $\mathfrak{h} \triangleq \min_{s\geq 0}(s^2 + \alpha(s))$. Indeed, $s^2 + \alpha(s)$ coincides with s^2 for $s \geq \gamma$, so, being positive, it has a minimum. By Corollary 6, $\mathbf{T}_{p,1}$ is admissible because $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}_{p,1} = \frac{\mathrm{d}}{\mathrm{d}t}f_p$. It also trivially satisfies (41). Finally, $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}_{p,2} = -2Z^2\tau_p$, where $Z^2\tau_p^2 \leq \|\eta_p\|^2 + \alpha(\|\eta_p\|)$ with $\|\eta_p\|$ bounded by (54). Equation (41) holds for $\mathbf{T}_{p,2}$ because $\mathbf{T}_{p,2} \leq \mathbf{T}_{p,1}$ by Lemma 6. ∎

[15]Similarly to Condition (‡), imposing the identity $\rho(\mathbf{x}, \boldsymbol{\eta}, \boldsymbol{\tau}) = \rho(\mathbf{x} + \boldsymbol{\varrho}, \boldsymbol{\eta}, \boldsymbol{\tau})$ on a perturbation $\rho : \mathcal{X}_\# \to \mathbb{R}_{\geq 0}$ is not overly restrictive in practice, since, in the current problem, measurement uncertainties and computational errors are not expected to depend on the displacement of the MAS relative to any particular choice of origin for the global coordinate system.

[16]Once again, the choice of system $\mathcal{H}$, $\mathcal{K}$, or $\mathcal{L}$ does not matter, since Lemma 6 only discusses the evolution of solutions over flow intervals.

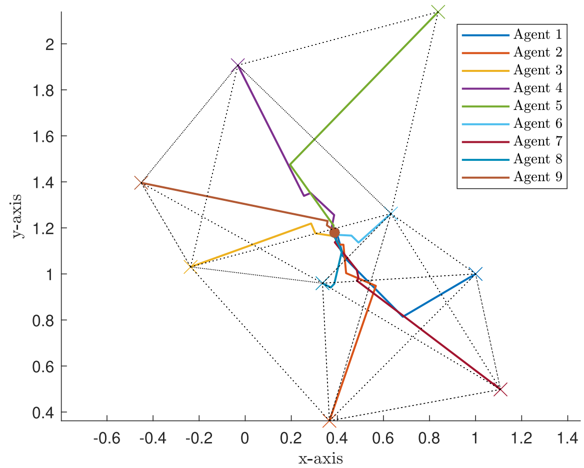

Fig. 3. Trajectories of a MAS with $N = 9$ agents, communication radius $R = 1$, a random initial configuration (denoted by the $\times$'s), and a connected initial graph $\mathcal{G}$. The nodes and edges of $\mathcal{G}$ are represented by the $\times$'s and black-dashed lines, respectively. $\mathcal{G}$ has $|\mathcal{E}| = 20$ and $\lambda_2(\mathcal{G}) = 1.578$. The final configuration (represented by the $\bullet$'s) were required to be in ν-approximate rendezvous with $\nu = 0.1$. The parameter $\widetilde{R}$ is selected as the length of the longest edge of $\mathcal{G}$ plus the machine epsilon of our computer (10^{-16}), which yields $\widetilde{R} = 0.9392$, corresponding to $\varepsilon = 0.0608$. Note that $\omega = 2541.5$ and $\sigma = 1.8863 \times 10^{-6}$ (see Theorem 2). With the gain $\mu = 1$, approximate rendezvous was achieved within four time units. The triggers $\mathbf{T}_{p,1}$ were used with $\beta = 3.8298 \times 10^{-13}$ and γ set to $\gamma = 0.1$ [see (57)]. Note how each cusp along an agent's trajectory coincides with a jump time for that agent. The simulation was executed in MATLAB, using [25]. Also note that the parameter β is well above machine epsilon for a single-precision float. Moreover, the plots in Figs. 4 and 5 demonstrate that the functions $\alpha(\|\eta_p\|)$ from (56) repeatedly make nonnegligible positive contributions to the values of the triggers $\mathbf{T}_p$ in spite of the seemingly small size of β, whose contribution to the trigger is additive.

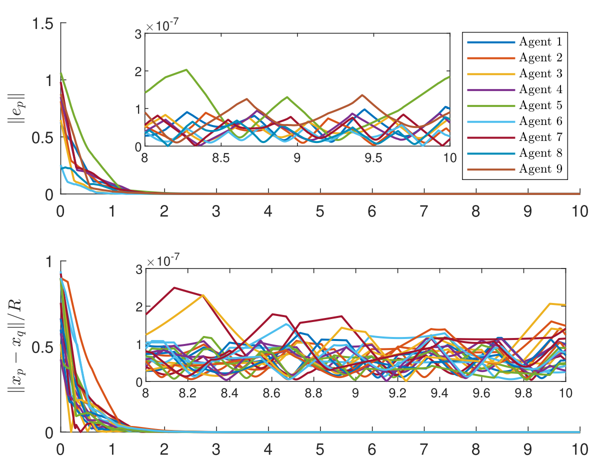

Fig. 4. Evolution of the rendezvous error $\|e_p\| = \|x_p - \frac{1}{N}\sum_{q\in\mathcal{V}} x_q\|$ for each agent $p \in \mathcal{V}$ (top), and edge lengths normalized by the communication radius R for each $pq \in \mathcal{E}$ (bottom). Note that the ν-approximate rendezvous objective is achieved, where the steady state values of the normalized edge lengths are several orders of magnitude smaller than $\nu = 0.1$. Hence, the control strategy can produce better performance than the expected theoretical result, highlighting the conservativeness of the Lyapunov analysis. Observe that all normalized edge lengths are strictly below 1, implying that no edge of the initial graph is ever broken. Note that the bottom plot is not color-matched to the other plots: since the colors correspond to communication edges and not agents.

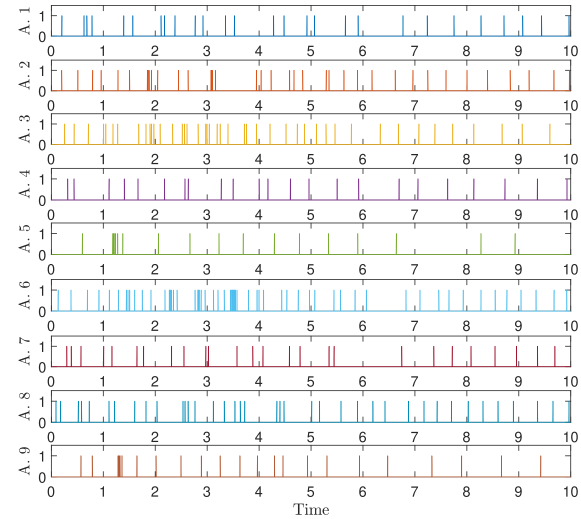

Fig. 5. Flow intervals and jump times for each agent. Note that "A. 1" means Agent 1, "A. 2" means Agent 2, etc. Spikes represent jump times, while white spaces correspond to flow intervals. Recall that, to update its controller at a jump time, an agent can either measure the positions of its neighbors and itself simultaneously or the displacement between itself and its neighbors. In comparison with approximate rendezvous strategies that require continuous measurements, our result employs much fewer measurements to achieve approximate rendezvous. Finally, note how the communication events seem less frequent in the second half of the simulation, after the objective had been reached. This, too, points to the fact that the value of β may not be regarded as negligibly small (see caption to Fig. 3).

On first glance, the event trigger $\mathbf{T}_{p,1}$ requires agent p to continuously measure $x_q - x_p$ for each $q \in \mathcal{N}_p$. This could be useful in settings where intermittent actuation is desired and continuous measurements are inexpensive, such as with satellite constellations where power is limited and visual measurements can be made over large distances. For other scenarios, an alternative communication protocol can remove the need for continuously monitoring the neighbors' states. Recalling that $\dot{x}_p = \eta_p$ is constant over $[t_i^p, t_{i+1}^p)$, observe that $x_p(t) = \eta_p\tau_p + x_p(t_i^p)$ for all $t \in [t_i^p, t_{i+1}^p)$. Under the assumption of instantaneous communication, it suffices for each agent $q \in \mathcal{N}_p$ to broadcast η_q and x_q at each jump time of agent q. Consequently, agent p can compute ζ_p by using the solution for $x_q(t)$ over the appropriate time interval for each $q \in \mathcal{N}_p \cup \{p\}$.

IX. Implementation and Simulation

Putting together the theoretical results, suppose that we are given N agents with $\mathbf{x}(0)$ satisfying Assumption 1, and $\nu > 0$. At the initial time, $\mathcal{G} = \mathcal{G}_R(\mathbf{x}(0))$, $\lambda_2(\mathcal{G})$, and $|\mathcal{E}|$ are computed. The parameter ε is selected so that $\widetilde{R} > \|\Delta\mathbf{x}(0)\|_\infty$, and ω is selected to satisfy (10). For any choice of $\mu, \gamma > 0$, σ is selected according to Theorem 2, guaranteeing that *any* collection $\{\mathbf{T}_p\}_{p\in\mathcal{V}}$ of admissible triggers generates a controller driving the MAS to ν'-approximate rendezvous at an exponential rate, uniformly over initial conditions $\phi(0,0) \in \mathcal{C}_{\widetilde{R}}(\mathcal{G})$ satisfying (24). Theorem 1 guarantees that no edges of $\mathcal{G}$ are broken at any time for any of the above initial conditions.

Fig. 3 shows the trajectories of a MAS with $N = 9$ agents in the Euclidean plane $\mathbb{R}^2$, with randomized initial positions and with $\mathbf{T}_{p,1}$ (with the default choice of $\beta = \sigma K \widetilde{R}^2$) serving as the event trigger function for each agent. Fig. 4 depicts the evolution of the rendezvous error for each agent (top plot) and the edge lengths normalized by the communication radius R for each edge of the initial graph $\mathcal{G}$ (bottom plot). The rendezvous error of agent p is given by $e_p \triangleq x_p - \frac{1}{N}\sum_{q\in\mathcal{V}} x_q$. Fig. 5 illustrates the event times for each agent, i.e., the times agent p updated their auxiliary control variable η_p for each $p \in \mathcal{V}$. Notice that the event times are intermittent and occur asynchronously in the MAS.

X. Conclusion

Distributed controllers with intermittent distance-limited communication and communication graph maintenance were developed for the rendezvous problem. Applying the HDI framework of [19] enables an especially simple and systematic approach to the analysis of the closed-loop system, tying together personal clocks, triggering conditions, stability, and topological properties of solutions. This results in a family of controllers, which yield complete, Zeno-free solutions while maintaining the initial graph structure and keeping the prescribed approximate rendezvous set exponentially stable with respect to continuous time. In contrast with [12], where the growth of the edge potentials is insufficient for providing a graph maintenance guarantee for all initial configurations with a connected communication graph, and where this guarantee shrinks as the number of agents grows, our design fits any such configuration with a controller capable of maintaining the initial graph. In addition, a number of questions regarding the modeling of ETC systems with multiple independent triggers were considered, in an extension of the work in [21], establishing the sufficiency of no-flow conditions for ensuring that jumps do occur at event times. The proposed alternative models were shown to have many of the desired characteristics of the [21]-style model (e.g., non-Zeno behavior) even without introducing no-flow conditions, with solutions differing only in their jump structures. Namely, the desired attractor is robustly stable for all the considered designs under mild conditions, and solutions only differ in whether or not the corresponding hybrid time domains contain sequences of zero-time jumps and in the ordering of such sequences, resulting in solutions (for the different models) that coincide everywhere, with respect to continuous time, except for the set of jump times. These differences are, therefore, the only source of the failure of nominal well-posedness in the alternative models, motivating a search for a coarser topology[17] on the space of maximal solutions that would treat all the considered models as equivalent models of the same control design, at least in the presence of no-flow conditions. Further motivation for such research is provided by the results of Section VII on the retention of robustness properties by controller designs modified in the way proposed in this article (at the expense of satisfying HBC).

Returning to the network control problem, future research may address less conservative triggers, heterogeneous and higher order agent dynamics, as well as modeling exogenous disturbances by set-valued flow maps.

[17]That is, coarser than the graphical convergence topology.

Appendix A
Proof of (21)

Proof: Denoting $\tilde{\mathbf{x}} \triangleq (y)_{p\in\mathcal{V}}$, there exist $p, q \in \mathcal{E}$ such that

$$\begin{aligned}\|\Delta\mathbf{x}\|_\infty^2 &= \|x_p - x_q\|^2 = \|x_p - y + y - x_q\|^2\\ &\le (\|x_p - y\| + \|y - x_q\|)^2\\ &\le \|x_p - y\|^2 + 2\|x_p - y\|\|y - x_q\| + \|y - x_q\|^2\\ &\le 2\|x_p - y\|^2 + 2\|y - x_q\|^2 \le 2\sum_{i\in\mathcal{V}}\|x_i - y\|^2\\ &\le 2\|\mathbf{x} - \tilde{\mathbf{x}}\|^2 = 2\|\mathbf{x}^\perp\|^2.\end{aligned}$$

Next, we prove the claim that $y = \frac{1}{N}\sum_{p\in\mathcal{V}} x_p \triangleq z$. Let $\mathbf{L}$ be the vector Laplacian for the complete graph on $\mathcal{V}$. Then,

$$(\mathbf{L}\mathbf{x})_p = \sum_{q\in\mathcal{V}}(x_p - x_q) = N x_p - \sum_{q\in\mathcal{V}} x_q = N(x_p - z).$$

We conclude that $\mathbf{L}\mathbf{x} = N(\mathbf{x} - \mathbf{w})$, where $\mathbf{w} = (z)_{p\in\mathcal{V}}$. Therefore, since $\mathbf{L}$ is self-adjoint, we have that $(\mathbf{x} - \mathbf{w}) \in \ker(\mathbf{L})^\perp$. Since $\mathbf{w} \in \ker(\mathbf{L})$, $\mathbf{x} = (\mathbf{x} - \mathbf{w}) + \mathbf{w}$ is the orthogonal decomposition of $\mathbf{x}$ that we seek by the uniqueness of orthogonal decompositions. In particular, $z = y$, as required. Finally, to prove the last inequality, use our formula for $\tilde{\mathbf{x}}$:

$$\begin{aligned}\|\mathbf{x}^\perp\|^2 &= \|\mathbf{x} - \tilde{\mathbf{x}}\|^2 = \textstyle\sum_{p\in\mathcal{V}}\|x_p - y\|^2\\ &= \textstyle\sum_{p\in\mathcal{V}}\left\|\frac{1}{N}\sum_{q\in\mathcal{V}}(x_p - x_q)\right\|^2\\ &= \textstyle\frac{1}{N^2}\sum_{p\in\mathcal{V}}\sum_{q,q^*\in\mathcal{V}}\langle x_p - x_q, x_p - x_{q^*}\rangle\\ &\le \textstyle\frac{1}{N^2}\sum_{p\in\mathcal{V}}\sum_{q,q^*\in\mathcal{V}}|\langle x_p - x_q, x_p - x_{q^*}\rangle|\\ &\le \textstyle\frac{1}{N^2}\sum_{p\in\mathcal{V}}\sum_{q,q^*\in\mathcal{V}}\|x_p - x_q\|\|x_p - x_{q^*}\|\\ &= \textstyle\frac{1}{N^2}\sum_{p\in\mathcal{V}}\sum_{q\in\mathcal{V}}\|x_p - x_q\|\sum_{q^*\in\mathcal{V}}\|x_p - x_{q^*}\|\\ &\le \textstyle\frac{1}{N^2}\sum_{p\in\mathcal{V}}(N\cdot\max_{q\in\mathcal{V}}\|x_p - x_q\|)^2\\ &\le \textstyle\frac{1}{N^2}\cdot N^3\cdot\max_{p,q\in\mathcal{V}}\|x_p - x_q\|^2\\ &= N\|\Delta\mathbf{x}\|_\infty^2.\end{aligned}$$

■

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